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OSCILLATION OF ODD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATION

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1. Introduction

Biological models have provided many examples of differential equations with retarded arguments. The construction of a biological model using delays has been paralleled by the mathematical investigation of the nonlinear delay equations. Such equations also appear in control theory, economics, physics, ecology, etcetera. Recently the study of oscillatory behaviour of solutions of odd order differential equations is a subject of great interest in which the retardation is only responsible for oscillation. For example, the first order delay differential equation

$$x'(t) - x(t - \tfrac{\pi}{2}) = 0$$

has oscillatory solutions $x_1(t) = \sin t$ and $x_2(t) = \cos t$, although all solutions of the corresponding ordinary differential equation

$$x'(t) + x(t) = 0$$

are nonoscillatory.

The results of this paper was motivated by certain recent results due to Gopalsamy, et al. [3], Graef et al. [4] and Zhang [8].

In [3], authors proved that if

$$(\mathrm{C}_1) \qquad Q \in C([a,\infty), \mathbb{R}^+), \quad \mathbb{R}^+ = (0,\infty), \quad \tau, \sigma \in (0,\infty), \quad p \in [0,1)$$

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and

$$(\mathrm{C}_2) \qquad \liminf_{t\to\infty} \int_{t-\sigma}^{t} (t-s)^{n-1} Q(s)\,\mathrm{d}s > (1-p)(n-1)!,$$

then every solution of the *odd order* neutral delay differential equation

$$(\mathrm{E}_1) \qquad \big(x(t) - px(t-\tau)\big)^{(n)} + Q(t)x(t-\sigma) = 0,$$

oscillates, that is, every solution $x(t)$ of (E_1) has zeros for arbitrarily large t. When $Q(t) = q \in (0,\infty)$, then (C_2) reduces to

$$(\mathrm{C}_3) \qquad q\sigma^n > (1-p)(n)!.$$

In this paper, we prove the same result in which (C_2) may be replaced by the conditions

$$(\mathrm{C}_4) \qquad \liminf_{t\to\infty} \int_{t-\sigma}^{t} Q(s)\,\mathrm{d}s > \frac{1-p}{\mathrm{e}} \quad \text{for } n = 1,$$

$$\liminf_{t\to\infty} \int_{t-\frac{\sigma}{n}}^{t} \sigma^{n-1} Q(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}(1-p)\Big(\frac{n}{n-1}\Big)^{n-1}(n-1)! \quad \text{for } n > 1.$$

Clearly, when $Q(t) = q \in (0,\infty)$, (C_4) becomes

$$q\sigma > (1-p)/\mathrm{e} \quad \text{for } n = 1 \qquad \text{and}$$
$$q\sigma^n > \frac{1}{\mathrm{e}}(1-p)\Big(\frac{n}{n-1}\Big)^{n-1}(n)!, \quad \text{for } n > 1,$$

which in view of the inequality

$$\begin{aligned} \frac{1}{\mathrm{e}}\Big(\frac{n}{n-1}\Big)^{n-1} &= \frac{1}{\mathrm{e}}\Big(\sum_{r=0}^{n-1} C(n-1,r)\Big(\frac{1}{n-1}\Big)^{r}\Big) \\ &\leqslant \frac{1}{\mathrm{e}}\Big(\mathrm{e} - \frac{1}{(n)!}\Big) < 1, \end{aligned}$$

where $C(n-1,r)$ is the $(r+1)$ th binomial coefficient of the expansion $\big(\frac{n}{n-1}\big)^{n-1}$, is a weaker condition than that of (C_3).

In [4], Graef et al. proved that if (C_1) is satisfied, $f\colon \mathbb{R} \to \mathbb{R}$ such that $xf(x) > 0$ for $x \neq 0$ and f is increasing, then every solution of

$$\big(x(t) - px(t-\tau)\big)' + Q(t)f\big(x(t-\sigma)\big) = 0,$$

oscillates if

$$\int^{\infty} Q(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^{\pm\alpha} \frac{\mathrm{d}s}{f(s)} < \infty \quad \text{for every } \alpha > 0.$$

Here we prove the same result for a general odd order neutral delay differential equation with several deviating arguments. A result of Zhang has been generalized here.

