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PERIODIC BOUNDARY VALUE PROBLEM FOR CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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Recently, B. Mehri [1] established conditions under which the equation

$$x''' + ax'' + bx' + h(t, x) = p(t) \tag{1}$$

admits a periodic solution. His proving technique consists of defining a rather complicated Green function to the given periodic boundary value problem and verifying the conditions of Schauder fixed point theorem.

Here a similar method is applied to the equations

$$x''' + ax'' + g(t, x') + cx = p(t), \tag{2}$$

resp.

$$x''' + f(t, x'') + bx' + cx = p(t), \tag{3}$$

but with the Green function, constructed in an extremely simple way, when $\operatorname{sgn} a = \operatorname{sgn} c \neq 0$, resp. $b^3 < -27c^2/4$; however, it is practically impossible to employ the same Green function to the equation (1) with an oscillatory restoring term (see Corollary).

In what follows let us assume that the following conditions are satisfied:

1) *a, b, c are real constants,*
2) $f(t, z)$, $g(t, y) \in C^0(R^1 \times R^1)$, $p(t) \in C^0(R^1)$,
3) *$f(t, z)$, $g(t, y)$, $p(t)$ are ω-periodic in a t-variable for all t, x, y, $z \Rightarrow \exists P$: $|p(t)| \leq P$ for $t \in \langle 0, \omega \rangle$ /ω, P — const./.*

Consider the periodic boundary value problem:

$$x''' + ax'' + bx' + cx = 0, \tag{4}$$

$$x(0) = x(\omega), \quad x'(0) = x'(\omega), \quad x''(0) = x''(\omega) \tag{B}$$

and to it the corresponding Green function, defined in the following way:

$$t \leqslant \tau \ldots G(t, \tau) :=$$

$$= \frac{e^{\varrho_1(t+\omega-\tau)}(\varrho_3 - \varrho_2)}{(1 - e^{\varrho_1 \omega})\Delta(\varrho_1, \varrho_2, \varrho_3)} + \frac{e^{\varrho_2(t+\omega-\tau)}(\varrho_1 - \varrho_3)}{(1 - e^{\varrho_2 \omega})\Delta(\varrho_1, \varrho_2, \varrho_3)} + \frac{e^{\varrho_3(t+\omega-\tau)}(\varrho_2 - \varrho_1)}{(1 - e^{\varrho_3 \omega})\Delta(\varrho_1, \varrho_2, \varrho_3)},$$

$$t \geqslant \tau \ldots G(t, \tau) :=$$

$$= \frac{e^{\varrho_1(t-\tau)}(\varrho_3 - \varrho_2)}{(1 - e^{\varrho_1 \omega})\Delta(\varrho_1, \varrho_2, \varrho_3)} + \frac{e^{\varrho_2(t-\tau)}(\varrho_1 - \varrho_3)}{(1 - e^{\varrho_2 \omega})\Delta(\varrho_1, \varrho_2, \varrho_3)} + \frac{e^{\varrho_3(t-\varrho)}(\varrho_2 - \varrho_1)}{(1 - e^{\varrho_3 \omega})\Delta(\varrho_1, \varrho_2, \varrho_3)},$$

where

$$\Delta(\varrho_1, \varrho_2, \varrho_3) := \det \begin{vmatrix} 1 & 1 & 1 \\ \varrho_1 & \varrho_2 & \varrho_3 \\ \varrho_1 & \varrho_2 & \varrho_3 \end{vmatrix}$$

and $\varrho_1, \varrho_2, \varrho_3$ is a triad of solution of the characteristic equation [cf. (4)]:

$$\varrho^3 + a\varrho^2 + b\varrho + c = 0. \tag{4'}$$

In order such a definition to be correct, it is enough to show that the discriminant $D(\varrho_1, \varrho_2, \varrho_3)$ of the equation (4′) is not equal to zero, because we have for the Vandermond determinant that

$$\Delta(\varrho_1, \varrho_2, \varrho_3) = \prod_{k=1}^{2} \prod_{i=k+1}^{3} (\varrho_i - \varrho_K) \neq 0$$

holds if and only if $\varrho_1, \varrho_2, \varrho_3$ differ one from another.

