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GRAPHIC MATRICES

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Introduction. The results of this paper have been announced in [12].

We shall consider finite simple graphs.

A sequence $d = (d_1, d_2, \ldots, d_n)$ *of non-negative integers is called graphic if there exists a graph* $G = (V, E)$ *such that* $V = \{v_1, v_2, \ldots, v_n\}$ *and* $\deg(v_i) = d_i$ *for* $i = 1, 2, \ldots, n$. *Then the graph G is called a realization of d.*

Two characterizations of graphic sequences are known in literature: Erdös – Gallai's criterion (see [6]) of a combinatorial character, and Havel – Hakimi's criterion (see [9], [11]) of the recursive form. Since we use these criteria in the sequel, we quote them.

Erdös – Gallai's criterion.

Let $d = (d_1, d_2, \ldots, d_n)$ *be a monotonic sequence of non-negative integers with the maximum term* d_1. *Then d is graphic iff*:

$$\sum_{r=1}^{n} d_r \equiv 0 (\mathrm{mod}\, 2),$$

$$\sum_{r=1}^{m} d_r \leqslant m(m-1) + \sum_{r=m+1}^{n} \min\{m, d_r\}, \quad \text{for } m = 1, 2, \ldots, n.$$

Havel – Hakimi's criterion.

Let $d = (d_1, d_2, \ldots, d_n)$ *be a monotonic sequence of non-negative integers with the maximum term* d_1. *Then d is graphic iff the modified sequence*

$$d' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} \dot{-} 1, d_{d_1+2}, \ldots, d_n)$$

is graphic.

In the investigations of the realizations of d which are graphs of some special kind, it is necessary to know not only the degrees of all vertices but also the degrees of the neighbours of any vertex, for example if we study Γ-regular graphs, Γ^--regular graphs (see [15, 16, 12, 17]) and semi-regular graphs, which have applications in chemistry (see [1, 7]). Considering such a problem we observe first that to every graph G we can assign a matrix $\mathbf{M}_G$ of non-negative integers which informs as about the degrees of the neighbours of each vertex. In Section 1 we define more exactly this matrix and we call it the distribution matrix of G.

A matrix $\mathbf{M}$ *is graphic if* $\mathbf{M} = \mathbf{M}_G$ *for some simple graph* G. In this paper we characterize graphic matrices int two ways — combinatorially and recursively (Sections 3 and 4, respectively). Solving this problem we use Erdös–Gallai's and Havel–Hakimi's criteria with some modifications for bipartite graphs.

In Section 5 we consider the set $R_V(\mathbf{M}^*)$ of all graphs with the same vertex set, having the same distribution matrix $\mathbf{M}^*$. We define an operation $(*)$-switching which is a restriction of the switching used by Eggleton in [3] and [4]. We prove that the set $R_V(\mathbf{M}^*)$ can be generated by a single graph $G \in R_V(\mathbf{M}^*)$ using $(*)$-switching operations finitely many times (Theorem 3, Corollary 1).

1. The distribution matrix of a graph

Let $G = (V, E)$ be a finite simple graph. For $v \in V$ we denote:

$$\Gamma(v) = \{u \in V: \{u, v\} \in E\},$$

$$\deg_G(v) = |\Gamma(v)|,$$

$$D(G) = \{\deg_G(v): v \in V\}.$$

Assume that for the graph G we have $D(G) = \{d_1, d_2, \dots, d_k\}$, where $d_1 > d_2 > \dots > d_k$. Then for $i, j \in \{1, 2, \dots, k\}$ we define:

$$V_i = \{v \in V: \deg_G(v) = d_i\},$$

$$E_{ij} = \{\{u, v\} \in E: u \in V_i, v \in V_j\},$$

$$t^i(v) = |V_i \cap \Gamma(v)| \quad \text{for } v \in V.$$

For a graph G we define a function $t_G: V \to N^k$ as follows:

$$t_G(v) = (t^1(v), t^2(v), \dots, t^k(v)) \quad \text{for } v \in V.$$

(Here N denotes the set of non-negative integers.)

The function t_G will be called the distribution function of the vertices of G.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$. We define a $(k \times n)$-matrix $\mathbf{M}_G$ as follows:

$$\mathbf{M}_G = [t_G(v_1), t_G(v_2), \dots, t_G(v_n)].$$

The column $t_G(v_s)$ for $s \in \{1, 2, \dots, n\}$ will be called the distribution of v_s, the matrix $\mathbf{M}_G$ will be called the distribution matrix of G.

2. Graphic matrices

Let $\mathbf{M}$ be a matrix of non-negative integers. $\mathbf{M}$ will be called graphic if there exists a simple graph G such that $\mathbf{M}$ is the distribution matrix of G. Then we shall say that the graph G is a realization of $\mathbf{M}$.

Observe that the problem of characterizing graphic matrices is essentially more complicated than that of characterizing graphic sequences. In fact, two graphs can have the same degree-sequence and different distribution matrices. For example compare the graphs G_1 and G_2 in Fig. 1.

