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SEQUENTIAL CONVERGENCES ON CYCLICALLY ORDERED GROUPS

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In the paper presented it will be shown that on each cyclically ordered group there are at most two sequential convergences which are compatible with its inner structure.

The results of the paper were announced at the Conference on Convergence held in 1985 at Szczyrk.

A cyclically ordered group $(G, +, C)$ is a group $(G, +)$ with a cyclic order C, i.e. with a set C of ordered triplets of pairwise distinct elements of G fulfilling the following conditions:

(1) if $(a, b, c) \in C$, then $(b, c, a) \in C$;

(2) if $(a, b, d) \in C$ and $(b, c, d) \in C$, then $(a, c, d) \in C$;

(3) if $(a, b, c) \in C$, then $(x + a + y, x + b + y, x + c + y) \in C$ for all $x, y \in G$;

(4) either $(a, b, c) \in C$ or $(a, c, b) \in C$.

The group operation in a cyclically ordered group will be written additively although the commutativity will not be assumed. For notions non-defined here we refer to [2].

Every subroup of a cyclically ordered group will be considered to be cyclically ordered by the induced cyclic order.

Every linearly ordered group can be considered as a cyclically ordered group ([2]). The multiplicative group of complex numbers of absolute value one, denoted by K, equipped with a natural cyclic order, is an example of a cyclically ordered group which cannot be linearly ordered. If we have a linearly ordered group M, then $K \times M$ can be cyclically ordered by the natural induced cyclic order; similarly, every subgroup of $K \times M$ can be understood as a cyclically ordered group (for details see [2]).

Some results of the theory of cyclically ordered groups, developed by L. S. Rieger in [10], were applied by S. Świerczkowski in order to prove the following representation theorem.

Theorem 1. (S. Świerczkowski, [11].) *Let $(G, +, C)$ be a cyclically ordered group. Then there exists a linearly ordered group M such that $(G, +, C)$ is isomorphic to some subgroup of $K \times M$.*

A. I. Zabarina and G. G. Pestov have obtained a certain improvement upon this theorem in [13]. Cyclically ordered groups were studied also in [6, 9].

We recall that a cyclically ordered group $(G, +, C)$ is Archimedean if it contains no elements x, y such that, for each positive integer n, $(0, nx, y) \in C$. From Świerczkovski's theorem above it follows that G is Archimedean if and only if G is isomorphic to some subgroup of K with the cyclic order on G carried over from K.

The following notation is adopted: N denotes the set of all positive integers; G^N is the set of all sequences with all members belonging to G; $G^N \times G$ is the set of all pair (S, s) with $S \in G^N$ and $s \in G$; *Mon* denotes the set of all monotone mappings from N to N; if $S \in G^N$ and $w \in Mon$, then $S \circ w$ is the subsequence of S with the n-th member $S(w(n))$; $\mathrm{const}(s)$ denotes the constant sequence with all members equal to s.

In accordance with the notion of a positive cone of a cyclically ordered group introduced and investigated in [12] we introuce $|x|$, the absolute value of x, for $x \in G$ such that

$$\text{if } x = -x, \text{ then } |x| = x,$$

$$\text{if } x \neq -x \text{ and } (-x, 0, x) \in C, \text{ then } |x| = x,$$

$$\text{if } x \neq -x \text{ and } (x, 0, -x) \in C, \text{ then } |x| = -x.$$

It is easy to verify the following assertion.

Lemma 1. *Let x and y be elements of G. Then*

(i) $|x| = 0$ *if and only if* $x = 0$;
(ii) $|x| = |-x|$;
(iii) $||x|| = |x|$;
(iv) *if* $x \neq -x$, *then* $(-|x|, 0, |x|) \in C$;
(v) *if* $x \neq -x$ *and* $y = -y \neq 0$, *then* $(|x|, y, -|x|) \in C$;
(vi) *if* $(-|x|, y, |x|) \in C$, *then* $(-|x|, \pm|y|, |x|) \in C$.

Let us introduce the notion of a convergence on a cyclically ordered group.