We have, however, for $b = 0$ [cf. (2)]:

$$D(\varrho_1, \varrho_2, \varrho_3) = \left[\frac{4}{27} a^6 - 27 \left(c - \frac{2}{27} a^3\right)^2\right] \neq 0 \Leftrightarrow c \neq 0$$

and for $a = 0$ [cf. (3)]:

$$D(\varrho_1, \varrho_2, \varrho_3) = [-4b^3 - 27c^2] \neq 0 \Leftrightarrow b^3 \neq -\frac{27}{4} c^2.$$

Consequently, if we consider the operator equations

$$x(t) = \int_0^\omega G(t, \tau)[p(\tau) - g(\tau, x'(\tau))]d\tau, \tag{2'}$$

resp.

$$x(t) = \int_0^\omega G(t, \tau)[p(\tau) - f(\tau, x\ (\tau))]d\tau, \tag{3'}$$

then each of their solutions represents on the interval $\langle 0, \omega \rangle$ also the solution of (2), resp. (3), satisfying (B).

Deriving (2′) once and (3′) twice, we obtain:

$$(2'') \qquad x'(t)=\int_0^{\omega}\frac{\partial G(t,\tau)}{\partial t}[p(\tau)-g(\tau,x'(\tau)]d\tau \quad /:=T_1(x')/,$$

resp.

$$(3'') \qquad x''(t)=\int_0^{\omega}\frac{\partial^2 G(t,\tau)}{\partial t^2}[p(\tau)-f(\tau,x\ (\tau)]d\tau \quad /:=T_2(x\ )/.$$

Hence the expressions

$$\int_0^t \bar{x}'(\tau)d\tau+C_1 \quad \text{resp.} \quad \int_0^t\int_0^{\tau}\bar{x}''(s)ds\,d\tau+C_1t+C_2$$

with suitable constants C_1, C_2 must satisfy the operator equations (2′) resp. (3′) if and only if $\bar{y}=\bar{x}'(t)$ resp. $\bar{z}=\bar{x}''(t)$ are appropriate fixed points of operators $T_1(y)$ resp. $T_2(z)$. Therefore our purpose is to prove existence of such fixed points.

In order to apply Schauder's fixed point theorem for this goal, it is necessary to prove that for some closed and convex subset $\mathscr{K}$ of the Banach space $\mathscr{B}$ of all continuous functions on the interval $\langle 0,\omega\rangle$ with the norm

$$\|\cdot\|:=\max_{t\in\langle 0,\omega\rangle}|\cdot|$$

the conditions

$$\begin{array}{ll} \text{I)} & T_i(\mathscr{K})\subset\mathscr{K} \\ \text{II)} & T_i(\mathscr{K}) \textit{ is compact} \end{array} \quad \textit{for} \quad i=1,2$$

and

are satisfied.

If we choose $\mathscr{K}$ as $\mathscr{K}:=\{x\in B \mid \|x\|\leqslant D,\ D\textit{-const.}\}$, then it is clear that $\mathscr{K}$ is a closed and convex set. Thus, for satisfying I) it is enough to prove that for every $y\in\mathscr{K}$ there is $\|T_1(y)\|\leqslant D$, resp. for every $z\in\mathscr{K}$, there is $\|T_2(z)\|\leqslant D$.

This can be fulfilled, when there exist such constants G^*, G_1 resp. F^*, G_2 that there is for $\operatorname{sgn} a=\operatorname{sgn} c\neq 0$:

$$\|T_1(y)\|\leqslant\omega G_1(P+G^*)\leqslant D_1,$$

resp. for $b^3<-27c^2/4\neq 0$:

$$\|T_2(z)\|\leqslant\omega G_2(P+F^*)\leqslant D_2,$$

where

$$G^*=\max_{\substack{t\in\langle 0,\omega\rangle\\ |y|\leqslant D_1}}|g(t,y)| \quad \text{and} \quad \left|\frac{\partial G(t,\tau)}{\partial t}\right|\leqslant G_1, \quad (t,\tau)\in\langle 0,\omega\rangle\times\langle 0,\omega\rangle,$$

resp.

$$F^*=\max_{\substack{t\in\langle 0,\omega\rangle\\ |z|\leqslant D_2}}|f(t,z)| \quad \text{and} \quad \left|\frac{\partial^2 G(t,\tau}{\partial t^2}\right|\leqslant G_2 \quad (t,\tau)\in\langle 0,\omega\rangle\times\langle 0,\omega\rangle.$$

