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A STRONG LAW OF LARGE NUMBERS FOR IDENTICALLY DISTRIBUTED VECTOR LATTICE-VALUED RANDOM VARIABLES

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For over twenty years many authors have devoted their attention to the laws of large numbers and the similar convergence theorems, i.e. the results on the convergence of weighted sums of random variables in one or another sense. While a number of interesting results has been produced for the norm topology, the theory is much less developed for the weak topology and almost completely neglected is the convergence with respect to the order. It is the latter case I intend to discuss now. The main reason for doing so is that in a number of spaces the order convergence is stronger than the topological one. (e.g. L^p-spaces, $1 \leqslant p < \infty$. See [3]).

In what follows I shall consider functions with values in an Archimedean vector lattice E.

Definition 1. *Let (Z, S, P) be a probability space. A sequence (f_n) of functions from Z to E converges to a function f almost uniformly if for every $\varepsilon > 0$ there exists a set $A \in S$ such that $P\{A\} < \varepsilon$ and (f_n) converges relatively uniformly to f uniformly on $Z - A$; i.e. there exists a sequence (a_n) of real numbers converging to 0 and an element $r \in E$ such that $|f_n(z) - f(z)| \leqslant a_n r$ for each $z \in Z - A$.*

Definition 2. *A function $f\colon Z \to E$ is called a random variable if there exists a sequence (f_n) of countably valued random variables such that (f_n) converges to f almost uniformly.*

The question whether or not the set of all random variables is closed with respect to the almost uniform convergence can be answered in the affirmative provided E has the so-called σ-property. (For the definition see [3], [5]).

Proposition 1. *Let E be an Archimedean vector lattice with the σ-property. Then the vector lattice of all random variables is closed with respect to the almost uniform convergence.*

Proof. Let a sequence (f_n) of random variables almost uniformly converge to a function f and for each n let (f_n^k) be a sequence of countably valued random variables almost uniformly converging to f_n. Choose a real sequence (a_n) such that $a_n \downarrow 0$. Then for each n there exists a set $A_n \in S$ such that $P\{A_n^C\} < a_n 2^{-n}$ and

$$|f_n^k(z) - f_n(z)| \leqslant \alpha_n^k r_n$$

for each $z \in A_n$, some $r_n \in E$ and $\alpha_n^k \xrightarrow{k} 0$.

Since E has the σ-property, there is an $u \in E$ such that $r_n \leqslant K(n)\, u$ for each n, where $K(n)$ is a function from N to N, N the set of natural numbers. Denoting $\alpha_n^k K(n)$ by b_n^k we have $|f_n^k(z) - f_n(z)| \leqslant b_n^k u$ for each $z \in A_n$. As the set of real numbers has the diagonal property there exists a sequence $b_n^{k(n)}$ converging to 0. Hence we have

$$|f_n^{k(n)}(z) - f_n(z)| \leqslant b_n^{k(n)} u, \qquad z \in A_n,\ n = 1, 2, \ldots$$

Given $\varepsilon > 0$, put $\varepsilon_0 = 2^{-1}\varepsilon$. Since (f_n) converges to f almost uniformly there exists a set $A_0 \in S$ such that $P\{A_0^C\} < \varepsilon_0$ and $|f_n(z) - f(z)| \leqslant \alpha_n r$ for each $z \in A_0$ and some $r \in E$, $\alpha_n \to 0$. There exists a natural number n_0 such that $a_{n_0} \leqslant 2^{-1}\varepsilon$. Put $Z_\varepsilon = \bigcap_{n=n_0}^{\infty} A_n \cap A_0$. We have $P\{Z_\varepsilon^C\} \leqslant P\{A_0^C\} + \sum_{n=n_0} P\{A_n^C\} \leqslant \sum_{n=n_0}^{\infty} 2^{-n} a_n + 2^{-1}\varepsilon \leqslant 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon$ and

$$|f_n^{k(n)}(z) - f(z)| \leqslant b_n^{k(n)} u + \alpha_n^r \text{ for each } z \in Z_\varepsilon.$$

From now on E is equipped with a locally solid linear metrizable topology (i.e. E has a base of neighbourhoods of 0 consisting of solid sets), P means a complete probability measure.