2. Sufficient conditions for oscillation

Consider the differential equations

$$\left(x(t) - \sum_{j=1}^{K} p_j x(t-\tau_j)\right)^{(n)} + \sum_{i=1}^{m} Q_i(t) x(t-\sigma_i) = 0, \tag{E_2}$$

and

$$\left(x(t) - \sum_{j=1}^{K} p_j x(t-\tau_j)\right)^{(n)} + \sum_{i=1}^{m} Q_i(t) f_i\big(x(t-\sigma_i)\big) = 0, \tag{E_3}$$

where

(C_5) p_j, $\tau_j \in [0,\infty)$ $(j = 1,2,\ldots,K)$, $Q_i \in C([a,\infty),(0,\infty))$, $a \in \mathbb{R}$, $Q_i(t) > 0$ almost everywhere $\sigma_i \in (0,\infty)$ $(i = 1,2,\ldots,m)$, n is odd and $\sum_{j=1}^{K} p_j < 1$. Denote $p = \sum_{j=1}^{K} p_j$,
$\tau_0 = \min_{j=1}^{K} \tau_j$, $\tau = \max_{j=1}^{K} \tau_j$, $\sigma_0 = \min_{i=1}^{m} \sigma_i$ and $\sigma = \max_{i=1}^{m} \sigma_i$.

(C_6) $f_i\colon \mathbb{R} \to \mathbb{R}$ such that f_i are nondecreasing and $x f_i(x) > 0$ for $x \neq 0$ $(i = 1,2,\ldots,m)$.

By a solution of (E_2) ((E_3)) in $[a,\infty)$ with initial function θ, we mean a function $x \in C([a-\alpha,\infty),\mathbb{R})$, $\alpha = \max\{\tau,\sigma\}$, such that $x(t) = \theta(t)$ for $t \in [a-\alpha, a]$, $x(t)$ is continuable in $[a,\infty)$ and satisfies (E_2) ((E_3)). Such a continuable solution $x(t)$ is said to be oscillatory if it has zeros for arbitrarily large t. Otherwise, we call it nonoscillatory.

Lemma 1. [2, Lemma 1]. *Suppose that $f \in C^{(n)}([T,\infty),(0,\infty))$ such that $f^{(i)}(t)$ has no zeros in $[T,\infty)$ for $i = 1,2,\dots,(n-1)$ and $f^{(n)}(t) \leqslant 0$, $t \geqslant T$. If $\alpha > 0$ then*

$$f(t-\alpha) \geqslant \frac{\alpha^{n-1}}{(n-1)!} f^{(n-1)}(t), \quad t \geqslant T + 2\alpha.$$

Theorem 1. *Suppose that* (C_5) *hold. If $n = 1$ and all solutions of*

$$x'(t) + \lambda_1 \sum_{i=1}^{m} Q_i(t)x(t-\sigma_i) = 0 \tag{1}$$

are oscillatory for some

$$\lambda_1 \in \big(0, 1/(1-p)\big), \tag{2}$$

then all solutions of (E_2) *are oscillatory. If $n > 1$ and all solutions of*

$$x'(t) + \lambda_2 \sum_{i=1}^{m} \sigma_i^{n-1} Q_i(t)x(t-\sigma_i/n) = 0 \tag{3}$$

are oscillatory for some

$$\lambda_2 \in \left(0, \frac{1}{(n-1)!(1-p)}\left(\frac{n-1}{n}\right)^{n-1}\right), \tag{4}$$

then all solutions of (E_2) *are oscillatory.*

Proof. Contrary to the theorem, suppose that $x(t)$ is a nonoscillatory solution of (E_2). Without any loss of generality, assume that $x(t) > 0$ for $t \geqslant t_0$. Set

$$z(t) = x(t) - \sum_{j=1}^{K} p_j x(t-\tau_j). \tag{5}$$

From (E_2) it follows that $z^{(n)}(t) \leqslant 0$, $t \geqslant t_0 + \sigma$ almost everywhere and consequently, there exists a $T \geqslant t_0 + \sigma$ such that $z^{(i)}(t)$ has no zeros in $[T,\infty)$ for $i = 0,1,2,\dots,(n-1)$. Therefore, $z(t) < 0$, or $z(t) > 0$, $t \geqslant T$. We shall reach to some contradiction in both cases. Suppose that $z(t) < 0$, $t \geqslant T$. Since n is odd, it follows that $z'(t) < 0$, $t \geqslant T$. Let $\limsup\limits_{t\to\infty} x(t) = \mu$. If $\mu = \infty$, then $x(t)$ is unbounded and hence there exists a sequence $\langle t_n\rangle_{n=1}^{\infty}$ such that $t_n \to \infty$, $x(t_n) \to \infty$ as $n \to \infty$ and $x(s) < x(t_n)$ for $s < t_n$. From (5) we have $z(t_n) \geqslant (1-p)x(t_n)$ and consequently, $z(t_n) \to \infty$ as $n \to \infty$, a contradiction. If $0 \leqslant \mu < \infty$, there exists a sequence $\langle t_n\rangle_{n=1}^{\infty}$

