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ON COMPLETE MV-ALGEBRAS

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Though the number of published papers on MV-algebras is rather large (the fundamental source are Chang's articles [1] and [2]), the terminology and notation in this field seem to be far from being unified. We will apply the terminology from [5], [6].

It is well-known that MV-algebras are term equivalent to Wajsberg algebras (called also W-algebras); cf., e.g., Cignoli [3]. Further, MV-algebras are categorically equivalent to bounded commutative BCK-algebras (cf. Mundici [8]); such BCK-algebras were studied by Traczyk [10].

Cignoli [3] studied the structure of MV-algebras which are complete and atomic. His main result is the following theorem:

(∗) ([3], Theorem 2.6.) An MV-algebra is complete and atomic if and only if it is a direct product of finite linearly ordered MV-algebras.

An MV-algebra $\mathscr{A}$ which is a direct product of MV-algebras $\mathscr{A}_i$ $(i \in I)$ is complete if and only if all $\mathscr{A}_i$ are complete. Further, a complete linearly ordered MV-algebra is atomic if and only if it is finite (cf. 1.3 below). Thus (∗) can be expressed as follows:

(∗∗) An MV-algebra is complete and atomic if and only if it is a direct product of complete atomic linearly ordered algebras.

Let $\mathscr{A} = (A; \oplus, *, \neg, 0, 1)$ be an MV-algebra. We can introduce lattice operations $\vee, \wedge$, and hence also the corresponding partial order $\leqslant$ on A (cf. Section 1 below). Let $0 < x \in A$ and let $\alpha > 1$ be a cardinal. The element x will be called an α-atom of $\mathscr{A}$ if the interval $[0, x]$ is a chain having cardinality α. Hence the notion of the 2-atom coincides with the usual notion of the atom. The MV-algebra $\mathscr{A}$ is said to be α-atomic if for each $0 < y \in A$ there exists an α-atom x of $\mathscr{A}$ with $x \leqslant y$.

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Let R be the additive group of all reals with the natural linear order. For each MV-algebra $\mathscr{A}$ there exists a lattice ordered group G with a strong unit such that $\mathscr{A}$ can be constructed by means of G (cf. $(*_2)$ and $(*_3)$ in Section 1 below). If G is isomorphic to R, then $\mathscr{A}$ will be said to be of type R.

By applying the results of [6] the following will be proved in the present paper:

(A) Let $\mathscr{A}$ be an MV-algebra and let α be a cardinal.

- (i) $\mathscr{A}$ is complete and α-atomic if and only if it is isomorphic to a direct product of complete α-atomic linearly ordered MV-algebras.
- (ii) Let $\alpha > 2$. An MV-algebra is complete, α-atomic and linearly ordered if and only if it is of type R.
- (iii) If $\mathscr{A}$ is a complete α-atomic MV-algebra with $A \neq \{0\}$, then either $\alpha = 2$ or $\alpha = c$ (the cardinality of the continuum).

(B) Let $\mathscr{A}$ be a complete MV-algebra. Then $\mathscr{A}$ is isomorphic to a direct product $\mathscr{A}_1 \times \mathscr{A}_2 \times \mathscr{A}_3$ such that

- (i) $\mathscr{A}_1$ is atomic;
- (ii) $\mathscr{A}_2$ is c-atomic;
- (iii) for each cardinal α, there are no α-atoms in $\mathscr{A}_3$.

Let us remark that for each infinite cardinal α there exists a non-complete MV-algebra $\mathscr{A}$ such that, whenever x is a nonzero element of A, then x is an α-atom of $\mathscr{A}$.

1. Preliminaries and auxiliary results

For the notion of the MV-algebra we introduce the following definition (cf. [5] and [6]):

$(**)$ An MV-algebra is a system $\mathscr{A} = (A;\oplus,*,\neg,0,1)$ (where $\oplus,*$ are binary operations, $\neg$ is a unary operation and $0,1$ are nulary operations) such that the following identities are satisfied:

(m$_1$) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(m$_2$) $x \oplus 0 = x$;
(m$_3$) $x \oplus y = y \oplus x$;
(m$_4$) $x \oplus 1 = 1$;
(m$_5$) $\neg\neg x = x$;
(m$_6$) $\neg 0 = 1$;
(m$_7$) $x \oplus \neg x = 1$;
(m$_8$) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$;
(m$_9$) $x * y = \neg(\neg x \oplus \neg y)$.

We recall the following results $(*_i)$ $(i = 1, 2, 3)$ (for $(*_1)$ cf. [5]; for $(*_2)$ and $(*_3)$ cf. [7] 2.5 and 3.8; cf. also [6], 1.2, 1.3 and 1.4).

$(*_1)$ Let $\mathscr{A}$ be an MV-algebra. For each $x, y \in A$ put $x \vee y = (x * \neg y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Then $\mathscr{L}(\mathscr{A}) = (A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1.

