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DISCREPANCIES OF POINT SEQUENCES ON THE SIERPIŃSKI CARPET

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ABSTRACT. Several types of discrepancies of finite point sequences on the Sierpiński carpet C_s $(s \geq 2)$ are introduced. Various estimates relating these discrepancies are proven.

1. Introduction

The (2-dimensional) Sierpiński carpet is a well-known planar fractal set, which can be constructed as follows. Let A_0 be the unit square of vertices $P_0(0,0)$, $P_1(0,1)$, $P_2(1,0)$, $P_3(1,1)$. Let A_1 be the set that one gets by dividing A_0 into 9 congruent squares with side length $1/3$ and "deleting" the open "central" square. A_1 is the union of 8 squares of side length $1/3$. By repeating this operation for each of these eight squares successively one gets the sets $A_2, A_3, \dots$. The set A_n is the union of 8^n squares with side length 3^{-n}, called *elementary squares of level n*. In the following, for simplicity, we will also call them simply *n-squares*.

DEFINITION 1. The set $C = \bigcap\limits_{n=0}^{\infty} A_n$ is called the (*planar*) *Sierpiński carpet.*

Figure 1 shows the set $\bigcap\limits_{n=0}^{4} A_n$.

The definition can be extended in higher dimension.

The s-dimensional Sierpiński carpet ($s \geq 2$) is a fractal set, embedded in $\mathbb{R}^s$ and can be obtained as follows. Let $A_0 = [0,1]^s$ be the unit cube in $\mathbb{R}^s$. We denote by A_1 the set obtained by dividing A_0 into 3^s congruent cubes with side

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length 1/3 and "deleting" the open "central" cube, i.e $A_1 = [0,1]^s \setminus \left(\frac{1}{3}, \frac{2}{3}\right)^s$. A_1 is the union of $3^s - 1$ cubes with side length $1/3$. By repeating this operation for each of these $3^s - 1$ cubes successively one gets $A_2, A_3, \ldots, A_n, \ldots$. Thus A_n is the union of $(3^s - 1)^n$ cubes with side length 3^{-n}, called *elementary cubes of level* n, or, simply, n*-cubes*.

$$A_n = A_{n-1} \setminus \left(\bigcup_{k=0}^{3^{n-1}-1} \left(\frac{k}{3^{n-1}} + \frac{1}{3^n}, \frac{k}{3^{n-1}} + \frac{2}{3^n} \right) \right)^s .$$

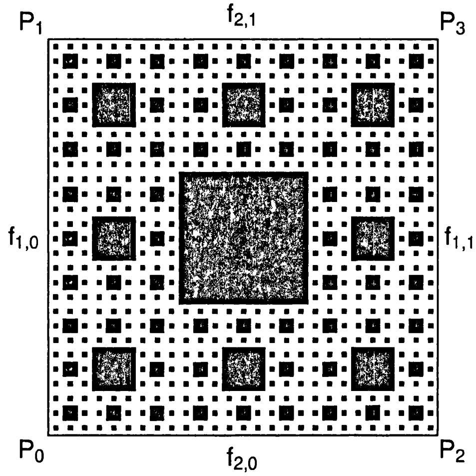

FIGURE 1. $\bigcap_{n=0}^{4} A_n$. The black squares are deleted.

DEFINITION 2. The set $C_s = \bigcap_{n=0}^{\infty} A_n$ is called the *s-dimensional Sierpiński carpet.*

Remarks.

1. Obviously we have $C_2 = C$.

2. C_s is a fractal set with Hausdorff dimension $\alpha_s = \frac{\log(3^s-1)}{\log 3}$, as it can be shown e.g. by using techniques of [Fal90] and [Fal97].

If we regard C_s as the attractor of an *IFS* (*I*terated *F*unctions *S*ystem) and observe that it verifies the open set condition, it can be shown, e.g. using techniques of [Fal97], that $0 < \mathcal{H}^{\alpha_s}(C_s) < \infty$, where $\mathcal{H}^{\alpha_s}(C_s)$ is the α_s-dimensional Hausdorff measure μ on C_s. Hence we introduce the normalized Hausdorff measure μ on C_s ($\mu(A) = \frac{\mathcal{H}^{\alpha_s}(A)}{\mathcal{H}^{\alpha_s}(C_s)}$ for all Borel sets $A \subset C_s$). We will use, for simplicity, the notation α instead of α_s.

If we denote the set of all vertices of the n-cubes building up A_n by V_n and the set of all edges of these cubes by E_n, we get a finite graph $F_n = (V_n, E_n)$ for every $n \in \mathbb{N}\setminus\{0\}$. We will refer to this graph later, when we define the geodesic metric on C_s.

In the present paper we study different types of discrepancies of point sequences on C_s. The notion of *discrepancy* is closely related to that of *uniform distribution*.

In a compact metric space X endowed with a normed Borel measure ν a sequence (x_n) is said to be ν*-uniformly distributed* if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f(x)\, \mathrm{d}\nu(x) \tag{1}$$

holds for all continuous real-valued functions f on X.

It has been proven ([KN74]) that in order that (1) holds, it is necessary and sufficient that the above condition is satisfied for all functions $f = \chi_A$, where $A \subset X$ is any Borel set with $\nu(\partial A) = 0$.