$D(G_1)=D(G_2)=\{3,2,1\}$, $k=3$, $d_{G_1}=d_{G_2}=(3,2,2,2,1)$

$$M_{G_1}=\left[\begin{array}{c|ccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad M_{G_2}=\left[\begin{array}{c|ccc|c} 0 & 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Fig. 1

The aim of this chapter is to introduce some notions and to prove some lemmas used in the sequel.

Let $a=(a_1, a_2, \ldots, a_n)$, $b=(b_1, b_2, \ldots, b_m)$ be two sequences of non-negative integers. A pair (a, b) will be called graphic if there exists a bipartite graph $G=(V_1, V_2, E)$ such that $V_1=\{v_1, v_2, \ldots, v_n\}$, $V_2=\{u_1, u_2, \ldots, u_m\}$ and $\deg_G(v_i)= a_i$, $\deg_G(u_j)=b_j$ for $i=1, 2, \ldots, n$, and $j=1, 2, \ldots, m$.

The bipartite graph G is called a realization of the pair (a, b).

Let $\mathbf{M}=[\alpha_1, \alpha_2, \ldots, \alpha_n]$ be a $(k\times n)$-matrix of non-negative integers, where

$$\alpha_i=\begin{bmatrix} a_i^1 \\ \vdots \\ a_i^k \end{bmatrix}.$$

By $\mathbf{M}^*$ we denote a matrix with the same columns as in $\mathbf{M}$, but ordered as follows:

$$\alpha_i \text{ precedes } \alpha_j \text{ if } \left(\sum_{s=1}^{k} a_i^s > \sum_{s=1}^{k} a_j^s\right) \text{ or } \left(\sum_{s=1}^{k} a_i^s = \sum_{s=1}^{k} a_j^s \text{ and } i<j\right).$$

Let for some matrix $\mathbf{M}$ of non-negative integers the matrix $\mathbf{M}^*$ be of the form:

$$(1) \qquad \mathbf{M}^* = \begin{bmatrix} t_{11}^1 \dots t_{1s_1}^1 \dots t_{i1}^1 \dots t_{is_i}^1 \dots t_{k1}^1 \dots t_{ks_k}^1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ t_{11}^k \dots t_{1s_1}^k \dots t_{i1}^k \dots t_{is_i}^k \dots t_{k1}^k \dots t_{ks_k}^k \end{bmatrix},$$

where for every $i = 1, 2, \dots, k$, $q = 1, 2, \dots, s_i$ we have:

$$t_{iq}^1 + t_{iq}^2 + \dots + t_{iq}^k = d_i \qquad (d_1 > d_2 > \dots > d_k).$$

Lemma 1. *Let $G = (V, E)$ be a realization of a matrix $\mathbf{M}^*$ of the form* (1). *Then the graph G can be decomposed into graphs $G_{ii} = (V_i, E_{ii})$ and bipartite graphs $G_{ij} = (V_i, V_j, E_{ij})$ for $i < j$, $i, j \in \{1, 2, \dots, k\}$, where:*
1° *any of the graphs G_{ii} is a realization of the sequence*

$$t_i^i = (t_{i1}^i, t_{i2}^i, \dots, t_{is_i}^i),$$

2° *any of the graphs G_{ij} is a realization of the pair (t_i^j, t_j^i) of the sequences*

$$t_i^j = (t_{i1}^j, t_{i2}^j, \dots, t_{is_i}^j),$$

$$t_j^i = (t_{j1}^i, t_{j2}^i, \dots, t_{js_j}^i).$$

Proof. Obviously the graphs G_{ii} and G_{ij} are edge-disjoint. Denote by $\mathbf{M}_i$ ($i = 1, 2, \dots, k$) the submatrix of $\mathbf{M}^*$ consisting of all columns for which the sum of elements is equal to d_i. Then $\mathbf{M}_i$ is of the form:

$$(2) \qquad \mathbf{M}_i = \begin{bmatrix} t_{i1}^1 \dots t_{is_i}^1 \\ \cdots\cdots\cdots \\ t_{i1}^i \dots t_{is_i}^i \\ \cdots\cdots\cdots \\ t_{i1}^k \dots t_{is_i}^k \end{bmatrix}.$$

So the columns from $\mathbf{M}_i$ are distributions of the vertices from V_i. Note that the ith row in $\mathbf{M}_i$ is the degree-sequence of the graph G_{ii}. Analogously for $i \neq j$ the jth row in $\mathbf{M}_i$ and the ith row in $\mathbf{M}_j$ is the pair of degree-sequences of the bipartite graph G_{ij}.

Lemma 2. *Let $\mathbf{M}$ be a matrix of non-negative integers such that the matrix $\mathbf{M}^*$ is of the form* (1). *Let*

$$U = \{v_{11}, \dots, v_{1s_1}, \dots, v_{i1}, v_{is_i}, \dots, v_{k1}, \dots, v_{ks_k}\}$$

and $U_i = \{v_{i1}, \dots, v_{is_i}\}$ for $i = 1, 2, \dots, k$. Further, let a simple graph $H_{ii} = (U_i, F_{ii})$ be a realization of the sequence $t_i^i = (t_{i1}^i, \dots, t_{is_i}^i)$ and a bipartite graph $H_{ij} = (U_i, U_j, F_{ij})$ be a realization of the pair of sequences

$$t_i^j = (t_{i1}^j, \ldots, t_{is_i}^j), \quad t_j^i = (t_{j1}^i, \ldots, t_{js_j}^i)$$

where $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$.