$(*_2)$ Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each a and b in A we put

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a, \quad 1 = u, \quad a * b = \neg(\neg a \oplus \neg b).$$

Then $\mathscr{A} = (A; \oplus, *, \neg, 0, 1)$ is an MV-algebra.

If G and $\mathscr{A}$ are as in $(*_2)$, then we put $\mathscr{A} = \mathscr{A}_0(G, u)$.

$(*_3)$ Let $\mathscr{A}$ be an MV-algebra. Then there exists an abelian lattice ordered group G with a strong unit u such that $\mathscr{A} = \mathscr{A}_0(G, u)$.

In what follows, $\mathscr{A}$ and G are as in $(*_2)$ and $(*_3)$.

1.1. Lemma. *$\mathscr{A}$ is complete if and only if G is complete.*

Proof. Let $\mathscr{A}$ be complete. Hence the interval $[0, u]$ is complete. The fact that u is a strong unit of G implies that for proving the completeness of G it suffices to verify that for each positive integer n the lattice $[0, nu]$ is complete.

We proceed by induction on n. The case $n = 1$ is trivial. Suppose that $n > 1$ and that the interval $[0, (n-1)u]$ is complete. Since $[(n-1)u, nu]$ is isomorphic to $[0, u]$, we obtain that $[(n-1)u, nu]$ is complete as well.

Let $X = \{x_i\}_{i \in I}$ be a nonempty subset of $[0, nu]$. For each $i \in I$ we put

$$x_i^1 = x_i \wedge (n-1)u, \quad x_i^2 = x_i \vee (n-1)u.$$

In view of the assumption the elements

$$x^1 = \bigvee_{i \in I} x_i^1, \quad x^2 = \bigvee_{i \in I} x_i^2$$

exist. For each $i \in I$ the relation

$$x_i = x_i^1 + (x_i^2 - a)$$

is valid, where $a = (n-1)u$. Put

$$x = x^1 + (x^2 - a).$$

Then $x \geqslant x_i$ for each $i \in I$. Let $y \in [0, nu], y \geqslant x_i$ for each $i \in I$. Put $y^1 = y \wedge (n-1)u, y^2 = y \vee (n-1)u$. Then $y^1 \geqslant x_i^1$ and $y^2 \geqslant x_i^2$ for each $i \in I$. At the same time we have

$$y = y^1 + (y^2 - a).$$

Therefore $y \geqslant x$. Thus $x = \sup X$ is valid in $[0, nu]$. Similarly we can verify that $\inf X$ does exist in $[0, nu]$. Hence $[0, nu]$ is complete.

The converse implication is obvious. □

1.2. Lemma. *$\mathscr{A}$ is linearly ordered if and only if G is linearly ordered.*

Proof. If G is linearly ordered, then clearly $\mathscr{A}$ is linearly ordered as well. Suppose that G fails to be linearly ordered. Then there are $g_i \in G$ with $0 < g_i$ $(i = 1, 2)$, $g_1 \wedge g_2 = 0$. Since u is a strong unit in G we infer that $u_i = g_i \wedge u > 0$ $(i = 1, 2)$. We have $u_1 \wedge u_2 = 0$ and $u_1, u_2 \in A$. Hence $\mathscr{A}$ is not linearly ordered. □

Let Z be the additive group of all integers with the usual linear order. It is well-known that if $H \neq \{0\}$ is a complete linearly ordered group, then H is isomorphic either to Z or to R; hence if $0 < h \in H$, then the interval $[0, h]$ is atomic if and only if $[0, h]$ is finite. Hence $(*_1)$, $(*_2)$, 1.1 and 1.2 yield

1.3.1. Corollary. *Let $\mathscr{A}$ be an MV-algebra, $A \neq \{0\}$. Suppose that $\mathscr{A}$ is linearly ordered and complete. Then* (i) *$\mathscr{A}$ is finite if and only if it is atomic, and* (ii) *$\mathscr{A}$ is infinite if and only if it is c-atomic.*

1.3.2. Corollary. *Let $\mathscr{A}$ be as in* 1.3.1. *Then* (i) *$\mathscr{A}$ is atomic if and only if G is isomorphic to Z;* (ii) *$\mathscr{A}$ is c-atomic if and only if it is of type R.*

For each nonempty subset X of a lattice ordered group H we denote

$$X^\delta = \{y \in H : |y| \wedge |x| = 0 \quad \text{for each} \quad x \in X\}.$$

X^δ will be said to be the polar in H generated by the set X. For a thorough theory of polars in lattice ordered groups cf. Šik [9]. Each polar is a convex ℓ-subgroup of H; if $0 \in X$ and X is a linearly ordered convex subset of H, then $X^{\delta\delta}$ is linearly ordered as well.