Definition. *A set $\mathscr{L} \subseteq G^N \times G$ is said to be a convergence on $(G, +, C)$ if the following conditions are satisfied*:

(F) $(S, s) \in \mathscr{L}$ *implies* $(S \circ w, s) \in \mathscr{L}$ *for each* $w \in Mon$.
(L) $(S, s) \in \mathscr{L}$ *and* $(R, r) \in \mathscr{L}$ *imply* $(S + R, s + r) \in \mathscr{L}$ *and* $(-S, -s) \in \mathscr{L}$.
(U) *If* $S \in G^N$ *and if for each* $u \in Mon$ *there exists* $v \in Mon$ *such that* $(S \circ u \circ v, s) \in \mathscr{L}$, *then* $(S, s) \in \mathscr{L}$.
(S) $(\mathrm{const}(s), s) \in L$ *for each* $s \in G$.
(H) $(S, a) \in \mathscr{L}$ *and* $(S, b) \in \mathscr{L}$ *imply* $a = b$.
(C) *If* $(S, a) \in \mathscr{L}$ *and if* $(0, |T(n) - a|, |S(n) - a|) \in C$ *for each* $n \in N$, *then* $(T, a) \in \mathscr{L}$.

The conditions (FLUSH) define a FLUSH-convergence structure for G cf. [7], i.e. a convergence group (cf. [8]); the last condition concerns a relation between the convergence and the cyclic order on C. Problems of FLUSH-convergence structures were investigated by many authors (cf. the survey paper [1]).

Example 1. The discrete convergence $d(G)$ on G, defined by

$$d(G) = \{(S, s) \in G^N \times G;\ S(n) \neq s \text{ for finitely many } n \in N\},$$

is convergence on $(G, +, C)$.

Example 2 (see [5], § 3). Let $o(G)$ be defined by
$o(G) = d(G)$ if $\operatorname{card} G \leqslant 2$ and
$o(G) = \{(S, s) \in G^N \times G$; if $x \in G$ and $x \neq -x$,
then $(-|x|, S(n) - s, |x| \notin C$ for finitely
many $n \in N\}$ if $\operatorname{card} G > 2$.
The set $o(G)$ is a convergence on $(G, +, C)$.

In the particular case when G is linearly ordered, $o(G)$ coincides with the order convergence on G.

The following assertion is rather easy to verify (for detailed proof cf. [5]):

Proposition. *Let G be isomorphic to some subgroup of K and let $\mathscr{L}$ be a convergence on G. Then $\mathscr{L} = d(G)$ or $\mathscr{L} = o(G)$.*

The following theorem is the main result of this paper.

Theorem 2. *Let $(G, +, C)$ be an arbitrary cyclically ordered group and let $\mathscr{L}$ be a convergence on $(G, +, C)$. Then $\mathscr{L} = d(G)$ or $\mathscr{L} = o(G)$.*

In view of Proposition we may assume that G is not isomorphic to any subgroup of K. We shall prove the theorem by means of a sequel of lemmas. Note that if $S \in G^N$, then $S - s$ and $|S|$ are defined pointwise. One can easily prove the following lemma.

Lemma 2. *Let $\mathscr{L}$ be a subset of $G^N \times G$ satisfying the conditions* (FLUSH). *Then $(S, s) \in \mathscr{L}$ if and only if $(|S - s|, 0) \in \mathscr{L}$.*

Let us denote
$L^+ = \{x \in G;\ (-nx, 0, nx) \in C$ whenever $n \in N\}$,
$L^- = \{x \in G;\ (nx, 0, -nx) \in C$ whenever $n \in N\}$,
$L = L^+ \cup \{0\} \cup L^-$.

For $a, b \in L$ we define $a \leq b$ if and only if $a = b$ or $(a - b, 0, b - a) \in C$.

Theorem 3. (A. I. Zabarina — G. G. Pestov, [13], Thms. 2.6 and 2.7.) *L is both a normal subgroup of G and a linearly ordered group with respect to the above defined order $\leq$. Moreover, if $h \in L$ and* $(0, x, |h|) \in C$ *for some $x \in G$, then $x \in L$.*

Lemma 3. *If $\mathscr{L}$ is a convergence on G and $(S, 0) \in \mathscr{L}$, then there is a final segment of S belonging to L^N.*

Proof. Suppose it is not. Then there is a sequence $w \in Mon$ such that

$S(w(n)) \notin L$ whenever $n \in N$. By the condition (F) and Lemma 2 we have $(|S \circ w|, 0) \in \mathscr{L}$. Since G is not Archimedean, there exists an $l \in L$, $l \neq 0$ (see [13]). Applying Theorem 3 we get $(0, |l|, |S(w(n))|) \in C$ for each $n \in N$. It follows that $(\text{const}(|l|), 0) \in \mathscr{L}$. Finally, the conditions (S), (H) and Lemma 1 (i) imply $l = 0$, a contradiction.