Then the graph

$$H = \Big(\bigcup_{i \in \{1, \ldots, k\}} U_i, \bigcup_{\substack{i, j \in \{1, \ldots, k\} \\ i \leqslant j}} F_{ij} \Big)$$

is a realization of the matrix **M**.

Proof. Let us fix a vertex v from the set U. Let $v \in U_{i_0}$ for some $i_0 \in \{1, 2, \ldots, k\}$. So $v = v_{i_0 q}$ for some $q \in \{1, 2, \ldots, s_{i_0}\}$. Observe that v belongs to all graphs $H_{i_0 j}$, and to all graphs $H_{j i_0}$ for $j = 1, 2, \ldots, k$, and to no other. Since $F_{ij} = F_{ji}$ for every $i, j \in \{1, 2, \ldots, k\}$, so

$$\deg_{H_{i_0 j}}(v_{i_0 q}) = \deg_{H_{j i_0}}(v_{i_0 q}) = t_{i_0 q}^j .$$

All the graphs $H_{i_0 j}$ are edge-disjoint, thus

$$t_H(v_{i_0 q}) = (t_{i_0 q}^1, \ldots, t_{i_0 q}^k).$$

The sequence $(t_{i_0 q}^1, \ldots, t_{i_0 q}^k)$ is the qth column in the matrix $\mathbf{M}_{i_0}$ of the form:

$$\begin{bmatrix} t_{i_0 1}^1 \ldots t_{i_0 s_{i_0}}^1 \\ \ldots\ldots\ldots\ldots \\ t_{i_0 1}^k \ldots t_{i_0 s_{i_0}}^k \end{bmatrix}.$$

As the vertex v has been chosen arbitrarily, so the graph H is a realization of the matrix $\mathbf{M}^*$ and consequently of **M**.

3. A combinatorial characterization of graphic matrices

Let a matrix $\mathbf{M}^*$ be of the form (1). Put

$$\underline{t}_j^i = (\underline{t}_{j1}^i, \underline{t}_{j2}^i, \ldots, \underline{t}_{js_j}^i),$$

where $(\underline{t}_{j1}^i, \ldots, \underline{t}_{js_j}^i)$ is a permutation of the sequence t_j^i such that

$$\underline{t}_{j1}^i \geqslant \ldots \geqslant \underline{t}_{js_j}^i, \qquad i, j \in \{1, 2, \ldots, k\}.$$

Theorem 1. *Let* **M** *be a matrix of non-negative integers such that the matrix* $\mathbf{M}^*$ *is of the form* (1). *Then* **M** *is graphic iff for every* $i, j \in \{1, 2, \ldots, k\}$ *the following conditions* (i)—(iv) *are satisfied*:

(i) $$\sum_{r=1}^{s_i} \underline{t}_{ir}^i \equiv 0 \pmod 2,$$

(ii) $$\sum_{r=1}^{m} \underline{t}_{ir}^{i} \leqslant m(m-1) + \sum_{r=m+1}^{s_i} \min\{m, \underline{t}_{ir}^{i}\} \quad \text{for } m = 1, 2, \ldots, s_i,$$

(iii) $$\sum_{r=1}^{s_i} \underline{t}_{ir}^{j} = \sum_{r=1}^{s_j} \underline{t}_{jr}^{i},$$

(iv) $$\sum_{r=1}^{m} \underline{t}_{ir}^{j} \leqslant \sum_{r=1}^{s_j} \min\{m, \underline{t}_{jr}^{i}\} \quad \text{for } m = 1, \ldots, s_i, \ i < j.$$

Proof. From Erdös–Gallai's criterion we infer that (i) and (ii) are necessary and sufficient conditions for the sequence t_i^i to be graphic. The conditions (iii) and (iv) are necessary and sufficient conditions for the representability of the pair (t_i^j, t_j^i) of sequences by a bipartite graph ([6], also [1, Chapter 6 Theorems 1, 7]).

Finally, by Lemmas 1 and 2, we conclude that conditions (i)—(iv) can be put all together, i.e. that they form a necessary and sufficient condition for the matrix $\mathbf{M}^*$ to be graphic.

4. A recursive characterization of graphic matrices

First we introduce some notions.

Let $a = (a_1, a_2, \ldots, a_n)$ be a non-increasing sequence of non-negative integers such that $a_1 \leqslant n - 1$.

Denote:

$$\text{red}(a) = (a_2 - 1, \ldots, a_{a_1+1} - 1, a_{a_1+2}, \ldots, a_n).$$

Let (a, b) be a pair of non-increasing sequences of non-negative integers such that $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_m)$ and $a_1 \leqslant m$.

Denote:

$$\text{red}(a, b) = ((0, a_2, \ldots, a_n), (b_1 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, \ldots, b_m)).$$

Theorem 2. *A matrix* $\mathbf{M}^*$ *of the form* (1) *is graphic iff for every* $i, j \in \{1, 2, \ldots, k\}$ *we have*:

1° *the sequence* $\text{red}(\underline{t}_i^i)$ *is graphic,*

2° *the pair* $\text{red}(\underline{t}_i^j, \underline{t}_j^i)$ *is graphic for* $i < j$.

