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AN EXTENSION OF BRUNOVSKÝ'S SCORZA DRAGONI TYPE THEOREM FOR UNBOUNDED SET-VALUED FUNCTIONS*

C. J. HIMMELBERG—F. S. VAN VLECK

1. Introduction

In 1948 G. Scorza Dragoni [S] showed that a function $f: T \times X \to E$ which is measurable in t and continuous in x has the property that, given an $\varepsilon > 0$, there is a closed subset T_ε of T, with the measure of $T - T_\varepsilon$ less than ε, such that $f|_{T_\varepsilon \times X}$ is continous. Recently many people ([B], [C-1, 2], [G], [H-1], [HV], [J-1], [K]) have extended Scorza Dragoni's theorem in several directions — in many cases for use in control theory problems. Several of these papers extend the result to set-valued functions (hereafter called multifunctions). Most of the results for a multifunction F have required that $F(t, x)$ be compact and that $F(t, .)$ be continuous with respect to the Hausdorff metric on the space of nonempty compact subsets of the range space E. In [H-1] and [HV] the situation when $F(t, x)$ is not necessarily compact valued was treated but the continuity hypothesis was retained (in this case with respect to the Hausdorff pseudometric on the space of all nonempty subsets of E). On the other hand, Brunovský [B, Theorem 2.6] weakened the continuity requirement slightly to what he called $\tilde{\alpha}$-continuity and also did not require the values to be compact. However, he had to assume that the values $F(t, x)$ were closed and *convex*, and could only deduce a weaker form of continuity (which he called $\tilde{\beta}$-continuity). In this note we will generalize Brunovský's result in three ways: we can allow the spaces T, X and E to be more general; we do not assume that the values $F(t, x)$ are convex; and we deduce that $F|_{T_\varepsilon \times X}$ has the same continuity property as $F(t, .)$. When F has closed values, this result, Theorem 4.1, extends the earlier result [HV, Theorem 1] of the authors where continuity with respect to the Hausdorff pseudometric was assumed.

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2. Definitions and preliminary results

Let T be a compact Hausdorff space with positive Radon measure μ, let X be a Polish (= complete, separable metric) space, and let E be a locally compact separable metric space with metric d. If Y is a metric space with metric ϱ, define the Hausdorff pseudometric H_ϱ on the set $S(Y)$ of all nonempty subsets of Y by

$$H_\varrho(A, B) = \text{lub}\,\{\text{d}(x, B), \text{d}(y, A);\ x \in A,\ y \in B\}\ .$$

The closed (compact) nonempty subsets of Y will be denoted by $2^Y(\mathscr{C}(Y))$ A multifunction $G : X \to Y$ (i. e., for each $x \in X$, $G(x) \in S(Y)$) is *continuous* iff G is continuous as a function from X to $S(Y)$, when $S(Y)$ is topologized by H_ϱ. (Brunovský calls this type of continuity α-continuity apparently since he denotes the Hausdorff metric H_ϱ by α.) G is *upper (lower) semicontinuous* iff $G^{-1}(B) = \{x \mid G(x) \cap B \neq \emptyset\}$ is closed (open) for each closed (open) subset B of Y. G has *closed graph* iff G is a closed subset of $X \times Y$. (This is called $\tilde{\beta}$-continuity by Brunovský.) If $G : T \to Y$ is a multifunction, then G is *measurable (weakly measurable)* iff $G^{-1}(B)$ is μ-measurable for each closed (open) subset B of E.

We next recall some well-known facts about multifunctions:

1. If $F : X \to Y$ is upper semicontinuous and if $F(x)$ is closed for each $x \in X$, then F has closed graph.

2. Conversely, if Y is compact and if F has closed graph, then F is upper semicontinuous.

3. $F : X \to Y$ is lower semicontinuous iff, for each $x_0 \in X$ and each open set V such that $V \cap F(x_0) \neq \emptyset$, there is a neighborhood $U(x_0)$ of x_0 such that $V \cap F(x) \neq \emptyset$ for each $x \in U(x_0)$.

4. If Y is compact and if $F : X \to Y$ has closed values $F(x)$ for each x, then F is continuous iff F is both upper and lower semicontinuous.

5. $F : X \to Y$ has closed graph iff for sequences (x_n), (y_n) such that $(x_n) \to x \in X$ and $(y_n) \to y$ with $y_n \in F(x_n)$ we have $y \in F(x)$.

6. If $F : T \to Y$ has compact values and if Y is separable then F is measurable iff F considered as a function from T into $\mathscr{C}(Y)$ is measurable. (Cf. [B, Proposition 2.4] or [C-1, Theoreme 4.3]).