such that $t_n \to \infty$ and $x(t_n) \to \mu$ as $n \to \infty$. Clearly, the bounded sequence of real numbers $\langle x(t_n - \tau_1)\rangle_{n=1}^{\infty}$ admits a convergent subsequence say, $\langle x(t_{n(1)} - \tau_1)\rangle_{n(1)=1}^{\infty}$ which converges to $\mu_1 \leqslant \mu$. Again $\langle x(t_{n(1)} - \tau_2)\rangle$ is a bounded sequence of real numbers, so it admits a convergent subsequence say, $\langle x(t_{n(2)} - \tau_2)\rangle_{n(2)=1}^{\infty}$, which converges to some $\mu_2 \leqslant \mu$. Since $\langle t_{n(2)}\rangle$ is a subsequence of $\langle t_{n(1)}\rangle$, it follows that $\langle x(t_{n(2)} - \tau_1)\rangle$ converges to μ_1. Proceeding in this way upto K time we can find a subsequence $\langle x(t_{n(K)} - \tau_K)\rangle_{n(K)=1}^{\infty}$ which converges to a number $\mu_K \leqslant \mu$. Clearly $\langle x(t_{n(K)} - \tau_j)\rangle_{n(K)=1}^{\infty}$ converges to $\mu_j \leqslant \mu$ $(j = 1, 2, \ldots, K)$. From (5) we see that $z(t_{n(K)}) \to \mu - \sum\limits_{j=1}^{K} p_j\mu_j$ as $n(K) \to \infty$. But $\mu - \sum\limits_{j=1}^{K} p_j\mu_j \geqslant \mu(1-p) > 0$ leads a contradiction to the fact that z is negative and decreasing in $[T, \infty)$. Hence $z(t) < 0$, $t \geqslant T$ is not possible.

Next, suppose that $z(t) > 0$, $t \geqslant T$. Here we claim that $z'(t) \leqslant 0$, $t \geqslant T$. If not, assume that $z'(t) > 0$, $t \geqslant T$. Consequently,

$$\liminf_{t\to\infty} x(t) = \liminf_{t\to\infty} \left(z(t) + \sum_{j=1}^{K} p_j x(t - \tau_j) \right) \geqslant \liminf_{t\to\infty} z(t) > 0. \tag{6}$$

From Theorem 2.7.4 [6] it follows that all solutions of either of the equations (1) or (3) are oscillatory implies that

$$\int_a^{\infty} \left(\sum_{i=1}^{m} Q_i(s) \right) \mathrm{d}s = \infty. \tag{7}$$

Integrating (E_2) from T to t and then using (6) and (7) we see that $z^{(n-1)}(t) \to -\infty$ as $t \to \infty$. Consequently, $z(t) \to \infty$ as $t \to \infty$. This contradiction proves our claim.

Denote $J = \{1, 2, \ldots, K\}$. From (5), it follows that

$$\begin{aligned} x(t) &= z(t) + \sum_{j=1}^{K} p_j x(t - \tau_j) \\ &\geqslant z(t) + p x(t - \tau_{k_1}), \quad t \geqslant T + \tau, \end{aligned} \tag{8}$$

for some $k_1 \in J$ which depends on t. Replacing t by $t - \tau_{k_1}$ in (8) and then using the resulting inequality in (8) we see that for $t \geqslant T + 2\tau$,

$$\begin{aligned} x(t) &\geqslant z(t) + p z(t - \tau_{k_1}) + p \sum_{j=1}^{K} p_j x(t - \tau_j - \tau_{k_1}) \\ &\geqslant z(t) + p z(t - \tau_{k_1}) + p^2 x(t - \tau_{k_2} - \tau_{k_1}), \end{aligned}$$