Proof. By Lemmas 1 and 2 the statement "the matrix $\mathbf{M}^*$ is graphic" is equivalent to the statement "Every sequence $\underline{t}_i^i$ is graphic and for $i < j$, each pair $(\underline{t}_i^j, \underline{t}_j^i)$ of sequences is graphic". Now for the sequences $\underline{t}_i^i$ we use Havel–Hakimi's theorem. For the pair $(\underline{t}_i^j, \underline{t}_j^i)$, where $i < j$, the idea of the proof is the same as in the proof of Havel–Hakimi's theorem given by F. Harary in [10], hence we do not present the details here.

Remark 1. Using the last theorem one can formulate an algorithm for testing whether a matrix $\mathbf{M}^*$ is graphic, moreover using Havel—Hakimi's method (see [10, p. 58]) with some modifications for bipartite graphs, we can construct a graph realizing a matrix $\mathbf{M}^*$ if this matrix is graphic.

5. A set of all realizations of a matrix $\mathbf{M}^*$.

Let $\mathbf{M}^*$ be a matrix of the form (1). Denote by $R_V(\mathbf{M}^*)$ the set of all (labelled) graphs with the same vertex set $V = \{v_1, v_2, \ldots, v_n\}$ which are realizations of $\mathbf{M}^*$. Our aim is to characterize the set $R_V(\mathbf{M}^*)$.

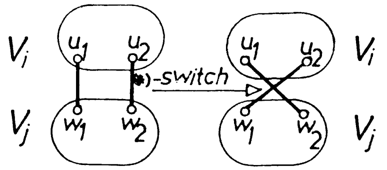
We introduce come auxiliary definitions.

Let $G = (V, E)$ be a simple graph. Let $D(G) = \{d_1, d_2, \ldots, d_k\}$ be defined as above, and (u_1, w_1, u_2, w_2) be a sequence of different vertices such that:
1° $u_1, u_2 \in V_i, \quad w_1, w_2 \in V_j$,
2° $\{u_1, w_1\}, \{u_2, w_2\} \in E$,
3° $\{u_1, w_2\}, \{u_2, w_1\} \notin E$.
We define a new graph $G_{(u_1, w_1, u_2, w_2)} = (V, E')$, where

$$E' = (E \setminus \{\{u_1, w_1\}, \{u_2, w_2\}\}) \cup \{\{u_1, w_2\}, \{u_2, w_1\}\}.$$

The graph $G_{(u_1, w_1, u_2, w_2)}$ will be called a $(*)$-*switching* of G, and the operation which leads from G to $G_{(u_1, w_1, u_2, w_2)}$ will be called a $(*)$-*switching operation.*

The $(*)$-switching operation is a restriction of the *elementary d-invariant transformation* introduced by S. L. Hakimi in [9] and called the switching operation by R. B. Eggleton in [4], namely in the definition of switching one does not require the condition 1°, and there are two possibilities of exchanging the edges, as shown in Fig. 2.



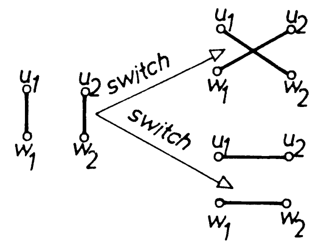


Fig. 2

For two graphs $G = (V, E)$, $H = (V, F)$ put $G \dot{-} H = (V, E \dot{-} F)$, where $\dot{-}$ is the symmetrical difference.
Let $u_1 w_1 u_2 w_2 \ldots u_m w_m u_{m+1}$, where $u_{m+1} = u_1$, be a cycle in the graph $G \dot{-} H$. This cycle will be called alternating (or briefly a-cycle) if $\{u_s, w_s\} \in E$ and $\{w_s, u_{s+1}\} \in F$ for every $s \in \{1, 2, \ldots, m\}$.

Lemma 3. *Let G, $H \in R_V(\mathbf{M}^*)$, $G = (V, E)$, $H = (V, F)$, and for $i, j \in \{1, 2, \ldots, k\}$ $G_{ij} \dot{-} H_{ij} = (V_i, V_j, E_{ij} \dot{-} F_{ij})$, where $i \neq j$, $G_{ij} \dot{-} H_{ij} = (V_i, E_{ii} \dot{-} F_{ii})$ for $i = j$. Then every non 1-element component of the graph $G_{ij} \dot{-} H_{ij}$ is an alternating cycle of the form*

$$(3) \qquad u_1, w_1, u_2, w_2, \ldots, u_m, w_m, u_{m+1},$$

where $u_{m+1} = u_1$, $u_s \in V_i$, $w_s \in V_j$, $\{u_s, w_s\} \in E_{ij} \backslash F_{ij}$, and $\{w_s, u_{s+1}\} \in F_{ij} \backslash E_{ij}$ for $s \in \in \{1, 2, \ldots, m\}$.