1.4. Lemma. *Let $\mathscr{A}$ be a complete MV-algebra. Let $0 < x \in G$ and suppose that the interval $[0, x]$ is linearly ordered. Then either $[0, x]$ is finite or $[0, x]$ has cardinality c.*

Proof. Put $X = [0, x]$. Then X is, at the same time, an interval of G. Thus $X^{\delta\delta}$ is linearly ordered. According to 1.1, G is complete. Hence by the Riesz Theorem (cf., e.g., Fuchs [4], Chap. V), $X^{\delta\delta}$ is a direct factor of G. Therefore in view of [6], 3.2, $X_1 = X^{\delta\delta} \cap [0, u]$ is a direct factor of $\mathscr{A}$. Moreover, $X^{\delta\delta}$ is linearly ordered and hence X_1 is linearly ordered as well. Each direct factor of a complete MV-algebra must be complete. Now it suffices to apply 1.3.2. $\square$

1.5. Corollary. *Let α be a cardinal and let $\mathscr{A}$ be a complete MV-algebra, $A \neq \{0\}$. If $\mathscr{A}$ is α-atomic, then either $\alpha = 2$ or $\alpha = c$.*

The notion of an α-atom of a lattice ordered group can be defined in the same way as in the case of MV-algebras. A lattice ordered group G is said to be α-atomic if for each $g \in G$ with $0 < g$ there exists an α-atom g_1 in G such that $g_1 \leqslant g$.

By a similar argument as above we obtain

1.5'. Lemma. *Let α be a cardinal and let G be a complete nonzero lattice ordered group. If G is α-atomic, then either $\alpha = 2$ or $\alpha = c$.*

1.6. Example. Let α be an infinite cardinal. Next, let I be a linearly ordered set which is isomorphic to the first ordinal having the power α. For each $i \in I$ let G_i be a linearly ordered group isomorphic to Z. Put $G' = \Gamma_{i \in I} G_i$, where Γ denotes the operation of lexicographic product (cf., e.g., [4]). For $g' \in G'$ and $i \in I$ let g'_i be the component of g' in G_i. Denote $I(g') = \{i \in I \colon g'_i \neq 0\}$. Let G'' be the subgroup of G consisting of all $g' \in G'$ for which the set $I(g')$ is finite; G'' is linearly ordered by the inherited order. G'' is a non-complete linearly ordered group such that whenever $x, y \in G$ and $x < y$, then the power of the interval $[x, y]$ in G is α. Choose $u \in G''$ with $0 < u$ and let G be the convex ℓ-subgroup of G'' generated by the element u. Hence u is a strong unit in G. Let $\mathscr{A}$ be as in $(*_2)$. Thus each strictly positive element of A is an α-atom in $\mathscr{A}$.

2. Proofs of (A) and (B)

The assertion (ii) and (iii) of (A) were already proved (cf. 1.3.2 and 1.5); the remaining part of (A) will be proved as follows. The case $A = \{0\}$ being trivial we can suppose that $A \neq \{0\}$.

a) Suppose that an MV-algebra $\mathscr{A}$ is a direct product of MV-algebras $\mathscr{A}_i$ ($i \in I$). Without loss of generality we can suppose that the direct decomposition under consideration is internal (in the sense of [5]). Assume that all $\mathscr{A}_i$ are linearly ordered, complete and α-atomic. For each $x \in A$ and $i \in I$ we denote by x_i the component of x in $\mathscr{A}_i$. Let $x > 0$. Then there exists $i \in I$ such that $x_i > 0$. There is an α-atom y^i of $\mathscr{A}_i$ with $y^i \leqslant x_i$. The element y^i is, at the same time, an α-atom in $\mathscr{A}$ and $y^i \leqslant x$. Thus $\mathscr{A}$ is α-atomic.

b) Suppose that $\mathscr{A}$ is a complete and α-atomic MV-algebra. Since $A \neq \{0\}$, the set of all α-atoms of $\mathscr{A}$ is nonempty. For each α-atom x of $\mathscr{A}$ let X be as in the proof of 1.4. Hence $X^{\delta\delta}$ is a direct factor of G; let $\{G_i\}_{i\in I}$ be the set of all $X^{\delta\delta}$ which can be constructed in this way. Each G_i is linearly ordered, complete and α-atomic.

For each $y \in G$ and $i \in I$ let y_i be the component of y in G_i. It is well-known that if $y \geqslant 0$, then y_i is the greatest element of the set $G_i \cap [0, y]$.