Let $\mathcal{D}$ be a system of Borel sets A ($\subset X$) in the mentioned metric space X, such that $\nu(\partial A) = 0$ for each $A \in \mathcal{D}$.

DEFINITION 3. The (*volume*) *discrepancy* of the sequence $\{x_1, x_2, \dots, x_N\} \subset X$ *with respect to* $\mathcal{D}$ is defined by

$$D_N(x_n) = D_N^{\mathcal{D}}(x_n) = \sup_{A\in\mathcal{D}} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_A(x_n) - \nu(A) \right|,$$

where χ_A is the characteristic function of the set A.

By changing the system $\mathcal{D}$ of Borel sets one gets different types of discrepancies.

Uniform distribution of point sequences occurs and plays a role e.g. when numerical integration has to be done. As it can easily be seen from the definitions, the discrepancy of a point sequence reflects the quantitative measure of "not being uniform distributed". The existence of uniform distributed sequences of points on the fractal C_s follows from [KN74; Chap. 3, Theorem 2.2].

Discrepancies of point sequences on fractals have already been studied for the planar Sierpiński gasket [GT98] and also for the Sierpiński gasket in higher dimension [Kli98]. The equivalence of the studied discrepancies on the gasket has been proven in both cases. In [AMT00] a tight bound for the L_2-discrepancy with respect to halfspaces is found for point sequences on self-similar fractals that fulfill the open set condition.

2. Discrepancies on C_s

In the following we will define and compare different discrepancies on C_s for $s \geq 2$. The measure that we are using here is the normalized Hausdorff measure μ already mentioned.

NOTATIONS. Given two integers $k \leq l$, we shall write $i = \overline{k,l}$ instead of $i = k, \dots, l$ (all integers i which satisfy $k \leq i \leq l$).

In some situations when we define certain cuboids we use the notation $\leq^*$. This has to be read as one of the inequality symbols $<$ and $\leq$, depending on whether we consider the defined cuboid together with its corresponding boundary or not.

The following remarks are useful for the study of C_s.

Remarks.

1. The pairwise parallel faces of A_0 are:

$$f_{i,c} = \{(x_1, \dots, x_s) : \ x_j \in [0,1], \ j \in \{1, \dots, s\} \setminus \{i\}, \ x_i = c\},$$

where $c \in \{0,1\}$ (see Figure 1 for the case $s = 2$).

2. We call *elementary face of a given type* (i, c_i) of an n-cube its face parallel to the face f_{i,c_i} of A_0 and closer (in the Euclidean sense) to f_{i,c_i} than to $f_{i,1-c_i}$.

3. Every n-cube has a *vertex of type* h, $h \in \{0, \dots, 2^s - 1\}$, $h = \sum_{i=1}^{s} c_i \cdot 2^{s-i}$, namely the intersection of the n-cube's faces of type (i, c_i), $i = \overline{1, s}$.

2.1. Some definitions.

Let $\mathcal{D}$ be a system of Borel sets A ($\subset C_s$) such that the boundary ∂A satisfies $\mu(\partial A) = 0$. By taking $X = C_s$ in Definition 3 and by choosing different systems $\mathcal{D}$ of Borel sets we get different discrepancies on C_s. In all these considerations we consider C_s endowed with the Euclidean metric, unless we explicitly mention an other metric (the geodesic metric).

The *elementary discrepancy* $D_N^{\mathcal{E}}$ is the discrepancy defined by the system $\mathcal{E}$ of all elementary cubes (intersected with C_s). We consider each n-cube ($n \geq 1$) together with its faces of type $(i, 0)$, $i = 1, 2, \dots, s$ (i.e. with "half" of its boundary). Moreover, if for some $i \in \{1, 2, \dots, s\}$ a face of type $(i, 1)$ of the n-cube lies on the face of type $(i, 1)$ of an m-cube (containing the n-cube, $m < n$) which is also a face of a deleted m-cube or if the face of type $(i, 1)$ of the considered n-cube lies on $f_{i,1}$, then we take the n-cube together with its face of type $(i, 1)$ (i.e. with more than "half" of its boundary).

Furthermore we consider $\mathcal{D}$ to be the system $\mathcal{S}$ of all sets which are intersections of C_s with cuboids whose faces are parallel to the faces of A_0 and whose vertices belong to C_s and thus we define the *carpet discrepancy* $D_N^{\mathcal{S}}$. The cuboids mentioned here are sets of the form $R = \prod_{i=1}^{s} [a_i, b_i)$, $a_i, b_i \in [0,1]$ for all $i \in \{1,\dots,s\}$ and if for some $i \in \{1,\dots,s\}$ we have $b_i = 1$, then we take $[a_i, b_i]$ instead of $[a_i, b_i)$ in the above product.

The last system $\mathcal{D}$ to be considered here is the one denoted by $\mathcal{C}$ which consists of all sets which are intersections of C_s with cuboids of $\mathcal{S}$ having as a vertex one of the vertices of A_0. For $\mathcal{D} = \mathcal{C}$ we get the *corner discrepancy* $D_N^{\mathcal{C}}$.