Proof. Let $i, j \in \{1, 2, \ldots, k\}$. For $v \in V_i$ we denote:

$$\Gamma_{G_{ij}}(v) = \{u \in V_j: \{u, v\} \in E_{ij}\}, \quad \Gamma_{H_{ij}}(v) = \{u \in V_j: \{u, v\} \in F_{ij}\}.$$

If $\Gamma_{G_{ij}}(v) = \Gamma_{H_{ij}}(v)$, then v is an isolated vertex in $G_{ij} \dot{-} H_{ij}$. Let $\Gamma_{G_{ij}}(v) \neq \Gamma_{H_{ij}}(v)$. Then the degree of the vertex v in the graph $G_{ij} \dot{-} H_{ij}$ is an even number, different from zero. In fact, there exists an edge $e' \in E_{ij} \backslash F_{ij}$ incident to v iff there existst an edge $e'' \in F_{ij} \backslash E_{ij}$ incident to v (since $t_G(v) = t_H(v)$). So v belongs to a cycle. This is an a-cycle, since the number of edges of $G_{ij} \dot{-} H_{ij}$ incident to v and belonging to E_{ij} is equal to the number of edges incident to v and belonging to F_{ij}.

Let G, $H \in R_V(\mathbf{M}^*)$, $G = (V, E)$, $H = (V, F)$. For $i, j \in \{1, 2, \ldots, k\}$ denote by ζ_{ij} a set of all alternating cycles formed from the edges of the graph $G_{ij} \dot{-} H_{ij}$ in such a way that every edge of $G_{ij} \dot{-} H_{ij}$ belongs exactly to one cycle.

The set ζ_{ij} will be called an a-cyclic partition of the graph $G_{ij} \dot{-} H_{ij}$. Obviously such a set need not be unique.

The number

$$\delta(G_{ij}, H_{ij}, \zeta_{ij}) = \tfrac{1}{2}|E(G_{ij} \dot{-} H_{ij})| - |\zeta_{ij}|$$

will be called the distance of the graphs G_{ij}, H_{ij} with respect to the set ζ_{ij}.

Lemma 4. *Let G, $H \in R_V(\mathbf{M}^*)$, $G = (V, E)$, $H = (V, F)$ and for $i, j \in \{1, 2, \ldots, k\}$ let ζ_{ij} be an a-cyclic partition of the graph $G_{ij} \dot{-} H_{ij}$. Further, let $\delta(G_{ij}, H_{ij}, \zeta_{ij}) = p$ and $p > 0$. Then there exists a sequence $G_{ij} = G_{ij}^0, G_{ij}^1, \ldots, G_{ij}^m = H_{ij}$ of graphs and a sequence $\zeta_{ij}^1, \zeta_{ij}^2, \ldots, \zeta_{ij}^m$ of a-cyclic partitions of the graphs $G_{ij}^1 \dot{-} H_{ij}$, $G_{ij}^2 \dot{-} H_{ij}, \ldots, G_{ij}^m \dot{-} H_{ij}$, respectively — such that $m \leqslant p$ and for every $r \in \{1, 2, \ldots, m\}$ the following two conditions are satisfied:*

$$G_{ij}^r \text{ is a } (*)\text{-switching of } G_{ij}^{r-1},$$

$$\delta(G_{ij}^r, H_{ij}, \zeta_{ij}^r) < \delta(G_{ij}^{r-1}, H_{ij}, \zeta_{ij}^{r-1}).$$

Proof. We use induction on the number p.

1° For $p = 1$ we have only one possibility:

$$|E(G_{ij} \dot{-} H_{ij})| = 4 \quad \text{and} \quad |\zeta_{ij}| = 1.$$

Let $C \in \zeta_{ij}$ and $C = u_1 w_1 u_2 w_2 u_1$. We apply the following (*)-switching operation:

$$G_{ij}^1 = G_{ij(u_1,\, w_1,\, u_2,\, w_2)}\,.$$

Then we have: $E(G_{ij}^1 \dot{-} H_{ij}) = \emptyset$, $\zeta_{ij}^1 = \emptyset$ since $\{u_1, w_1\}, \{u_2, w_2\} \notin E_{ij}^1 \cup F_{ij}$ and $\{w_1, u_2\}, \{u_1, w_2\} \in E_{ij}^1 \cap F_{ij}$. So

$$\delta(G_{ij}^1, H_{ij}, \zeta_{ij}^1) = \tfrac{1}{2}\cdot 0 - 0 = 0.$$

2° Assume that the statement is true for all graphs G_{ij}, H_{ij} and all a-cyclic partitions ζ_{ij} such that

$$\delta(G_{ij}, H_{ij}, \zeta_{ij}) < p, \qquad p > 1.$$

3° Assume that for G_{ij}, H_{ij}, ζ_{ij} we have

$$\delta(G_{ij}, H_{ij}, \zeta_{ij}) = p.$$

Denote $|E(G_{ij} \dot{-} H_{ij})| = e$, $|\zeta_{ij}| = c$. Then $p = \frac{1}{2}\cdot e - c$.

