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OSCILLATION THEOREMS OF COMPARISON TYPE OF DELAY DIFFERENTIAL EQUATIONS WITH A NONLINEAR DAMPING TERM

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ABSTRACT. In this paper, we study the oscillatory behaviour of the solutions of delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_{n-1}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\cdots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}x(t)+f\big(t,x(t-g),\frac{\mathrm{d}}{\mathrm{d}t}x(t-h)\big)=0\,,\qquad n \text{ is even}$$

by comparing with certain differential equations of the same or lower order whose oscillatory character is known. The obtained results can be applied to the delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_{n-1}(t)}\cdots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}x(t) + q(t)\big(|x(t-g)|^{m_1}\big)\Big(\Big|\frac{\mathrm{d}}{\mathrm{d}t}x(t-h)\Big|^{m_2}\Big)\operatorname{sgn}x(t-g)=0\,,$$

where m_1 and m_2 are positive constants.

1. Introduction

We consider the functional differential equation

$$L_n x(t) + f\big(t, x(t-g), \overset{\prime}{x}(t-h)\big) = 0\,, \qquad n \text{ is even}, \ \big(' = \frac{\mathrm{d}}{\mathrm{d}t}\big)\,, \tag{E}$$

where $L_0x(t) = x(t)$, $L_kx(t) = \frac{1}{a_k(t)}\big(L_{k-1}x(t)\big)'$, $k = 1,2,\dots,n$, $a_n = 1$, $a_i\colon [t_0,\infty) \to (0,\infty)$, $i = 1,2,\dots,n-1$, $f\colon [t_0,\infty)\times\mathbb{R}^2 \to \mathbb{R} = (-\infty,\infty)$ are continuous, g and h are positive constants and $h \ge g$. We assume that:

(1) $\int^{\infty} a_i(s)\,\mathrm{d}s = \infty$, $i = 1,2,\dots,n-1$,

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(2) there exist a continuous function $q\colon [t_0,\infty)\to(0,\infty)$ and real constants m_1 and m_2, $m_1>0$ and $m_2\ge 0$ such that

$$f(t,x_1,x_2)\operatorname{sgn} x_1 \ge q(t)\big(|x_1|^{m_1}\big)\big(|x_2|^{m_2}\big) \qquad \text{for} \quad x_1\ne 0 .$$

The oscillatory behaviour of functional differential equations has been intensively studied in recent years. Most of the literature on this subject has been concerned with equations of type (E) and/or related equations, specially when f satisfies condition (2) with $m_2=0$, see [1], [5], [7] and [8], and the references cited therein. It seems that very little is known regarding the oscillation of equation (E) when f satisfies condition (2) with $m_2\ne 0$, see [2]; [4], [10] and [12], and the references cited therein. In this paper, we proceed further in this direction to establish some new oscillation results for equation (E). Theorems 1 and 2 are concerned with the oscillation of equation (E) via comparison with the oscillatory behaviour of two equations of order n and $n-1$, and in Theorem 3, we reduce the problem of the oscillation of equation (E) to the problem of the oscillation of a certain set of first order equations and the oscillation of all bounded solutions of certain retarded equation of order $n-1$.

The *domain* of L_n $D(L_n)$ is defined to be the set of functions $x\colon [T_x,\infty)\to\mathbb{R}$ such that $L_jx(t)$, $j=0,1,\dots,n$, exist and are continuous on $[T_x,\infty)$, $T_x\ge t_0$. In what follows, we consider only the "nonconstant" solutions in $D(L_n)$, of equation (E). A *solution* of equation (E) is called *oscillatory* if it has arbitrary large zeros, otherwise, it is called *nonoscillatory*. *Equation* (E) is said to be *oscillatory* if all its solutions are oscillatory.

2. Main results

We begin by formulating preparatory results which are needed in proving our main results.

For functions $p_i\colon [t_0,\infty)\to\mathbb{R}$, $i=1,2,\dots$, we define

$$I_0 = 1 ,$$

$$I_i(t,s;p_i,\dots,p_1) = \int_s^t p_i(u) I_{i-1}(u,s;p_{i-1},\dots,p_1)\,\mathrm{d}u , \qquad i=1,2,\dots .$$

It is easy to verify that for $i=1,2,\dots,n-1$

$$I_i(t,s;p_1,\dots,p_i) = (-1)^i I_i(s,t;p_i,\dots,p_1)$$

and

$$I_i(t,s;p_1,\dots,p_i) = \int_s^t p_i(u) I_{i-1}(t,u;p_1,\dots,p_{i-1})\,\mathrm{d}u .$$

The following two lemmas will be needed in the proofs of the main results.