for some $k_2 \in J$. Proceeding successively, for any positive integer N, we get

$$x(t) \geqslant z(t) + \sum_{i=1}^{N-1} p^i z\Big(t - \sum_{j=1}^{i} \tau_{k_j}\Big) + p^N x\Big(t - \sum_{j=1}^{N} \tau_{k_j}\Big), \tag{9}$$

for some $k_j \in J$ $(j = 1, 2, \ldots, N)$ and $t \geqslant T + N\tau$. From (9) and the fact that z is decreasing we obtain

$$x(t) \geqslant z(t)\{1 + p + p^2 + \ldots, + p^{N-1}\}, \quad t \geqslant T + N\tau,$$

that is,

$$x(t) \geqslant z(t)\Big\{\frac{1 - p^N}{1 - p}\Big\}, \quad t \geqslant T + N\tau. \tag{10}$$

We shall obtain separate contradictions for the cases $n = 1$ and $n > 1$, respectively.

If $n = 1$, choose N large enough such that $x(t) > \lambda_1 z(t)$ and then from (E_2) we see that $z(t)$ is an eventually positive solution of the differential inequality

$$z'(t) + \lambda_1 \sum_{i=1}^{m} Q_i(t) z(t - \sigma_i) \leqslant 0,$$

which by Theorem 2.2 [3] implies that Eq. (1) has a nonoscillatory solution, a contradiction to our assumption.

Suppose that $n > 1$. From Lemma 1, it follow that

$$z(t - \sigma_i) > \frac{1}{(n-1)!}\Big(\frac{n-1}{n}\sigma_i\Big)^{n-1} z^{(n-1)}(t - \sigma_i/n), \tag{11}$$

for $i = 1, 2, \ldots, m$ and $t \geqslant T + 2\sigma$. Set

$$\beta = \lambda_2 \Big(\frac{n}{n-1}\Big)^{n-1} (n-1)! \tag{12}$$

From (4), it follows that $\beta < 1/(1-p)$ and hence in (10) choose N large enough such that $(1 - p^N)/(1 - p) > \beta$. Consequently, from (10) we obtain

$$x(t) > \beta z(t), \quad t \geqslant T + N\tau,$$

that is,

$$x(t - \sigma_i) > \beta z(t - \sigma_i), \quad t \geqslant T + N\tau + \sigma_i \quad (i = 1, 2, \ldots, m). \tag{13}$$

From (11), (12) and (13) we obtain

$$x(t-\sigma_i) > \lambda_2(\sigma_i^{n-1})z^{(n-1)}(t-\sigma_i/n), \quad t \geqslant T + N\tau + 3\sigma. \tag{14}$$

Using (14) in (E_2) we see that $z^{(n-1)}(t)$ is an eventually positive solution of the first order differential inequality.

$$y'(t) + \lambda_2 \sum_{i=1}^{m} \sigma_i^{n-1} Q_i(t) y(t-\sigma_i/n) \leqslant 0. \tag{15}$$

Consequently, by Theorem 2.2 of [3] and the remark following it, we see that the equation associated with (15), that is, Eq. (3) admits a nonoscillatory solution, which again is a contradiction to our assumption. This completes the proof. □

Remark 1. When (C_5) hold, $n = 1$, $Q_i(t) = q_i \in \mathbb{R}^+$ $(i = 1, 2, \ldots, m)$ and $K = 1$, Zhang [8] proved that any one of the conditions

$$\sum_{i=1}^{m} q_i \sigma_i > \frac{(1-p_1)}{e} \quad \text{or} \quad \Big(\prod_{i=1}^{m} q_i\Big)^{1/m} \Big(\sum_{i=1}^{m} \sigma_i\Big) > \frac{(1-p_1)}{e}$$

implies that all solutions of (E_2) are oscillatory. In view of Proposition 3 due to Arino, et al [1], Theorem 1 is a generalization of the above observations.