Let $C \in \zeta_{ij}$, $C = u_1 w_1 u_2 w_2 \ldots w_n u_1$, $n \geqslant 2$. Then we have the following two cases:

1. $u_1 \neq w_2$ and $\{u_1, w_2\} \notin E_{ij}$,
2. $u_1 = w_2$ or $\{u_1, w_2\} \in E_{ij}$.

Case 1. We apply the following (*)-switching operation:

$$G_{ij}^1 = G_{ij(u_1,\, w_1,\, u_2,\, w_2)}\,.$$

We denote $e_1 = |E_{ij}^1 \dot{-} F_{ij}|$, $c_1 = |\zeta_{ij}|$, $p_1 = \frac{1}{2}\cdot e_1 - c_1$. Since the (*)-switching operation preserves the degrees of all vertices of the graph G_{ij}, so for the graph $G_{ij}^1 \dot{-} H_{ij}$ there exists an a-cyclic partition ζ_{ij}. The number of edges of the graph $G_{ij}^1 \dot{-} H_{ij}$ depends on whether the edge $\{u_1, w_2\}$ belongs to the set F_{ij} or it does not.

1.1. Let $\{u_1, w_2\} \notin F_{ij}$. Then

$$E_{ij}^1 \dot{-} F_{ij} = ((E_{ij} \dot{-} F_{ij}) \setminus \{\{u_1, w_1\}, \{u_2, w_2\}, \{w_1, u_2\}\}) \cup \{\{u_1, w_2\}\},$$

$$\zeta_{ij}^1 = (\zeta_{ij} \setminus \{C\}) \cup \{C'\}, \quad \text{where } C' = u_1 w_2 u_3 \ldots w_n u_1\,.$$

Thus $e_1 = e - 2$, $c_1 = c$ and $p_1 = \frac{1}{2}(e-2) - c = p - 1$.

1.2. Let $\{u_1, w_2\} \in F_{ij}$ and let $\{u_1, w_2\}$ be an edge of some cycle $C_k \in \zeta_{ij}$. Then

$$E_{ij}^1 \dot{-} F_{ij} = (E_{ij} \dot{-} F_{ij}) \setminus \{\{u_1, w_1\}, \{u_2, w_2\}, \{w_1, u_2\}, \{u_1, w_2\}\}.$$

If $C_k \neq C$, then, after removing the edges $\{u_1, w_1\}$, $\{u_2, w_2\}$ and $\{w_1, u_2\}$ from the cycle C, we get a chain L_1, and after removing the edge $\{u_1, w_2\}$ from the cycle C_k, we get a chain L_2. The chains L_1 and L_2 form an alternating cycle.

If $C_k = C$ and $|E(C)| > 4$, then we obtain two chains L_1 and L_2. Then L_1

forms an a-cycle, and L_2 forms an a-cycle, or both together form an a-cycle.

If $C_k = C$ and $|E(C)| = 4$, then the cycle C disappears.

Concluding, we get in the case 1.2.:

$$e_1 = e - 4, \quad c_1 \geqslant c - 1, \quad p_1 = \tfrac{1}{2}(e - 4) - c_1 \leqslant p - 1.$$

Case 2. Let $\{u_1, w_2\} \in E_{ij}$ or $u_1 = w_2$.
Let $s = \min\{k \in \{3, 4, \ldots, n\}: u_1 \neq w_k$ and $\{u_1, w_k\} \notin E_{ij}\}$. The number s always exists, since n satisfies the above condition. So

$$(u_1 \neq w_s \text{ and } \{u_1, w_s\} \notin E_{ij}) \quad \text{and} \quad (\{u_1, w_{s-1}\} \in E_{ij} \text{ or } u_1 = w_{s-1}).$$

2.1. Assume that $\{u_1, w_{s-1}\} \in E_{ij}$. We prove that the following $(*)$-switching operation can be used:

$$G_{ij}^1 = G_{ij(u_1, w_{s-1}, u_s, w_s)}.$$

In fact, $u_1 \neq w_{s-1}$ since $\{u_1, w_{s-1}\}$ is an edge of the graph G_{ij}, $u_1 \neq u_s$ since ($\{u_1, w_{s-1}\} \in E_{ij}$ and $\{w_{s-1}, u_s\} \in F_{ij}$), $u_1 \neq w_s$ by the definition of the number s. In the remaining cases the vertices are different, being three consecutive vertices of the alternating cycle. Further, $\{u_1, w_{s-1}\} \in E_{ij}$ by assumption, $\{u_s, w_s\} \in E_{ij}$ and $\{w_{s-1}, u_s\} \notin E_{ij}$ by the definition of the a-cycle, $\{u_1, w_s\} \notin E_{ij}$ by the definition of the number s.

The numbers e_1 and c_1 depend on whether $\{u_1, w_{s-1}\}$, $\{u_1, w_s\}$ are edges of the graph H_{ij} or are not.

Table 1 ilustrates the influence of the particular edges on the number $|E_{ij}^1 \dot{-} F_{ij}|$ and on the number $|\zeta_{ij}^1|$. The symbol $+$ denotes that a given edge belongs to the suitable set, the symbol $-$ denotes the opposite case.