The next result seems to be somewhat less well known and so we include a proof for completeness.

Proposition 2.1. *$F : X \to Y$ is a lower semicontinuous multifunction iff, for each $x \in X$, $y \in F(x)$, and sequence $(x_n) \to x$, there is a sequence $(y_n) \to y$ with $y_n \in F(x_n)$ for each n.*

Proof. Suppose F is not lower semicontinuous. Then by (3) there is an

$x_0 \in X$ and an open set G with $G \cap F(x_0) \neq \emptyset$ such that for each neighborhood$U(x_0)$ of x_0 there is an $x \in U(x_0)$ with $G \cap F(x) = \emptyset$. Let $B_{\frac{1}{n}}(x_0)$ be the open ball of radius $\frac{1}{n}$ about x_0. For each n, choose $x_n \in B_{\frac{1}{n}}(x_0)$ such that $G \cap F(x_n) = \emptyset$. This sequence (x_n) converges to x_0. Next choose $y_0 \in G \cap \cap F(x_0)$. If F has the second property given in the proposition, then for each n there is a $y_n \in F(x_n)$ such that $(y_n) \to y_0$. Hence $y_n \in G$ for n sufficiently large and so, for those n, $G \cap F(x_n) \neq \emptyset$, a contradiction.

To establish the "only if" part, assume that F is lower semicontinuous and let $x_0 \in X$, $y_0 \in F(x_0)$, and $(x_n) \to x_0$ be given. By lower semicontinuity of F at x_0, given $\varepsilon > 0$ there is a number $\delta = \delta(\varepsilon, x_0, y_0)$ such that $x \in B_\delta(x_0)$ implies $B_{\frac{\varepsilon}{2}}(y_0) \cap F(x) \neq \emptyset$.

For each n, choose $y_n \in F(x_n)$ such that $d(y_0, y_n) \leq d(y_0, F(x_n)) + \frac{1}{n}$. We claim that $(y_n) \to y_0$. If not, then there is a subsequence (y_{n_k}) and an $\varepsilon > 0$ such that $d(y_0, y_{n_k}) \geq \varepsilon$. For this $\varepsilon > 0$, let δ be as above, i. e. $x \in B_\delta(x_0)$ implies $B_{\frac{\varepsilon}{2}}(y_0) \cap F(x) \neq \emptyset$. For sufficiently large k, $\frac{1}{n_k} < \frac{\varepsilon}{2}$ and $x_{n_k} \in B_\delta(x_0)$. Hence $B_{\frac{\varepsilon}{2}}(y_0) \cap F(x) \neq \emptyset$; say $y \in B_{\frac{\varepsilon}{2}}(y_0) \cap F(x_{n_k})$. Then

$$d(y_0, y_{n_k}) \leq d(y_0, F(x_{n_k})) + \frac{1}{n_k} < d(y_0, y) + \frac{\varepsilon}{2} < \varepsilon ,$$

a contraction.

Remark 2.2. Brunovský says that a multifunction $F : X \to Y$ is $\tilde{\alpha}$-continuous if it is $\tilde{\beta}$-continuous and has the property given in Proposition 2.1. Thus, by Proposition 2.1, this means F is $\tilde{\alpha}$-continuous iff F has closed graph and is lower semicontinuous. In general this type of continuity is weaker than continuity with respect to the Hausdorff pseudometric. By (2) and (4) above these two kinds of continuity coincide if Y is compact and F has closed values.

The last proposition of this section is a result essentially due to Castaing [C-1], Thèorème 1.1].

Proposition 2.3. *Let $F : T \to E$ be a multifunction with $F(t)$ closed for each $t \in T$. F is measurable iff F is weakly measurable.*

3. Extending to the one-point compactification

Recall that E was assumed to be a locally compact separable metric space with metric d. Let $E^* = E \cup \{\omega\}$ be the one-point compactification of E.

E^* is metrizable with a metric d^* which also defines the topology of E. It makes no difference whether d and d^* agree on E.

If $F : Z \to E$ is a multifunction with closed values $F(z)$, define a multifunction $F^* : Z \to E^*$ by

$$F^*(z) = F(z) \cup \{\omega\} .$$

In the rest of this section we explore the relationship between the properties of F and F^*. The idea of considering E^* when working with multifunctions was used by Jacobs [J-2]. However he did not consider F^* as defined here, but rather considered the multifunction defined by taking the closure of $F(z)$ in E^*. The present definition of F^* was suggested by the work of Watson [W] who showed that if E is a locally compact separable metric space, then it is possible to topologize 2^E with a compact metric topology. We are essentially using that idea here.