If $p \in C_s$ is arbitrary, we denote by $\Delta_h(p)$, $h \in \{0,\dots,2^s-1\}$, the cuboid of $\mathcal{S}$ with p and P_h as diagonal opposite vertices. The points $y \in \Delta_h(p)$ (for $h = \sum_{i=1}^{s} c_i(h) \cdot 2^{s-i}$) are characterized as follows:

$$y \in \Delta_h(p) \iff y \in C_s \text{ and } y_i = \big(1 - q_i(y)\big) \cdot m_i + q_i(y) \cdot M_i\,, \quad 0 \le q_i(y) \le^* 1\,, \tag{3}$$

where

$$m_i = \min\{x_i(p), x_i(P_h)\} \quad \text{and} \quad M_i = \max\{x_i(p), x_i(P_h)\}\,, \qquad i = \overline{1,s}\,. \tag{4}$$

In the above relations y_i, $i = \overline{1,s}$, are the Cartesian coordinates of y, $x_i(P_h)$, $i = \overline{1,s}$, those of P_h and $x_i(p)$, $i = \overline{1,s}$, those of p.

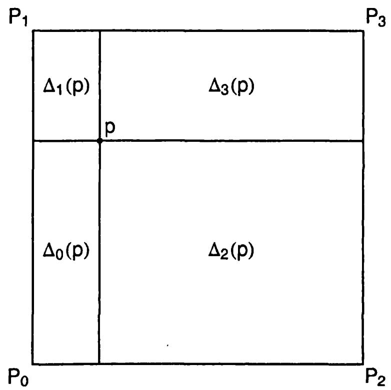

FIGURE 2. The sets defining corner discrepancy for $s = 2$.

Remark. We can define for any cuboid $A \in \mathcal{S}$ its face of type (i, c_i) and its vertex of type h, for $h = \sum_{i=1}^{s} c_i \cdot 2^{s-i}$, $h \in \{0,1,\dots,2^s-1\}$, which we denote by p'_h. With these notations it is easy to see that if $h \in \{0,1,\dots,2^s-1\}$, then $p'_{\overline{h}}$ with $\overline{h} = 2^s - 1 - h$ is the vertex diagonally opposite to p_h in A. Moreover, for A, as above, p'_h and $P_{\overline{h}}$ are diagonally opposite vertices in the cuboid $\Delta_h(p'_{\overline{h}})$.

2.2. Comparison of the elementary, carpet and corner discrepancy.

LEMMA 1. *Let $A \subset C_s$ be a Borel set and $\mathbf{x} = \{x_1, \dots, x_N\} \subset C_s$.*
We set

$$\#(A;N) = \sum_{n=1}^{N} \chi_A(x_n), \qquad D_N(A,\mathbf{x}) = \left| \frac{\#(A;N)}{N} - \mu(A) \right|.$$

If there are $A_1, A_2 \subset C_s$ (Borel sets) such that $A_1 \subset A \subset A_2$, $D_N(A_i,\mathbf{x}) \le \varepsilon$, $i = 1,2$, and $\max_{i=1,2} |\mu(A_i) - \mu(A)| \le \delta$, then $D_N(A,\mathbf{x}) \le \varepsilon + \delta$.

Proof. The inequalities

$$\frac{\#(A_1,N)}{N} - \mu(A_1) + \mu(A_1) - \mu(A) \le \frac{\#(A,N)}{N} - \mu(A)$$
$$\le \frac{\#(A_2,N)}{N} - \mu(A_2) + \mu(A_2) - \mu(A)$$

yield

$$\left| \frac{\#(A;N)}{N} - \mu(A) \right| \le \max_{i=1,2} \left| \frac{\#(A_i;N)}{N} - \mu(A_i) \right| + \max_{i=1,2} |\mu(A_i) - \mu(A)|.$$

Hence by the inequalities in the hypothesis we get $D_N(A,\mathbf{x}) \le \varepsilon + \delta$. □

PROPOSITION 2. *For any finite sequence of points $\{x_1, x_2, \dots, x_N\} \subset C_s$,*

$$D_N^{\mathcal{E}} \le D_N^{\mathcal{S}} \le c_{(s)} (D_N^{\mathcal{E}})^{1-\frac{s-1}{\alpha}} \qquad \textit{for all} \quad s \ge 2. \tag{5}$$

Proof. The left inequality follows directly from the definitions. Now we will prove the right inequality. Let R be a cuboid of $\mathcal{S}$ and let T_n be the union of all n-cubes contained in R. Then the number of n-cubes which intersect the boundary of R is less than $2s(3^{s-1})^n$. On the other hand, $T_n \backslash T_{n-1}$ includes less than $2s(3^{s-1})^{n-1}(3^s - 1 - 3^{s-1})$ n-cubes.

As $T_n = T_n \backslash T_{n-1} \cup T_{n-1} \backslash T_{n-2} \cup \dots \cup T_1 \backslash T_0 \cup T_0$, the number of elementary cubes of level $\le n$ contained in T_n is less than

$$2s(3^s - 1 - 3^{s-1}) \cdot \left((3^{s-1})^{n-1} + (3^{s-1})^{n-2} + \dots + (3^{s-1}) + 1\right)$$
$$= 2s(3^s - 1 - 3^{s-1}) \cdot \frac{(3^{s-1})^n - 1}{3^{s-1} - 1}.$$