Corollary 1. *Suppose that* (C_5) *holds and* $n > 1$. *If either of the conditions*

$$\liminf_{t\to\infty} \int_{t-\sigma_0/n}^{t} \Big(\sum_{i=1}^{m} \sigma_i^{n-1} Q_i(s)\Big)\,ds > \frac{1}{e}(1-p)\Big(\frac{n}{n-1}\Big)^{n-1}(n-1)! \tag{15}$$

or

$$\limsup_{t\to\infty} \int_{t-\sigma_0/n}^{t} \Big(\sum_{i=1}^{m} \sigma_i^{n-1} Q_i(s)\Big)\,ds > (1-p)\Big(\frac{n}{n-1}\Big)^{n-1}(n-1)! \tag{16}$$

are satisfied, then all solutions of (E_2) *are oscillatory.*

Proof. If (15) or (16) hold, then there exists $\mu \in (0,1)$ such that the term $\sum\limits_{i=1}^{m} \sigma_i^{n-1} Q_i(s)$ in (15) and (16) can be replaced by the term $\mu\big(\sum\limits_{i=1}^{m} \sigma_i^{n-1} Q_i(s)\big)$. By Theorem 2.7.1 and Remark 2.7.3 of [6], it follows that either of the resulting conditions implies that all solutions of

$$z'(t) + \lambda \sum_{i=1}^{m} \sigma_i^{n-1} Q_i(s) z(t-\sigma_i/n) = 0$$

are oscillatory, where

$$\lambda = \frac{\mu(\frac{n-1}{n})^{n-1}}{(1-p)(n-1)!}.$$

Hence by Theorem 1 all solutions of (E_2) are oscillatory. □

Remark 2. When $m = 1$ and $K = 1$, the motivation given in introduction follows from the above corollary.

Example 1. The equation

$$\left(x(t) - \tfrac{1}{2}x(t-1)\right)^{(3)} + 3(\tfrac{19}{20} + \mathrm{e}^{-t})x(t-1) = 0, \quad t \geqslant 1,$$

satisfies the hypotheses of Corollary 1 with (15). Hence every solution of it oscillates. Theorem 4.1 due to Gopalsamy, et al [3] is not applicable to it.

In the following we prove a theorem which generalizes a main result due to Zhang [8].

Theorem 2. *Suppose that* (C_5) *holds,* $Q_i(t) = Q_i \in (0, \infty)$ *and* $\tau \leqslant \sigma_0$. *If*

$$\left(\frac{\tau_0}{n}\right)^n \left(\sum_{i=1}^m Q_i\right) > F(\lambda), \tag{17}$$

where

$$F(\lambda) = \frac{1}{\lambda}(1 - p\lambda)^{n+1}$$

and λ *is the unique real root of the equation.*

$$n(1-p)y = \ln(y), \quad 1 \leqslant y \leqslant 1/p, \tag{18}$$

then every solution of (E_2) *oscillates.*

Proof. It is known from [7] that every solution of (E_2) oscillates if and only if the associated characteristic equation.

$$G(\mu) = -\mu^n \left(1 - \sum_{j=1}^K p_j \mathrm{e}^{\mu \tau_j}\right) + \sum_{i=1}^m Q_i \mathrm{e}^{\mu \sigma_i} = 0 \tag{19}$$

has no real roots. Clearly, $G(\mu) > 0$ for $\mu \leqslant 0$. We claim that $G(\mu) > 0$ for $\mu > 0$. If not, let $G(\mu) = 0$ for some $\mu > 0$. Multiplying (19) throughout by $\exp(-\mu\tau_0)(\tau_0/n)^n$ and simplifying we get

$$\begin{aligned} (\tau_0/n)^n \left(\sum_{i=1}^m Q_i\right) &\leqslant (\tau_0/n)\exp(-\mu\tau_0)\mu^n \left(1 - \sum_{j=1}^K p_j \mathrm{e}^{\mu\tau_j}\right) \\ &\leqslant (\tau_0/n)\exp(-\mu\tau_0)\mu^n (1 - p\mathrm{e}^{\mu\tau_0}). \end{aligned}$$

Put $x(t) = \exp(\mu\tau_0)$ in the above inequality to get

$$(20) \qquad (\tau_0/n)^n \Big(\sum_{i=1}^{m} Q_i\Big) \leqslant \Big(\frac{1}{n}\Big)^n \frac{(\ln x)^n (1-px)}{x}.$$

Setting

$$(21) \qquad R(x) = \frac{(\ln x)^n (1-px)}{x},$$

wee see that the maximum of $R(x)$ for $x > o$ attains at $x = \lambda$, where λ is the unique real root of (18). Hence

$$(22) \qquad R(x) \leqslant \frac{(\ln\lambda)^n (1-p\lambda)}{\lambda}.$$

Again from (18) we get

$$(23) \qquad \ln\lambda = n(1-p\lambda).$$

From (20), (21), (22) and (23) we obtain

$$(\tau_0/n)^n \Big(\sum_{i=1}^{m} Q_i\Big) \leqslant \frac{1}{\lambda}[1-p\lambda]^{n+1},$$

a contradiction to (17). Hence the proof is completed. □

Remark 2. When $n = 1$, $K = 1$, the above theorem was proved by Zhang in [8].