Let $y \in A$. We have $y_i \leqslant y$ for each $i \in I$; since $\mathscr{A}$ is complete, there exists $y' = \bigvee\limits_{i\in I} y_i$ in A. We shall verify that $y' = y$. By way of contradiction, suppose that $y'' = y - y' > 0$. Then $y'' \in A$ and hence there exists an α-atom x in A such that $x \leqslant y''$. Thus there is $i(1) \in I$ such that $G_{i(1)} = [0, x]^{\delta\delta}$. Clearly $y''_{i(1)} \geqslant x_{i(1)} = x > 0$. Since $0 \leqslant y_{i(1)} \in G_{i(1)} \cap [0, y']$ we obtain $y_{i(1)} \leqslant y'_{i(1)}$. On the other hand, the relation $y' < y$ gives $y'_{i(1)} \leqslant y_{i(1)}$. Thus $y_{i(1)} = y'_{i(1)}$. Therefore $y_{i(1)} = y'_{i(1)} + y''_{i(1)} = y_{i(1)} + x > y_{i(1)}$, which is a contradiction. Thus

$$y = \bigvee_{i\in I} y_i. \tag{1}$$

If $i(1)$ and $i(2)$ are distinct elements of I, then $G_{i(1)} \cap G_{i(2)} = \{0\}$. This implies that $y_{i(1)} \wedge y_{i(2)} = 0$.

Let φ be a mapping of A into the direct product $\prod\limits_{i\in I} A_i$ (where $A_i = [0, u_i]$ for each $i \in I$) defined by

$$\varphi(y) = (y_i)_{i\in I}.$$

We consider A_i to be partially ordered by the inherited partial order. Let y and z be elements of A. If $y \leqslant z$, then clearly $y_i \leqslant z_i$ for each $i \in I$. Conversely, assume that $y_i \leqslant z_i$ for each $i \in I$; then we infer from (1) that $y \leqslant z$. Thus if y and z are distinct, then $\varphi(y)$ and $\varphi(z)$ are distinct as well. Further, let $(t^i) \in \prod\limits_{i\in I} A_i$. There exists $t \in A$ with $t = \bigvee\limits_{i\in I} t^i$. For each $i(1) \in I$ we have

$$t_{i(1)} = t_{i(1)} \wedge t = t_{i(1)} \wedge \Big(\bigvee_{i\in I} t^i\Big) = t_{i(1)} \wedge t^{i(1)}$$

(since $t_{i(1)} \wedge t^i = 0$ whenever $i \neq i(1)$). Thus $t_{i(1)} \leqslant t^{i(1)}$. On the other hand, $t^{i(1)} \in G_{i(1)} \cap [0, t]$ and hence $t^{i(1)} \leqslant t_{i(1)}$. We obtain $t^{i(1)} = t_{i(1)}$ and therefore $\varphi(t) = (t^i)_{i\in I}$. We have verified that φ is an isomorphism of the lattice A onto $\prod\limits_{i\in I} A_i$.

For each $i \in I$ the element u_i is a strong unit in G_i, hence the MV-algebra $\mathscr{A}_i = (A_i; \oplus, *, \neg, 0, u_i)$ exists. From the construction of the isomorphism φ and from [6], 3.5 we infer that φ is, at the same time, an isomorphism of $\mathscr{A}$ onto $\prod_{i \in I} \mathscr{A}_i$. Each $\mathscr{A}_i$ is complete, linearly ordered and α-atomic. This completes the proof of (A).

Proof of (B). Let $\mathscr{A}$ be a complete MV-algebra and let G be as above. We denote by X_1 and X_2 the system of all atoms of $\mathscr{A}$ or the system of all c-atoms of $\mathscr{A}$, respectively. Put $G_i = X_i^{\delta\delta}$ $(i = 1, 2)$. By the Riesz Theorem, G_1 and G_2 are direct factors of G. For each $x_1 \in X_1$ and $x_2 \in X_2$ we have $x_1 \wedge x_2 = 0$. This yields that $G_1 \cap G_2 = \{0\}$. Therefore

$$G = G_1 \times G_2 \times G_3, \tag{2}$$

where

$$G_3 = (G_1 \cup G_2)^{\delta}. \tag{3}$$

All G_i $(i = 1, 2, 3)$ are complete. It follows from the definition of G_1 that it is atomic; analogously, G_2 is c-atomic. The relation (3) yields that for each cardinal α no α-atom exists in G_3.

For $i \in \{1, 2, 3\}$ let u_1 be the component of u in G_i. We can construct the MV-algebras $\mathscr{A}_i = (A_i; \oplus, *, \neg, 0, u_i)$ for $i = 1, 2, 3$, where $A_i = [0, u_i]$. Then all $\mathscr{A}_i$ are complete, $\mathscr{A}_1$ is atomic, $\mathscr{A}_2$ is c-atomic, and for each cardinal α, $\mathscr{A}_3$ has no α-atoms. Now we can apply [6], Lemma 3.2 (this lemma deals with direct decompositions having two factors, but by an obvious induction we can extend the validity of the lemma to direct decompositions having a finite number of direct factors); from (2) we infer that $\mathscr{A}$ is a direct product of $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$. □

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