Proposition 2. *Let E be a vector lattice equipped with a locally solid linear metrizable topology, P be a complete probability measure. Then each random variable is a measurable map from Z into E.*

Proof. There exists a sequence $\{A_k\}$, $A_k \in S$ such that $P\{A_k^C\} < k^{-1}$ and $|f_n(z) - f(z)| \leqslant a_n^k b_k$, $b_k \in E$ for all $z \in A_k$; $k = 1, 2 \ldots$ For each neighbourhood U of zero there exists a continuous monotonous Riesz pseudo-norm r such that $\{x \in E; r(x) < 1\} \subset U$ (see [5]). Because of this and the above inequalities we have that $f_n(z) \to f(z)$ in the topology for each $z \in Z$ except possibly a set of probability 0, since $P\{\bigcup A_k\}^C = 0$. Denote $\bigcup A_k$ by Z_0. An application of [1], prop. 2.2.3 implies that the restriction of f to Z_0 is a measurable function from Z_0 into E. Let B be any Borel subset of E. We have

$$f^{-1}(B) = \{z \in z_0; f(z) \in B\} \cup \{z \in Z - Z_0; f(z) \in B\} \in S.$$

It follows from proposition 2 that each random variable is a random element in the sense of [1]. Also definitions of independent, identically distributed and symmetric random variables coincide with the corresponding definitions in [1].

Definition 3. *A sequence (f_n) of random variables satisfies the strong law of large numbers if there exists an element $a \in E^+$ such that for every $\varepsilon > 0$*

$$\lim_n P\left\{\bigcap_{k=n}^{\infty}\left\{z;\left|k^{-1}\sum_{i=1}^{k} f_i(z)\right| \leqslant \varepsilon a\right\}\right\} = 1.$$

If (f_n) satisfies SLLN, then the consecutive arithmetic means converge to 0 relatively uniformly almost everywhere.

Definition 4. *An Archimedean vector lettice E is called σ-complete if every non-empty at most countable subset of E which is bounded from above has a supremum.*

Definition 5. *A linear topology on an Archimedean vector lattice E is said to be compatible with the ordering if the positive cone of E is closed in this topology.*

Theorem 1. *Let E be a σ-complete vector lattice with the σ-property equipped with a complete compatible metrizable locally solid linear topology. If f_n are independent, identically distributed, symmetric random variables in E, then the condition*

$$\sum_{n=1}^{\infty} P\{z; |f_1(z)| \leqslant na\}^C < \infty \quad \textit{for some } a \in E^+,$$

is necessary and sufficient for (f_n) to satisfy the strong law of large numbers.

Proof. For each n let (f_n^k) be a sequence of countably valued random variables converging almost uniformly to f_n. There exists a set of probability 1 such that $f_n^k \to f_n$ relatively uniformly on this set for each n. Consider now all f_n as functions defined on this set with values in E. Because of the inequality

$$|f_n| \leqslant |f_n - f_n^k| + |f_n^k|$$

which holds for each natural number n and each natural number k and the assumption that E has the σ-property we can regard all f_n as random variables in a principal ideal of E (i.e. ideal generated by a single element, say u, $u \in E^+)I_u$, $a \leqslant u$. Since E is equipped with a compatible topology, $I_u = \bigcup_{n=1}^{\infty} \langle -nu, nu \rangle$ is a Borel set in E. Hence f_n are independent, identically distributed and symmetric random variables in I_u.

Since E is σ-complete vector lattice, I_u equipped with the order-unit norm (i.e. the norm induced by u) is a Banach space (even Banach lattice). It will be denoted by $(I_u, \| \|_u)$. It is well-known that in such a lattice the norm-convergence and the relatively uniform convergence are equivalent. (see [4], p. 102).

Let us denote by $(y_n)_1^{\infty}$ the set of all values which the above mentioned countably valued random variables f_n^k take on. Put $y_0 = u$. Comsider the countable set

$A=\left\{\sum_{i=0}^{n} a_i y_i\,;\ n=0,1\ \dots\right\}$ of all linear combinations of y_i with the rational coefficients a_i. The set $B=\bigcap_{r\in Q}\bigcup_{a\in A}\{x\in I_u\,;\ |x-a|\leqslant ru\}$ is a linear subspace of I_u. (Q stands for the set of all rational numbers). This follows from the inequalities

$$|x+y|\leqslant|x|+|y| \text{ and}$$
$$|ax-by|\leqslant|a-b|\,|x|+|b|\,|x-y|.$$

It is obvious that all f_n take on only values in B. Equipped with the norm $\|\ \|_u$, B becomes a separable Banach space. Indeed for each $x\in B$ and each $\varepsilon>0$ there exists an element $a\in A$ such that $\|x-a\|_u<\varepsilon$. The completeness follows from the fact that B is closed in $(I_u,\|\ \|_u)$.