Tab. 1

	$\{u_1, w_{s-1}\}$		$\{w_{s-1}, u_s\}$	$\{u_s, w_s\}$	$\{u_1, w_s\}$	
E_{ij}	$+$		$-$	$+$	$-$	
F_{ij}	$+$	$-$	$+$	$-$	$+$	$-$
$\lvert E_{ij}^1 \dot{-} F_{ij}\rvert$	$+1$	-1	-1	-1	-1	$+1$
$\lvert\zeta_{ij}^1\rvert$	$+1$	$\geqslant 0$	0	0	$\geqslant 0$	$+1$

For example we discuss the case when $\{u_1, w_{s-1}\} \notin F_{ij}$ and $\{u_1, w_s\} \in F_{ij}$.

Let $\{u_1, w_{s-1}\} \in C_k$, $\{u_1, w_s\} \in C_m$ for some C_k, $C_m \in \zeta_{ij}$ and C_k, C_m, C are different. Since $G_{ij}^1 = G_{ij(u_1, w_{s-1}, u_s, w_s)}$, so

$$E_{ij}^1 \dot{-} F_{ij} = (E_{ij} \dot{-} F_{ij}) \setminus \{\{u_1, w_{s-1}\}, \{w_s, u_s\}, \{w_{s-1}, u_s\}, \{u_1, w_s\}\}.$$

Removing the edges $\{w_{s-1}, u_s\}$, $\{u_s, w_s\}$ from the cycle C we obtain two chains $L_1 = u_1 w_1 \dots w_{s-1}$, $L_2 = w_s u_{s+1} \dots w_n u_1$. After removing the edge $\{u_1, w_{s-1}\}$ from the cycle C_k we obtain a chain which together with L_1 forms an a-cycle C'. Similarly, after removing the edge $\{u_1, w_s\}$ from the cycle C_m we obtain a chain which together with L_2 forms an alternating cycle C''. So, if we remove the edges $\{u_1, w_{s-1}\}$ or $\{u_1, w_s\}$, we do not obtain fewer cycles.

Thus $\zeta_{ij}^1 = (\zeta_{ij}^1 \setminus \{C, C_k, C_m\}) \cup \{C', C''\}$ and $c_1 = c - 1$. Observe that $e_1 = e - 1 - 1 - 1 - 1 = e - 4$, so $p_1 = p - 1$.

Arguing similarly we conclude that in the case 2.1. we get always $p_1 < p$.

2.2. Assume that $u_1 = w_{s-1}$. Let us note that $u_1 \neq w_{s-2}$, hence $\{u_1, w_{s-2}\} \in E_{ij}$. Since $\{u_1, w_{s-2}\} \in E_{ij}$ and $\{u_1, u_s\} \notin E_{ij}$, so $u_s \neq w_{s-2}$.

2.2.1. Let $\{u_s, w_{s-2}\} \in E_{ij}$. Then put

$$G_{ij}^1 = G_{ij(w_{s-1},\, u_{s-1},\, w_{s-2},\, u_s)}.$$

If $\{u_s, w_{s-2}\} \in E_{ij} \cap F_{ij}$, then $e_1 = e - 1 - 1 + 1 - 1 = e - 2$, $c_1 = c$ and $p_1 = p - 1$.

If $\{u_s, w_{s-2}\} \in E_{ij} \setminus F_{ij}$, then $e_1 = e - 1 - 1 - 1 - 1 = e - 4$, $c_1 \geqslant c - 1$ and $p_1 \leqslant p - 1$.

2.2.2. Let $\{u_s, w_{s-2}\} \notin E_{ij}$. Then put

$$G_{ij}^1 = G_{ij(u_1,\, w_{s-2},\, u_s,\, w_s)}.$$

Table 2 ilustrates the changes of the numbers of edges and cycles.

Tab. 2

	$\{u_1, w_{s-2}\}$		$\{w_{s-2}, u_s\}$		$\{u_s, w_s\}$	$\{u_1, w_s\}$	
E_{ij}	+		−		+	−	
F_{ij}	+	−	+	−	−	+	−
$\lvert E_{ij}^1 \dot{-} F_{ij} \rvert$	+1	−1	−1	+1	−1	−1	+1
$\lvert \zeta_{ij}^1 \rvert$	+1	≥0	0	+1	0	≥0	+1

In every case from Table 2 we get $p_1 \leqslant p - 1$.

Finally, in each of the cases 1 and 2 we get the graph $G_{ij}^1 \dot{-} H_{ij}$ and the set ζ_{ij}^1 such that $\delta(G_{ij}^1, H_{ij}, \zeta_{ij}^1) < p$, so we can use the induction hypothesis.

Remark 2. From the proof of Lemma 4 we can obtain an algorithm for finding the sequence of (∗)-switching operations such that we get the graph H_{ij} from the graph G_{ij}. Details of this algorithm will be presented in [14].

Theorem 3. *Let G, $H \in R_V(\mathbf{M}^*)$ and $G \neq H$. Then there exists a sequence $G = G^0, G^1, \dots, G^m = H$ of graphs belonging to $R_V(\mathbf{M}^*)$ such that G^{s+1} is a (∗)-switching of G^s for $s \in \{0, 1, \dots, m-1\}$.*

Proof. We give a method of constructing the sequence $G^0, G^1, \ldots, G^m$.