Proposition 3.1. *$F : X \to E$ has closed graph iff $F^* : X \to E^*$ has closed graph. Hence F has closed graph iff F^* is upper semicontinuous.*

Proof. Since $F = F^* \cap (X \times E)$ it is clear that if F^* has closed graph, then F has closed graph.

Conversely, suppose F has closed graph. Let (x_n), (y_n) be sequences in X and E^*, respectively, such that $(x_n) \to x \in X$ and $(y_n) \to y \in E^*$ with $y_n \in F(x_n)$. If $y = \omega$, then $y \in F^*(x) = F(x) \cup \{\omega\}$. So suppose $y \neq \omega$. Then $y \in E$ and, since $(y_n) \to y$ and E is open in E^*, (y_n) must ultimately be in E. Since F has closed graph, $y \in F(x) \subseteq F^*(x)$.

Proposition 3.2. *$F : X \to E$ is lower semicontinuous iff F^* is lower semicontinuous.*

Proof. First assume that F is lower semicontinuous and let G^* be an open set in E^*. If $\omega \in G^*$, then $F^{*-1}(G^*) = X$. If $\omega \notin G^*$, then $G^* \subseteq E$ and is open in E. Hence, in this case, $F^{*-1}(G^*) = F^{-1}(G^*)$, which is open by the lower semicontinuity of F.

Next let F^* be lower semicontinuous. Since every open subset G of E is open in E^*, $F^{-1}(G) = F^{*-1}(G)$, and hence F is lower semicontinuous.

Putting Proposition 3.1 and 3.2 together with (4) of Section 2, we have the following corollary.

Corollary 3.3. *Let $F : X \to E$ be a multifunction with closed values. F has closed graph and is lower semicontinuous iff $F^* : X \to E^*$ is continuous.*

The next proposition relates the measurability of F and F^*.

Proposition 3.4. *Let $F : T \to E$ be a multifunction. F is measurable iff F^* is measurable.*

Proof. As in the proof of Proposition 3.2, F is weakly measurable iff F^* is weakly measurable. The result then follows from Proposition 2.3.

Remark 3.5. The results in Sections 2 and 3 are not given in full generality. For example, Proposition 2.1 is true for arbitrary topological spaces if nets are used instead of sequences. Proposition 2.3 is true for T a measurable space (cf. [H-2]). In Propositions 3.1 and 3.2, X can be a topological space and E only needs to be a locally compact Hausdorff space. Proposition 3.4 is likewise true much more generally.

4. A Scorza Dragoni theorem

We can now state and prove a Scorza Dragoni type theorem for multifunctions that essentially contains the result of Brunovský [B, Theorem 2.6], and, when F has closed values, a result of the authors [HV, Theorem 1]. The proof uses a Scorza Dragoni Theorem for functions due to Castaing [C-2, Théoréme] in addition to the results of Section 3.

Theorem 4.1. *Let T,X, and E be as in Section 2. Let $F : T \times X \to E$ be a multifunction with closed values such that*

(i) *$F(. , x)$ is measurable for each $x \in X$, and*

(ii) *$F(t , .)$ has closed graph and is lower semicontinuous for each $t \in T$.*

Then for each $\varepsilon < 0$ there is a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \varepsilon$ and $F|_{T_\varepsilon \times X}$ has closed graph and is lower semicontinuous.

Proof. Let E^* be the one-point compactification of E and let $F^* : T \times X \to$ $\to E^*$ be defined as in Section 3. By Corollary 3.3 and Proposition 3.4, F^* satisfies

(i)* $F^*(. , x)$ is measurable for each $x \in X$, and

(ii)* $F^*(t , .)$ is continuous for each $t \in T$.

From (i)*, (ii)* and (6) it follows that, as a function from $T \times X$ into $\mathscr{C}(E^*)$, the space of compact nonempty subsets of E^* with the Hausdorff metric H_{d^*}, F^* is measurable in t and continuous in x. Thus, by [C-2, Théorème], for each $\varepsilon > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \varepsilon$ and $F^*|_{T_\varepsilon \times X}$ is continuous. By Corollary 3.3 this means that $F|_{T_\varepsilon \times X}$ has closed graph and is lower semicontinuous.

Remark 4.2. Brunovský allows T to be a bounded measurable set (in a Euclidean space). Clearly, if T' were a measurable subset of T, we could approximate T' by a compact subset T'_ε such that $\mu(T' - T'_\varepsilon) < \varepsilon/2$ and then proceed as above.

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