Lemma 1 and the fact that the (normalized Hausdorff) measure of an n-cube is $(3^s - 1)^{-n}$ yield:

$$D_N^{\mathcal{S}} \le \left(2s(3^s - 1 - 3^{s-1}) \frac{(3^{s-1})^n - 1}{3^{s-1} - 1} + 2s(3^{s-1})^n \right) D_N^{\mathcal{E}} + 2s(3^{s-1})^n \frac{1}{(3^s - 1)^n},$$

which implies, for any $m \in \mathbb{N}\backslash\{0\}$,

$$D_N^{\mathcal{S}} \le 2s\Big((3^s-1-3^{s-1})\frac{(3^{s-1})^m-1}{3^{s-1}-1}+(3^{s-1})^m\Big)D_N^{\mathcal{E}}+2s(3^{s-1})^m\frac{1}{(3^s-1)^m}\,.$$

Inserting $m=\Big[\log_{3^s-1}\frac{1}{D_N^{\mathcal{E}}}\Big]$ we get:

$$\log_{3^s-1}\frac{1}{D_N^{\mathcal{E}}}-1<m\le\log_{3^s-1}\frac{1}{D_N^{\mathcal{E}}}\,,$$

$$\begin{aligned}
D_N^{\mathcal{S}} &\le \Big[2s\Big((3^s-1-3^{s-1})\frac{(3^{s-1})^m-1}{3^{s-1}-1}+(3^{s-1})^m\Big)+2s(3^{s-1})^m(3^s-1)\Big]D_N^{\mathcal{E}}\\
&=2s\Big[(3^s-1-3^{s-1})\frac{(3^{s-1})^m-1}{3^{s-1}-1}+(3^{s-1})^m(1+3^s-1)\Big]D_N^{\mathcal{E}}\\
&=2s\bigg(\frac{3^s-1-3^{s-1}+(3^{s-1}-1)3^s}{3^{s-1}-1}\cdot(3^{s-1})^m-\frac{2\cdot3^{s-1}-1}{3^{s-1}-1}\bigg)D_N^{\mathcal{E}}\\
&=2s\bigg(\frac{3^s-1-3^{s-1}+3^{2s-1}-3^s}{3^{s-1}-1}\cdot(3^{s-1})^m-\frac{2\cdot3^{s-1}-1}{3^{s-1}-1}\bigg)D_N^{\mathcal{E}}\\
&\le 2s\frac{3^{2s-1}-3^{s-1}-1}{3^{s-1}-1}(3^{s-1})^m D_N^{\mathcal{E}}\,.
\end{aligned}$$

On the other hand

$$m\le\log_{3^s-1}\frac{1}{D_N^{\mathcal{E}}}=\log_{3^{s-1}}\frac{1}{D_N^{\mathcal{E}}}\cdot\log_{3^s-1}3^{s-1}\,,$$

which implies

$$(3^{s-1})^m\le(3^{s-1})^{\log_{3^{s-1}}\frac{1}{D_N^{\mathcal{E}}}\cdot\log_{3^s-1}3^{s-1}}=\Big(\frac{1}{D_N^{\mathcal{E}}}\Big)^{\frac{\log 3^{s-1}}{\log 3}\cdot\frac{\log 3}{\log(3^s-1)}}=(D_N^{\mathcal{E}})^{-\frac{s-1}{\alpha}}\,,$$

thus

$$D_N^{\mathcal{E}}\le\underbrace{2s\cdot\frac{3^{2s-1}-3^{s-1}-1}{3^{s-1}-1}}_{c(s)}\cdot(D_N^{\mathcal{E}})^{1-\frac{s-1}{\alpha}}\,.$$

□

PROPOSITION 3. *For any finite sequence of points* $\{x_1,x_2,\dots,x_N\}\subset C_s$*,*

$$D_N^{\mathcal{C}}\le D_N^{\mathcal{S}}\le 2^s D_N^{\mathcal{C}}\qquad \textit{for all}\quad s\ge2\,. \tag{6}$$

P r o o f . Let A be a cuboid of $\mathcal{S}$. With the notations of the above remarks we have $A \subset \Delta_0(p'_{2^s-1})$.

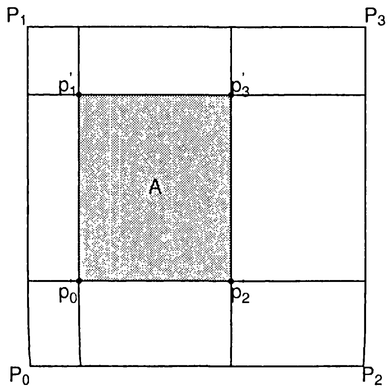

FIGURE 3. Relating the sets defining carpet and corner discrepancy.

We denote by X_i^1, $i \in \{1,2,\dots,s\}$, the Cartesian coordinates of the vertex p'_{2^s-1} of A and by X_i^0, $i \in \{1,2,\dots,s\}$, the Cartesian coordinates of the vertex p'_0 of A.

For $(i_1,i_2,\dots,i_s) \in \{0,1\}^s$ we define

$$S(i_1,i_2,\dots,i_s) = \{p \in C_s :\ x_k \leq^* X_k^{i_k},\ \ k = \overline{1,s}\},$$

where x_k, $k = \overline{1,s}$, are the Cartesian coordinates of the point p.

Every set $S(i_1,i_2,\dots,i_s)$ belongs to $\mathcal{C}$ (it is easy to see that it coincides with one of the sets $\Delta_0(p')$, where p' is one of the vertices of A).