Since it can be readily checked that the compactness of the operators T_1 resp. T_2 would be proved just like in [1], we may write the following

Theorem. *Let the above assumptions 1)—3) and the relations* $\operatorname{sgn} a = \operatorname{sgn} c \neq 0$ *resp.* $b^3 < -27c^2/4 \neq 0$ *be satisfied for (2) resp. (3). If there exist such constants* D_1 *resp.* D_2 *that for (2) resp. (3) the inequalities*

$$\omega G_1(P + G^*) \leq D_1 \quad \textit{resp.} \quad \omega G_2(P + F^*) \leq D_2$$

hold, then the equations (2) resp. (3) admit an ω*-periodic solution.*

Proof. It is obvious that the boundary value problems (2), (B) and (3), (B) are solvable and consequently the ω-periodic prolongation of their solutions must satisfy [cf. (3)] the equations (2) resp. (3) for all t. This completes the proof.

Corollary. *If* $G(t, \tau)$ *is the same Green function as before and* $c \neq 0$, $D(\varrho_1, \varrho_2, \varrho_3) = -4(b - a^2/3)^3 - 27(c - ab/3 + 2a^3/27)^2 > 0$, *then there is*

$$G_0 := \max_{(t, \tau) \in \langle 0, \omega \rangle \times \langle 0, \omega \rangle} |G(t, \tau)| \geq \frac{1}{\omega |c|}$$

valid.

Proof. If not, then $|c| < (\omega G_0)^{-1}$ holds.

Now let us consider (1) with $h(t, x) \equiv h(x) = cx + [h(x) - cx]$ and assume such positive constants h, H can be found that

$$h \leq h(x) \leq H \tag{5}$$

is satisfied for all x and

$$\left| \int_0^\infty p(t) dt \right| < \infty.$$

Then (1) cannot admit any ω-periodic solution $x(t)$, because after substituing $x(t)$ into (1) and integrating (1) over the period $k\omega$ (k-natural), we should come to the incorrect relation

$$\infty = \limsup_{k \to \infty} \left| \int_0^{k\omega} h(x(t)) dt \right| = \limsup_{k \to \infty} \left| \int_0^{k\omega} p(t) dt \right| < \infty.$$

But an easy calculation shows that (1) gives it under [cf. (5)]:

$$\omega G_0(P + H + |c| D_0) \leq D_0,$$

what is implied by the incorrect assumption $|c| < (\omega G_0)^{-1}$ contradictionally. Therefore $G_0 \geq (\omega |c|)^{-1}$ must be satisfied.

Consequence. *The application of the Green function* $G(t, \tau)$ *to the problem*

(1), (B) has not much meaning, when $h(t, x)$ is an oscillatory function in an x-variable for all t, x and

$$\int_0^\omega p(t)dt = 0.$$

Indeed. The sufficient condition should have been namely of the form

$$\max_{\substack{t \in \langle 0, \omega \rangle \\ |x| \leq D}} \omega G_0(P + |h(t, x) - cx|) \leq D,$$

resp. (see Corollary)

$$\max_{\substack{t \in \langle 0, \omega \rangle \\ |x| \leq D}} (P + |h(t, x) - cx|) \leq |c| D$$

at least, the validity of which is complicated by oscillations of the function $h(t, x)$.

REFERENCES

[1] MEHRI, B.: On the existence of periodic solution for certain non-linear differential equation, Rev. Roum. Math. Pures et Appl., 9, 1978, 1367—1373.

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ПЕРИОДИЧЕСКАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ НЕКОТОРЫХ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТРЕТЬЕГО ПОРЯДКА

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Резюме

В работе даются достаточные условия для существования периодических решений уравнений (2) и (3) при помощи теоремы Шаудера о неподвижной точке. Периодические решения уравнений (2) и (3) существуют, например, в случае, когда функции $f(t, z)$, $g(t, y)$ и $p(t)$ ограничены и непрерывны для $\operatorname{sgn} a = \operatorname{sgn} c \neq 0$, $b^3 < -27c^2/4 \neq 0$.

Показывается тоже, что если функция $h(t, x) \equiv h(x)$ из (1) колеблющаяся для всех значений своих аргументов, то использование данной функции Грина для (1) очень трудное и неэфективное.