Theorem 3. *Suppose that* (C_5), (C_6) *hold and for some* $k \in [1, 2, \dots, m]$,

$$(24) \qquad \int^{\infty} Q_k(s)\,\mathrm{d}s = \infty,$$

$$(25) \qquad \int_0^{\pm\alpha} \frac{\mathrm{d}s}{f_k(s)} < \infty \quad \textit{for every } \alpha > 0,$$

then every solution of (E_3) *oscillates.*

Proof. If possible, suppose that $x(t)$ is a nonoscillatory solution of (E_3). Suppose that $x(t) > 0$ for $t \geqslant t_0$. The case when $x(t) < 0$, $t \geqslant t_0$ may be treated similarly. Set $z(t)$ as in (5). From (E_3), it follows that $z^{(n)}(t) \leqslant 0$, $t \geqslant t_0 + \sigma$. Consequently, there exists a $T \geqslant t_0 + \sigma$ such that $z^{(i)}(t)$ has no zeros in $[T, \infty)$ for

$i = 0, 1, 2, \ldots, (n-1)$. If $z(t) < 0$, $t \geqslant T$ we proceed in the lines of Theorem 1 and reach at a contradiction. Thus $z(t) > 0$, $t \geqslant T$. By Lemma 1 we get

$$z(t-\sigma_i) \geqslant \frac{\sigma_i^{n-1}}{(n-1)!} z^{(n-1)}(t) \geqslant \mu z^{(n-1)}(t), \quad t \geqslant T + 2\sigma, \tag{26}$$

where

$$\mu = \min_{i=1}^{n} \left\{ \frac{\sigma_i^{n-1}}{(n-1)!} \right\}.$$

From (26) and the fact that $z(t) \leqslant x(t)$, it follows that

$$x(t-\sigma_i) \geqslant \mu z^{(n-1)}(t), \quad t \geqslant T + 2\sigma \quad (i = 1, 2, \ldots, m). \tag{27}$$

Using (E_3), (C_6) and (27) we get

$$z^{(n)}(t) + Q_k(t) f_k\big(\mu z^{(n-1)}(t)\big) \leqslant 0. \tag{28}$$

Set $T_0 = T + 2\sigma$. Dividing (28) throughout by $f_k\big(\mu z^{(n-1)}(t)\big)$ and then integrating the resulting inequality from T_0 to t we obtain

$$-\int_{\mu z^{(n-1)}(t)}^{\mu z^{(n-1)}(T_0)} \frac{\mathrm{d}x}{f_k(x)} + \int_{T_0}^{t} Q_k(s)\,\mathrm{d}s \leqslant 0. \tag{29}$$

We may note that $z(t) > 0$, $z^{(n)}(t) \leqslant 0$ in $[T, \infty)$ implies that $z^{(n-1)}(t)$ is positive and decreasing in $[T, \infty)$. Using (24) and (25) in (29) we lead to a contradiction when $t \to \infty$.

This completes the proof of theorem. □

Corollary 2. *Suppose that* (C_5) *and* (C_6) *are satisfied. If*

$$\int^{\infty} \left(\sum_{i=1}^{m} Q_i(s) \right) \mathrm{d}s = \infty$$

and

$$\int_0^{\pm\alpha} \frac{\mathrm{d}x}{f_i(x)} < \infty \quad \textit{for every } \alpha > 0 \quad (i = 1, 2, \ldots, m),$$

then every solution of (E_3) *oscillates.*

The proof of this theorem follows from Theorem 3.

Remark 3. In [4], Graef et al. proved the particular case of Corollary 2, that is, Corollary 2 when $n = 1$, $m = 1$ and $K = 1$.

The sublinearity condition on f_k assumed in (25) includes $f_k(x) = x^\alpha$, where α is the ratio of odd integers and $0 < \alpha < 1$.

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