We denote by $\#(i_1,\dots,i_s)$ the number of coordinates of the s-tuple $(i_1,\dots,i_s)$ which are equal 1.

By the inclusion-exclusion principle we obtain

$$\chi_A = \sum_{(i_1,\dots,i_s)\in\{0,1\}^s} (-1)^{s-\#(i_1,\dots,i_s)} \chi_{S(i_1,i_2,\dots,i_s)},$$

where by $-S$ we denote the complement of a set S.

By integration we derive

$$\mu(A) = \sum_{(i_1,\dots,i_s)\in\{0,1\}^s} (-1)^{s-\#(i_1,\dots,i_s)} \mu\big(S(i_1,i_2,\dots,i_s)\big).$$

Thus

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N}\chi_A(x_n) - \mu(A)\right| \\
&= \left|\frac{1}{N}\sum_{n=1}^{N}\sum_{(i_1,\dots,i_s)\in\{0,1\}^s}(-1)^{s-\#(i_1,\dots,i_s)}\chi_{S(i_1,i_2,\dots,i_s)}\right. \\
&\qquad \left. - \sum_{(i_1,\dots,i_s)\in\{0,1\}^s}(-1)^{s-\#(i_1,\dots,i_s)}\mu\big(S(i_1,i_2,\dots,i_s)\big)\right| \\
&\le \sum_{(i_1,\dots,i_s)\in\{0,1\}^s}\left|\frac{1}{N}\sum_{n=1}^{N}(-1)^{s-\#(i_1,\dots,i_s)}\chi_{S(i_1,i_2,\dots,i_s)}(x_n)\right. \\
&\qquad\qquad \left. - (-1)^{s-\#(i_1,\dots,i_s)}\mu\big(S(i_1,i_2,\dots,i_s)\big)\right| \\
&= \sum_{(i_1,\dots,i_s)\in\{0,1\}^s}\left|\frac{1}{N}\sum_{n=1}^{N}\chi_{S(i_1,i_2,\dots,i_s)}(x_n) - \mu\big(S(i_1,i_2,\dots,i_s)\big)\right| \le 2^s D_N^{\mathcal{C}}\,.
\end{aligned}$$

□

COROLLARY 4. *For any finite sequence of points* $\{x_1,x_2,\dots,x_N\}\subset C_s$,

$$\frac{1}{2^s}D_N^{\mathcal{E}} \le D_N^{\mathcal{C}} \le c(s)\big(D_N^{\mathcal{E}}\big)^{1-\frac{s-1}{\alpha}} \qquad \textit{for all} \quad s\ge 2\,. \tag{7}$$

Proof. It follows immediately from the last two propositions. □

2.3. The isotropic discrepancy on C_s.

In the following we introduce a new type of discrepancy on C_s: by taking $\mathcal{D}$ in Definition 3 to be the (much larger) class of sets

$$\mathcal{J} = \{C:\ C = A\cap C_s\,,\ A \text{ is a convex set contained in the unit cube } A_0\subset\mathbb{R}^s\}$$

we get the *isotropic discrepancy* $D_N^{\mathcal{J}}$ of a sequence $\{x_1,x_2,\dots,x_N\}\subset C_s$.

Remark. By simple arguments one can show that for any $C\in\mathcal{J}$, $C = A\cap C_s$, we have $\mathcal{H}^\alpha(\partial A) = 0$ and thus $\mu(\partial A\cap C_s) = 0$. In particular, the α-dimensional Hausdorff measure of any bounded region of an $(s-1)$-dimensional hyperplane in $\mathbb{R}^s$ is zero.

Our next aim is to compare $D_N^{\mathcal{J}}$ with $D_N^{\mathcal{S}}$. In order to do this we first give some definitions.

A *closed convex polytope* is defined as the convex hull of a finite number of points in $\mathbb{R}^s$. An *open convex polytope* is defined as the interior (with respect to the usual topology of $\mathbb{R}^s$) of a closed convex polytope.

Let us define

$$\mathcal{P} = \{C : C = P \cap C_s ,\ P \text{ is an open or closed convex polytope contained in } A_0 \text{ with vertices in } C_s\} .$$

Remark. It suffices to consider instead of $\mathcal{J}$ the smaller class $\mathcal{P}$ in order to compute the isotropic discrepancy of a finite sequence of points on C_s. The proof of this fact is verbally the same as that of [KN74; p. 94, Theorem 1.5], where the analogous result is established for the isotropic discrepancy on the unit cube in $\mathbb{R}^s$.

PROPOSITION 5. *For every sequence $\{x_1, x_2, \dots, x_s\} \subset C_s$ we have, with the above notations,*

$$D_N^{\mathcal{S}} \le D_N^{\mathcal{J}} \le \big(1 + 4s(3^s - 1)\big)\big(D_N^{\mathcal{S}}\big)^{1-\frac{s-1}{\alpha}} . \tag{8}$$

Proof. The first inequality follows immediately from the definitions, as $\mathcal{S} \subset \mathcal{J}$.

Now we prove the second inequality. Let C be a set of $\mathcal{J}$, $C = A \cap C_s$. By the previous remark, we may assume for simplicity that $C = P \cap C_s$, where P is an open convex polytope or a closed convex polytope contained in A_0 with vertices in C_s.