We recall that $\mathscr{L}$ is a convergence on a linearly ordered group $(L, +, \leq)$ if $\mathscr{L} \subseteq L^N \times L$, the conditions (FLUSH) are fulfilled and $(S, g) \in \mathscr{L}$ whenever $(R, g) \in \mathscr{L}$, $(T, g) \in \mathscr{L}$ and $R \leq S \leq T$ (cf. [3]). It is easy to observe that the following holds.

Lemma 4. *If $\mathscr{L}$ is a convergence on a cyclically ordered group $(G, +, C)$, then $\mathscr{L} \cap (L^N \times L)$ is a convergence on the linearly ordered subgroup $(L, +, \leq)$.*

Lemma 5. *Let $\mathscr{L}$ and $\mathscr{K}$ be convergences on G. Then $\mathscr{L} = \mathscr{K}$ if and only if $\mathscr{L} \cap (L^N \times L) = \mathscr{K} \cap (L^N \times L)$.*

Proof. It is sufficient to show that $\mathscr{L} \subseteq \mathscr{K}$ whenever $\mathscr{L} \cap (L^N \times L) = \mathscr{K} \cap (L^N \times L)$. Let $(R, r) \in \mathscr{L}$. From Lemma 2 it follows that for $S = |R - r|$, $(S, 0)$ belongs to $\mathscr{L}$. By Lemma 3 there is a final segment T of S belonging to L^N. Hence, by the condition (F), $(T, 0) \in \mathscr{L} \cap (L^N \times L) = \mathscr{K} \cap (L^N \times L) \subseteq \mathscr{K}$. From the condition (U) we obtain $(S, 0) \in \mathscr{K}$. So, by Lemma 2, we have $(R, r) \in \mathscr{K}$.

The following easy lemma enables us to prove the theorem.

Lemma 6. *Let $\mathscr{L}$ be a convergence on the linearly ordered group $(L, +, \leq)$ and let $\mathscr{L}^* = \{(S, s) \in G^N \times G$; there exists a final segment T of S such that $(T - s, 0) \in \mathscr{L}\}$. Then $\mathscr{L}^*$ is a convergence on the cyclically ordered group $(G, +, C)$ and $\mathscr{L}^* \cap (L^N \times L) = \mathscr{L}$.*

Corollary. *Let $\mathscr{L}, \mathscr{K}$ be convergences on the linearly ordered group $(L, +, \leq)$. Then $\mathscr{L} = \mathscr{K}$ if and only if $\mathscr{L}^* = \mathscr{K}^*$.*

Proof of Theorem 2. By Lemma 4, $\mathscr{L} \cap (L^N \times L)$ is a convergence on the linearly ordered group $(L, +, \leq)$. From [5], Thm. 2.10 (for a special case, if L is commutative, in [4], Thm. 3.9) it follows that $\mathscr{L} \cap (L^N \times L) = d(L)$ or $\mathscr{L} \cap (L^N \times L) = o(L)$. Applying Lemmas 3 and 6 to $\mathscr{L} = (\mathscr{L} \cap (L^N \times L))^*$ we have $\mathscr{L} = (d(L))^* = d(G)$ or $\mathscr{L} = (o(L))^* = o(G)$.

The following remark is a consequence of the above results (Thms. 1, 2 and Prop.).

Remark. If G is an infinite Archimedean cyclically ordered group, then $o(G) \neq d(G)$; thus G has exactly two convergences.

If G is not Archimedean and there is a decreasing sequence $S \in L^N$ (L defined as above) such that the infimum of the set $\{S(n); n \in N\}$ is zero, then G has two convergences as well.

Otherwise, G has only one convergence, namely $d(G)$.

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СЕКВЕНЦИАЛЬНЫЕ СХОДИМОСТИ НА ЦИКЛИЧЕСКИ УПОРЯДОЧЕННЫХ ГРУППАХ

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Резюме

В статье определено понятие секвенциальной сходимости на циклически упорядоченной группе. Показано, что всякая циклически упорядоченная группа имеет не более двух сходимостей.