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A NOTE ON A FUNCTION REPRESENTATION OF ORTHOMODULAR POSETS

JOSEF TKADLEC

In the papers [1], [2] the authors give axioms for a set of functions to characterize an orthomodular poset with "enough" states. In the attempt to improve the characterization, D. Strojewski [3] offers (seemingly) more lucid conditions and derives several consequences. However, his crucial auxiliary result does not seem to be correct. In this note we construct the appropriate counterexample and give the correct version of the representation theorem.

Let us first review the basic notions. By an *orthomodular poset* we mean a triple $(L, \leqslant, ')$ such that

(a) $(L, \leqslant)$ is a partially ordered set with a greatest element 1,

(b) the operation $': L \to L$ is an *orthocomplementation*, for every $a, b \in L$ we have $a'' = a$ and $a \leqslant b$ implies $b' \leqslant a'$,

(c) the least upper bound exists for every pair of orthogonal elements in L,

(d) $b = a \vee (b \wedge a')$ for every $a, b \in L$ with $a \leqslant b$.

By a *state* we mean a function $s: L \to [0, 1]$ such that $s(1) = 1$ and $s(a \vee b) = s(a) + s(b)$ for each pair of the orthogonal elements $a, b \in L$. Recall finally that a subset of states is called *full* if for every $a, b \in L$, $a \nleqslant b$ the subset contains such a state s that $s(a) \nleqslant s(b)$.

Theorem. *Let P be a nonvoid set and let $P \subset [0, 1]^S$ for a set S. Let further P satisfy*

(1) $(\forall f \in P, f \neq 0)(\exists s \in S) f(s) > 0.5$,

(2) $(\forall f \in P)\, 1 - f \in P$,

(3) $(\forall f, g \in P, f + g \leqslant 1)\, f + g \in P$,

(4) $(\forall f, g \in P, f + g \leqslant 1)(\exists h \in P, h \geqslant f, g)(\forall k \in P, k \geqslant f, g)\, k \geqslant h$.

Then $(P, \leqslant, ')$ with the pointwise ordering $\leqslant$ and the orthocomplementation given by $f' = 1 - f$ is an orthomodular poset with a full set of states $\bar{S} = \{\bar{s}: P \to [0, 1]; (\forall f \in P)\, \bar{s}(f) = f(s)\}$. Moreover, $f \vee g = f + g$ for the orthogonal elements $f, g \in P$.

Conversely, each orthomodular poset L with a full set S of states is orthoisomorphic to some subset $P \subset [0, 1]^S$ satisfying axioms (1)—(4) *with the ordering and the orthocomplementation given as above.*

Proof. One can easily verify that $(P, \leqslant, ')$ is a partially ordered set with orthocomplementation. Since the set P is nonempty, there is an $f \in P$ and we have $1 - f \in P$ (the condition (2)) and $1 = f + (1 - f) \in P$ (the condition (3)). Let $f, g \in P$ be orthogonal. Thus $f + g \leqslant 1$ and according to the condition (4) there exists $f \vee g \in P$. By the condition (3) we have $f + g \in P$. Since $f + g \geqslant f \vee g$, the conditions (2), (3) give $(f + g) - (f \vee g) = 1 - ((f \vee g) + (f + g)') \in P$. As further $(f + g) - (f \vee g) \leqslant \min(f, g) \leqslant \min(f, 1 - f) \leqslant 0.5$, we obtain $(f + g) - (f \vee g) = 0$ (the condition (1)). Hence $f \vee g = f + g$.

Let $f, g \in P$ and $f \leqslant g$. Then $f, g' \in P$ are orthogonal and also $f, (f \vee g')' \in P$ are orthogonal. Hence we infer that $f \vee (g \wedge f') = f \vee (g' \vee f)' = f + (1 - ((1 - g) + f)) = g$.

Let conversely L be an orthomodular poset with a full set S of states. Put

$$P = \{f_a \in [0, 1]^S;\ (\forall s \in S)\, f_a(s) = s(a),\ a \in L\}.$$

Then $a \mapsto f_a$ is obviously an orthoisomorphism between L and P with respect to the respective orderings and orthocomplementations. Hence the axioms (2), (4) hold. Let $f_a \in P$, $f_a \leqslant 0.5$. Then $0.5 \leqslant f_a'$ and therefore $f_a \leqslant f_a'$. Thus $a \leqslant a'$ in view of the orthoisomorphism. But it means that $a = 0$, which gives $f_a = 0$. This establishes the condition (1). Let $f_a, f_b \in P$ with $f_a + f_b \leqslant 1$. Then f_a is orthogonal to f_b and making use of the orthoisomorphism again we see that a is orthogonal to b. But it means that $(\forall s \in S)\ s(a \vee b) = s(a) + s(b)$. Hence $f_a + f_b = f_{a \vee b} \in P$. This completes the proof.

In the paper [3] the author states the same representation theorem without the condition (4). It turns out, however, that such a theorem is no longer valid as the following example shows (It should be noted that this example disproves also other results in [3] — Theorems 1, 2, etc.).

Example. *Put*

$$P = \left\{(a_0, a_1, a_2, a_3, a_4) \in \left\{0, \frac{1}{2}, 1\right\} \times \{0, 1\}^4;\right.$$
$$\left.\sum_{i=1}^{4} a_i \text{ is odd if and only if } a_0 = \frac{1}{2}\right\}.$$

Then P satisfies the axioms (1)—(3) *and it is not an orthomodular poset.*

Proof. The axioms (1)—(3) verify easily. P is not an orthomodular poset because the orthogonal elements $a = \left(\frac{1}{2}, 1, 0, 0, 0\right)$, $b = \left(\frac{1}{2}, 0, 1, 0, 0\right)$ do not have the least upper bound in P (we have three incomparable elements in P greater than a, b).

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О ФУНКЦИОНАЛЬНОМ ПРЕДСТАВЛЕНИИ ОРТОМОДУЛЯРНЫХ ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВ

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Резюме

М. И. Мочиньски и Т. Трачик установили условия для ортомодулярного частично упорядоченного множества чтобы оно имело «достаточное» количество состояний. Д. Строевски попытался улучшить эти условия, но это ему не совсем удалось.

В этой статье находится контрпример и исправление теоремы Д. Строевского.