LEMMA 1. *If $x \in D(L_n)$, then for $t, s \in [t_0, \infty)$ and $0 \le i < k \le n$*

$$\text{(i)} \quad L_i x(t) = \sum_{j=i}^{k-1} I_{j-i}(t, s; a_{i+1}, \dots, a_j) L_j(s) + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \dots, a_{k-1}) a_k(u) L_k x(u) \, \mathrm{d}u \,.$$

$$\text{(ii)} \quad L_i x(t) = \sum_{j=i}^{k-1} (-i)^{j-i} I_{j-i}(s, t; a_j, \dots, a_{i+1}) L_j x(s) + (-1)^{k-i} \int_t^s I_{k-i-1}(u, t; a_{k-1}, \dots, a_{i+1}) a_k(u) L_k(u) \, \mathrm{d}u \,.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

LEMMA 2. *Suppose conditions* (1) *and* (2) *hold. If $x \in D(L_n)$ is of constant sign and is not identically zero for all large t, then there exist a $t_x \ge t_0$ and an integer m, $0 \le m \le n$, with $n + m$ even for $x(t) L_n x(t)$ nonnegative, or $n + m$ odd for $x(t) L_n x(t)$ nonpositive, and such for every $t \ge t_x$*

$$m > 0 \qquad \textit{implies} \quad x(t) L_k x(t) > 0 \qquad (k = 1, 2, \dots, m)\,,$$

and

$$m \le n - 1 \quad \textit{implies} \quad (-1)^{m-k} x(t) L_k x(t) > 0 \qquad (k = m, m+1, \dots, n)\,.$$

This lemma generalizes a well-known lemma of Kiguradze (see [6]) and can be proved similarly.

Next, for $t \ge T \ge t_0$, we put

$$A_{j,i}[t, T] = \int_T^t I_{i-j}(t, s; a_j, \dots, a_{i-1}) a_i(s) I_{n-i-1}(t, s; a_{n-1}, \dots, a_{i+1}) \, \mathrm{d}s$$
$$\text{for} \quad i \ge j\,, \quad c = 1, 2 \quad \text{and} \quad i = 1, 2, \dots, n-1\,,$$

and

$$R[t, T] = \int_T^t a_1(s) \, \mathrm{d}s \,.$$

In the following theorem, we give a sufficient condition for the oscillation of the damped equation (E) via comparison with undamped equations of the form

$$L_n x(t) + c_1 \big(a_1(t-h)\big)^{m_2} q(t) \big(|x(t-g)|^{m_1}\big) \operatorname{sgn} x(t-g) = 0 \qquad (\mathrm{E}_1)$$

and

$$M_m y(t) + c_2\big(a_1(t-h)\big)^{m_2} q(t)\big(|y(t-h)|^{m_2}\big)\operatorname{sgn} y(t-h) = 0\,, \qquad (\mathrm{E}_2)$$

where $M_0 = y(t)$, $M_k y(t) = \dfrac{1}{b_k(t)}\big(M_{k-1}y(t)\big)'$, $k = 1,2,\dots,m$; $m = n-1$. $b_k(t) = a_{k+1}(t)$, $k = 1,2,\dots,n-1$ and c_1 and c_2 are positive constants.

THEOREM 1. *Let conditions* (1) *and* (2) *hold. If for every* $c_1 > 0$, *equation* (E_1) *is oscillatory, and for every* $c_2 > 0$, *every bounded solution of equation* (E_2) *is oscillatory, then equation* (E) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E). Assume $x(t) > 0$ and $x(t-g) > 0$ for $t \ge t_0$.