We show that we can find two sets P_1 and P_2, both of them finite unions of cuboids like those defining $\mathcal{S}$, such that $P_1 \subseteq P \subseteq P_2$. We construct P_1 and P_2 such that we can apply Lemma 1.

Let r be an arbitrary positive integer. For every lattice point $(h_1, h_2, \dots, h_s)$ with $0 \le h_j < 3^r$ for all $1 \le j \le s$ we define a cuboid

$$A^{(r)}_{h_1 h_2 \cdots h_s} = \Big\{(x_1, x_2, \dots, x_s) \in \mathbb{R}^s : \tfrac{h_j}{3^r} \le x_j \le^* \tfrac{h_j+1}{3^r} \text{ for } 1 \le j \le s\Big\} .$$

The collection $\mathcal{A}^{(r)}$ of those cuboids forms a partition of A_0. We take $P_1 := P_1^{(r)}$, where $P_1^{(r)}$ is the union of all cuboids of $\mathcal{A}^{(r)}$ that are entirely contained in P. We define $P_2 := P_2^{(r)}$ to be the union of all cuboids of $\mathcal{A}^{(r)}$ whose intersection with P is nonvoid.

It is easy to see that if we fix $s-1$ integers $h_1, \dots, h_{s-1}$ satisfying the above conditions, then the integers h, $0 \le h < 3^r$, with $A^{(r)}_{h_1 \cdots h_{s-1} h} \subseteq P$ are consecutive integers (because of the convexity of P). Thus the union of those cuboids $A^{(r)}_{h_1 \cdots h_{s-1} h}$ is a cuboid like those defining $\mathcal{S}$. These yield that P_1 can be written as the union of at most $3^{r(s-1)}$ such cuboids. One can show in the same way that P_2 can be written as the union of at most $3^{r(s-1)}$ cuboids like those mentioned before. Thus we have:

$$\max_{i=1,2}\left|\frac{\#(P_i \cap C_s; N)}{N} - \mu(P_i \cap C_s)\right| \le 3^{r(s-1)} D_N^{\mathcal{S}} .$$

Now we estimate $\left|\mu(P_i \cap C_s) - \mu(P \cap C_s)\right|$ for $i = 1,2$.

It is easy to see that the number of 3^{-r}-grid cubes intersecting the set $P_2 \setminus P$ is not greater than $2 \cdot 2s \cdot (3^r)^{s-1}$. Thus

$$\mu(P_2 \cap C_s) - \mu(P \cap C_s) \le 4 \cdot s \cdot \frac{(3^r)^{s-1}}{(3^s-1)^r} ,$$

since the normalized Hausdorff measure of an r-cube is $(3^s-1)^{-r}$.

Analogously one can show

$$\mu(P \cap C_s) - \mu(P_1 \cap C_s) \le 4 \cdot s \cdot \frac{(3^r)^{s-1}}{(3^s-1)^r} .$$

Thus

$$\left|\frac{\#(P \cap C_s; N)}{N} - \mu(P \cap C_s)\right| \le (3^{s-1})^r D_N^{\mathcal{S}} + 4s \cdot \left(\frac{3^{s-1}}{3^s-1}\right)^r .$$

Since this upper bound does not depend on P, we get

$$D_N^{\mathcal{J}} \le (3^{s-1})^r D_N^{\mathcal{S}} + 4s \cdot \left(\frac{3^{s-1}}{3^s-1}\right)^r .$$

This holds for any positive integer r. We take $r := \left\lfloor \log_{3^s-1} \frac{1}{D_N^{\mathcal{S}}} \right\rfloor$. It is easy to show that

$$(3^{s-1})^r \le \left(D_N^{\mathcal{S}}\right)^{-\frac{s-1}{\alpha}} \qquad \text{and} \qquad \left(\frac{3^{s-1}}{3^s-1}\right)^r \le (3^s-1) \cdot \left(D_N^{\mathcal{S}}\right)^{1-\frac{s-1}{\alpha}} .$$

Hence

$$D_N^{\mathcal{J}} \le \bigl(1 + 4s(3^s-1)\bigr)\left(D_N^{\mathcal{S}}\right)^{1-\frac{s-1}{\alpha}} .$$

□

2.4. Uniform distribution on C_s.

Propositions 2, 3, 5 and Corollary 4 show that the discrepancies $D_N^{\mathcal{E}}$, $D_N^{\mathcal{C}}$, $D_N^{\mathcal{S}}$ and $D_N^{\mathcal{J}}$ are equivalent in the following sense:

Proposition. *For all sequences $(x_n) \subset C_s$,*

$$\lim_{N\to\infty} D_N^{\mathcal{E}}(x_n) = 0 \iff \lim_{N\to\infty} D_N^{\mathcal{C}}(x_n) = 0$$
$$\iff \lim_{N\to\infty} D_N^{\mathcal{S}}(x_n) = 0 \iff \lim_{N\to\infty} D_N^{\mathcal{J}}(x_n) = 0 .$$

As C_s is a compact metric space, we can apply the general theory of uniform distribution.

Since a continuous function f on C_s is uniformly continuous, we can approximate f uniformly by characteristic functions of elementary cubes. Therefore, if any of the four discrepancies above tends to zero as $N \to \infty$ for some sequence (x_n), then (x_n) is uniformly distributed in the sense of Definition 1.