From now on this space will be denoted by $(B,\|\ \|_u)$. It remains to prove that f_n will maintain all the properties mentioned above. Since B is separable, its Borel sets are generated by open balls. Denote these Borel sets by W_S. For such a ball we have

$$\{x\in B\,;\ \|x-x_i\|_u<\varepsilon\}=\bigcup_n\{x\in B\,;\ \|x-x_i\|_u\leqslant\varepsilon(1-n^{-1})\}=$$
$$=\bigcup_n B\cap\{x\in I_u\,;\ \|x-x_i\|_u\leqslant\varepsilon(1-n^{-1})\}=$$
$$=B\cap\bigcup_n\{x\in I_u\,;\ |x-x_i|\leqslant\varepsilon(1-n^{-1})u\}$$

and this is a measurable set with respect to the topology induced on B by the original topology. The σ-algebra of these sets will be denoted by W_T. We have proved that $W_S\subset W_T$. It means that (f_n) are independent, identically distributed and symmetric random variables in $(B,\|\ \|_u)$.

Consider now the random variable f_1. We have

$$E\|f_1\|_u\leqslant 1+\sum_{n=1}^{\infty}P\{\|f_1\|_u>n\}=1+\sum_{n=1}^{\infty}P\{|f_1|\leqslant nu\}^C<\infty$$

(C stands for the set complement). It follows by using the well-known theorem on independent, identically distributed random variables in Banach spaces (see e.g. [1], th. 4.1.1.) that (f_n) satisfies SLLN in B and consequently in I_u. It means that

$$\lim_n P\left\{\bigcap_{k=n}^{\infty}\left\{z\in Z\,;\ \|k^{-1}\sum_1^k f_i(z)\|_u\leqslant\varepsilon\right\}\right\}=1$$

for each $\varepsilon > 0$, since f_n are symmetric. The above equality can be rewritten as follows

$$\lim_n P\left\{\bigcap_{k=n}^{\infty}\left\{z \in Z;\ \left|k^{-1}\sum_1^k f_i(z)\right| \leq \varepsilon u\right\} = 1\right.$$

owing to the property of the order-unit norm.

Necessity can be proved as follows. One can show by repeating step by step the first part of the proof that f_n are independent, identically distributed and symmetric random variables in a separable Banach space $(B, \|\ \|_u)$ with the norm induced by an element u, $a \leq u$. Since, by hypothesis, (f_n) satisfies SLLN with respect to the relatively uniform convergence, it satisfies SLLN with respect to the $\|\ \|_u$; i.e. $\left\|k^{-1}\sum_1^k f_i(z)\right\|_u \to 0$ a.e. Hence we have

$$\frac{f_n}{n} = \frac{\sum_1^n f_i}{n} - \frac{n-1}{n}\,\frac{\sum_1^{n-1} f_i}{n-1} \to 0 \text{ a.e.}$$

in the norm $\|\ \|_u$. If the series $\Sigma P\{\|f_1\|_u > n\}$ were divergent, $P\{\|f_1\|_u > n$ infinitely often$\}$ would be 1, by Borel—Cantelli lemma, a contradiction.

The theorem extends the results of [2].

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УСИЛЕННЫЙ ЗАКОН БОЛЬШИХ ЧИСЕЛ ДЛЯ ОДИНАКОВО РАСПРЕДЕЛЕННЫХ СЛУЧАЙНЫХ ВЕЛИЧИН СО ЗНАЧЕНИЯМИ В ВЕКТОРНОЙ РЕШЕТКЕ

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Резюме

В работе доказывается необходимое и достаточное условие для того, чтобы последовательность одинаково распределенных случайных величин со значениями в векторнои решетк удовлетворяла усиленному закону больших чисел.