1) We form the graph $G \dot{-} H$.
2) We decompose the graph $G \dot{-} H$ into the subgraphs $G_{ij} \dot{-} H_{ij}$ for $i, j \in \{1, 2, \ldots, k\}$, $i \leqslant j$.
3) By Lemma 4 for every graph $G_{ij} \dot{-} H_{ij}$ there exists a sequence $\sigma_{ij}^1, \ldots, \sigma_{ij}^r$ of (∗)-switchings such that from G_{ij} we obtain the graph $G_{ij}^r = H_{ij}$.
4) We order all (∗)-switchings in a sequence $\sigma_1, \ldots, \sigma_m$ in the lexicographic way, i.e. we put $\sigma_{ij}^u \prec \sigma_{kl}^w \Leftrightarrow (i, j, u) \prec (k, l, w)$, where $\prec$ denotes the lexicographic order.
5) We form the sequence of the graphs $G^0, G^1, \ldots, G^m$ where G^{s+1} is constructed from G^s by the (∗)-switching σ_{s+1} for $s \in \{0, 1, \ldots, m-1\}$.

From the above theorem we have the following

Corollary 1. *The set $R_V(\mathbf{M}^*)$ can be generated by single graph $G \in R_V(\mathbf{M}^*)$ using (∗)-switching operations finitely many times.*

Remark 3. Theorem 3 is analogous to Eggleton's result for the set $R_V(d)$ of all realizations of a given degree-sequence d (see [3]). However, Theorem 3 cannot be obtained from Eggleton's result, because switching does not preserve the distributions of vertices and the property *to be a bipartite graph* (see Fig. 3). The method used in the proofs of Lemmas 3 and 4 gives a constructive proof of Eggleton's theorem for the set $R_V(d)$. This proof is different from that presented by R. B. Eggleton and D. A. Holton in [5] and by R. Taylor in [18].

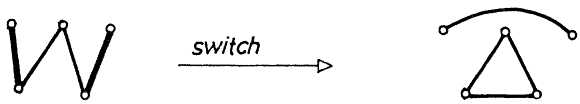


Fig. 3

REFERENCES

[1] BALABAN, A. T.—KEREK, F.: Graphs of parallel and or substitution reactions. Rev. Roum de Chemie 19, 1974, No. 4, 631—647.

[2] BERGE, C.: Graphs and Hypergraphs. North-Holland, Amsterdam 1973.

[3] EGGLETON, R. B.: Graphic sequences and graphic polynomials: a report. Infinite and Finite Sets. Colloqu. Math. Soc. J. Bolyai 10, 1973, 385—392.

[4] EGGLETON, R. B.—HOLTON, D. A.: Graphic sequences. Combinatorial Math. VI. Proc. 6th Australian Conf. on Combinatorial Math., Armidale, 1978, Springer-Verlag 1979, 1—10.

[5] EGGLETON, R. B.—HOLTON, D. A.: Simple and multigraphic realizations of degree sequences. Combinatorial Math. VIII. Proc. Geelong, Australia 1980, Springer-Verlag 1981, 155—172.

[6] ERDÖS, P.—GALLAI, T.: Graphs with prescribed degrees of vertices (in Hungarian). Mat. Lapok 11, 1960, 264—274.

[7] EVANS, C. W.: Some properties of semi-regular graphs. match 6, 1979, 117—135.

[8] GALE, D.: A theorem in folows in networks. Pacific J. Math. 7, 1957, 1073—1082.
[9] HAKIMI, S. L.: On the realizability of a set of integers as degrees of the vertices of a linear graph. I. J. SIAM 10, 1962, 496—506.
[10] HARARY, F.: Graph Theory. Addison-Vesley, Mass. 1969.
[11] HAVEL, V.: A remark on the existence of finite graphs (in Czech). Čas. pěst. mat. 80, 1955, 477—480.
[12] MAJCHER, Z.: Matrices representable by graphs.: Teubner-Texte zur Mathematik, Band 59, Proc. of the Third Czechoslovak Symposium on Graph Theory, Prague 1982, BSB B. G. Teubner Verlagsgesellschaft, 1983, 178—182.
[13] MAJCHER, Z.: On some regularities of graphs II. Čas. pěst. mat. 109, 1984, 380—388.
[14] MAJCHER, Z.: Algorithms for constructing paths in a graph of realizations of a degree sequence (to appear).
[15] PŁONKA, J.: On Γ-regular graphs. Colloquium Math. 46, 1982, 131—134.
[16] PŁONKA, J.: On some regularities of graphs I. Čas. pěst. mat. 107, 1982, 231—240.
[17] RAMA CHANDRAN, S.: Nearly regular graphs and their reconstruction. Graph Theory Newsletter B, 3, 1978.
[18] TAYLOR, R.: Constrained switchings in graphs. Combinatorial Math. VIII. Proc. Geelong, Australia 1980, Springer-Verlag 1981, 314—336.

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МАТРИЦЫ ГРАФОВ

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Резюме

В работе дана характеристика этих матриц и множества всех графов, которые имеют одно и тоже самое множество вершин и одинаковую матрицу распределения.