2.5. The geodesic metric on C_s.

We introduce a metric on C_s as follows: any points $x, y \in C_s$ are contained in k-cubes, $k \geq 1$, denoted by $Q_k(x)$ and $Q_k(y)$. Let x_k and y_k be vertices of type h_0 ($h_0 \in \{0, \dots, 2^s-1\}$, P_{h_0} is the reference vertex — for simplicity we may fix $h_0 = 0$) of $Q_k(x)$ and $Q_k(y)$, respectively, which are also vertices of the finite graph F_k. We define

$$d(x,y) = \lim_{k\to\infty} 3^{-k} d_k(x_k, y_k) ,$$

where d_k is the length of the shortest chain in F_k containing x_k and y_k. d is a metric on C_s, called the *geodesic metric*, $d(x,y)$ is the length of the shortest continuous curve in C_s connecting x and y.

Remark. It can be shown ([Cri02]) that the geodesic metric and the Euclidean metric on C_s are equivalent. This implies that both metrics lead to the same notion of (μ-)uniformly distributed sequences.

We will use this metric in order to define a new discrepancy on C_s for $s = 2$.

3. The planar case. Special ball discrepancy

3.1. Circles and balls on the carpet.

Let us analyse the circles and balls (with respect to the geodesic metric) having the centre $p_0 \in V_0$ and radius $r > 0$ or $p_0 \in V_k \setminus V_{k-1}$ and $0 < r < 3^{-(k-1)}$ for $k \geq 1$.

We call "elementary diagonal" any diagonal of some elementary square.

One can easily see that the circles mentioned above are intersections of C with the boundary of squares having the centre (intersection of their diagonals) in p_0, the diagonals parallel to the edges of A_0 and the length of their diagonals $2r$. Hence the circles are unions of Cantor sets. If the edges of the squares whose boundaries build the circles are elementary diagonals, then every such edge is in fact the image of the classical Cantor set on $[0,1]$ by an affine transformation having as linear part a contraction of ratio 3^{-k}.

The balls mentioned above are intersections of C with squares having the centre p_0, the diagonals parallel to the edges of A_0 and the length of their diagonals $2r$.

Remark. It is easy to notice that any point $p \in V_k$, $k \in \mathbb{N}$, is, if we relate it to an elementary square of level $k-1$ which contains it, a point of the type q_1 (vertex of exactly one of the eight k-squares contained in the $(k-1)$-square and thus necessarily vertex of the $(k-1)$-square), q_2 (common vertex of exactly two of the eight k-squares contained in the $(k-1)$-square) or q_3 (common vertex of exactly three of the eight k-squares contained in the $(k-1)$-square).

Figure 4 shows the boundaries of concentric balls (we have to intersect the lines shown with C) having their centre in q_1, q_2, or q_3.

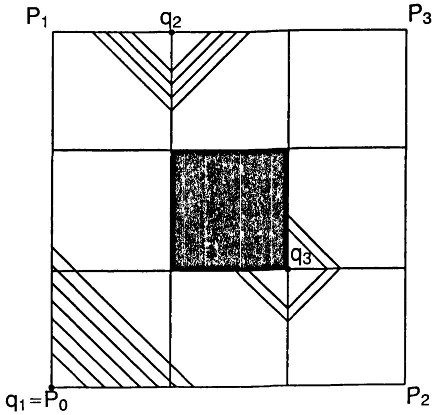

FIGURE 4. Circles on C.

As a conclusion may write

$$B(p_0, r) = C \cap \bigcup_{\substack{v_i \in \{\pm 1\} \\ i=1,2}} \underbrace{|p_0, p_0 + v_1 r \mathbf{e}_1, p_0 + v_2 r \mathbf{e}_2|}_{2\text{-simplex}}$$

where $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ (unit vectors), $p_0 \in V_k$, $k \in \mathbb{N}$, $r > 0$.

3.2. Discrepancies on C. The special ball discrepancy.

As we have seen in the previous section, we can define different discrepancies on C with respect to different systems $\mathcal{D}$. By fixing $s = 2$ we get from the previous section the elementary, the carpet, the corner and the isotropic discrepancy on C.

In the following we define and study an other notion of discrepancy on the planar Sierpiński carpet, the *special ball discrepancy*. We define *special balls* to be balls (with respect to the geodesic metric) having as their centre a point $p_0 \in \bigcup_{k\in\mathbb{N}} V_k$ and radius $r = 3^{-k}$ if $p \in V_k$.

We will compare $D_N^{\mathcal{B}_s}$ with the other four already mentioned discrepancies on C.

In the following we approach the special ball discrepancy $D_N^{\mathcal{B}_s}$ making use of the structure of the balls having as their centre a vertex of some elementary square of level n, $n \in \mathbb{N}$, and the radius 3^{-n}, $n \in \mathbb{N}\setminus\{0\}$.

Proposition 7. *We have*

$$\frac{1}{64} D_N^{\mathcal{E}} \leq D_N^{\mathcal{B}_s} \leq 12 D_N^{\mathcal{E}} + 64\left(D_N^{\mathcal{E}}\right)^{\frac{2}{3}} . \tag{10}$$

P r o o f . It is easy to notice that in a given k-square there are exactly four $(k+2)$-squares which do not have any common edge with any "deleted" elementary square, but necessarily have two common vertices with two "deleted" squares (see Figure 5).

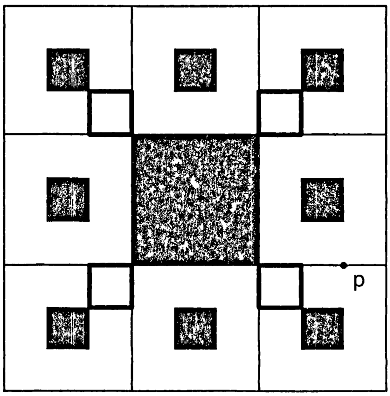

FIGURE 5. Particular $(k+2)$-squares inside a k-square, $k \geq 0$.

First we take an $(n-1)$-square of the same type (with respect to the $(n-3)$-square that contains it) as the four $(k+2)$-squares mentioned above.

It is easy to see that it can be approximated by eight balls of $\mathcal{B}_s$ with radius 3^{-n} and an error not greater than $7 \cdot 8^{-n}$ (see Figure 6).

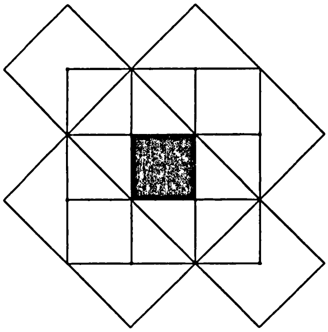

FIGURE 6. Covering an elementary square with special balls on C.

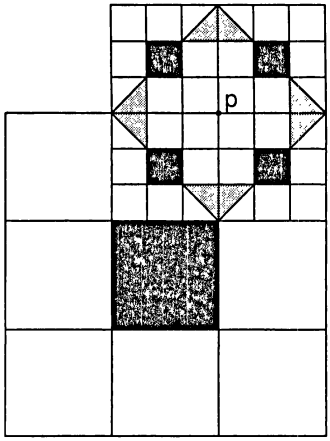

FIGURE 7. Covering a special ball with elementary squares on C.

These imply (as any other elementary square of level $(n-1)$ needs at most that many balls from $\mathcal{B}_s$ to be covered with)

$$D_N^{\mathcal{E}} \leq 8D_N^{\mathcal{B}_s} + 7 \cdot 8^{-m} \qquad \text{for all} \quad m \in \mathbb{N} \backslash \{0\}\,.$$

The deriving of this inequality can be done by using Lemma 1 (e.g. by letting A be the elementary square to be covered, A_1 the union of the two special balls included in A and A_2 the union of the mentioned special balls covering A).

For $m = \left[\log_8 \frac{1}{D_N^{\mathcal{B}_s}}\right]$ we get $D_N^{\mathcal{E}} \leq 64 D_N^{\mathcal{B}_s}$.

For the second inequality in (10) we choose a ball of $\mathcal{B}_s$ with the radius 3^{-k} and with the centre the common vertex of four "undeleted" k-squares, $k \geq 2$ (see Figure 5 and Figure 7).

It is easy to see that the ball is the union of twelve $(k+1)$-squares and eight rectangular triangles (each of them is a "half-square" of level $(k+1)$). Every such triangle is a union of three $(k+2)$-squares and two "half-squares" of level $k+2$.

Going on with this procedure we can conclude, after $L+1$ steps, that the given ball may be covered by not more than $12 + 8 \cdot 3 \cdot \left(1 + 2 + 2^2 + \cdots + 2^L\right)$ elementary squares of level $\leq k + L + 1$ with an error of $8 \cdot 2^{L+1} \cdot 8^{-(k+L+1)}$.

By applying Lemma 1, we get, for $L \in \mathbb{N}$,

$$D_N^{\mathcal{B}_s} \le \big(12 + 8 \cdot 3 \cdot (2^L - 1)\big) D_N^{\mathcal{E}} + 8 \cdot 2^{L+1} \cdot 8^{-(L+1)},$$

and for $L = \left[\log_8 \frac{1}{D_N^{\mathcal{E}}}\right]$ we have

$$D_N^{\mathcal{B}_s} \le \left(12 + 8 \cdot 3 \cdot 2 \cdot \left(D_N^{\mathcal{E}}\right)^{-\frac{1}{3}}\right) D_N^{\mathcal{E}} + 8 \cdot 2 \cdot (D_N^{\mathcal{E}})^{\frac{2}{3}} = 12 D_N^{\mathcal{E}} + 64 (D_N^{\mathcal{E}})^{\frac{2}{3}}.$$

□

Remark. (The case $s = 1$.) It is easy to see that the analogon of C_s in $\mathbb{R}$ is the well-known two-thirds-Cantor set, let us denote it here by C_1. Lemma 1 holds also for $s = 1$. The assertion of Proposition 2 becomes $D_N^{\mathcal{C}} \le D_N^{\mathcal{S}} \le 2 D_N^{\mathcal{C}}$, which is true ([KN74; Chap. 2, Theorem 1.3] states the analogous result on the unit interval). The assertion of Proposition 5 becomes $D_N^{\mathcal{S}} \le D_N^{\mathcal{J}} \le 5 D_N^{\mathcal{S}}$ and can be proven analogously. Problems occur when one tries to reconstruct the proof of Proposition 2 on C_1 in the same way.

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