By Lemma 2, there exist a $t_1 \ge t_0$ and an integer $N \in \{1,3,\dots,n-1\}$ such that

$$\begin{aligned} L_k x(t) &> 0 \qquad \text{for} \quad t \ge t_1\,, \quad (k = 1,2,\dots,N)\,,\\ (-1)^{N-k} L_k x(t) &> 0 \qquad \text{for} \quad t \ge t_1\,, \quad (k = N, N+1,\dots,n)\,. \end{aligned} \qquad (3)$$

Suppose that $N > 1$. From (3), we see that $L_1 x(t)$ is positive and increasing for $t \ge t_1$. There exist a $t_2 \ge t_1$ and a constant $A > 0$ such that

$$x'(t-h) \ge A a_1(t-h) \qquad \text{for} \quad t \ge t_2\,. \qquad (4)$$

Using (2) and (4) in equation (E), we get

$$L_n x(t) + A^{m_2}\big(a_1(t-h)\big)^{m_2} q(t)\big(|x(t-g)|\big)^{m_1}\operatorname{sgn} x(t-g) \le 0 \qquad \text{for} \quad t \ge t_2\,.$$

But, in view of [3] and [8], it follows that the equation

$$L_n x(t) + A^{m_2}\big(a_1(t-h)\big)^{m_2} q(t)\big(|x(t-g)|\big)^{m_1}\operatorname{sgn} x(t-g) = 0 \qquad \text{for} \quad t \ge t_2$$

has a positive nonoscillatory solution, a contradiction.

Next, let $N = 1$. Since $x(t)$ is an increasing function for $t \ge t_1$, there exist a $t_3 \ge t_1$ and a constant $B > 0$ so that

$$x(t-g) \ge B \qquad \text{for} \quad t \ge t_3\,. \qquad (5)$$

Using (2) and (5) in equation (E) we get

$$L_n x(t) + B^{m_1} q(t)\big(x'(t-h)\big)^{m_2} \le 0 \qquad \text{for} \quad t \ge t_3\,,$$

or

$$L_n x(t) + B^{m_1} q(t)\big(a_1(t-h)\big)^{m_2}\big(L_1 x(t-h)\big)^{m_2} \le 0 \qquad \text{for} \quad t \ge t_3 .$$

Setting $y(t) = L_1 x(t)$, $t \ge t_3$, we have

$$M_m y(t) + B^{m_1} q(t)\big(a_1(t-h)\big)^{m_2}\big(y(t-h)\big)^{m_2} \le 0 .$$

Clearly, $y(t)$ is a positive and decreasing function for $t \ge t_3$. Applying [11; Corollary 1'], we see that the equation

$$M_m y(t) + B^{m_1} q(t)\big(a_1(t-h)\big)^{m_2}\big(y(t-h)\big)^{m_2} \le 0 , \qquad \text{for} \quad t \ge t_3$$

has a bounded, eventually positive and decreasing solution, a contradiction. This completes the proof.

In the following result, we replace equation (E_2) in Theorem 1 by the equation

$$M_m w(t) + \big(a_1(t-h)\big)^{m_2}\big(R[t-g,T]\big)^{m_1} q(t)\big(|w(t-g)|^{m_1+m_2}\big)\operatorname{sgn} w(t-g) = 0 , \tag{E_3}$$

where M_m is defined as in equation (E_3).

THEOREM 2. *Let conditions* (1) *and* (2) *hold. If, for all* $c_1 > 0$, *the equation* (E_1) *is oscillatory and for all large* T *with* $t > T + g$ *all bounded solution of equation* (E_3) *are oscillatory, then equation* (E) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E), say $x(t) > 0$ and $x(t-g) > 0$ for $t \ge t_0$. As in the proof of Theorem 1, there exist a $t_1 \ge t_0$ and an integer $N \in \{1, 3, \dots, n-1\}$ such that (3) holds. Next, we consider the two cases: $N > 1$ and $N = 1$. The proof of the first case is similar to that of Theorem 1 and hence is omitted. Now, we consider the case $N = 1$. From (3) we see that the function $L_1 x$ is decreasing on $[t_1, \infty)$. Next, for $t \ge t_1$ we have

$$\begin{aligned} x(t) - x(t_1) &= \int_{t_1}^{t} \frac{a_1(s)}{a_1(s)} \overset{\prime}{x}(s)\, \mathrm{d}s \\ &= \left(\int_{t_1}^{t} a_1(s)\, \mathrm{d}s\right) L_1 x(t) - \int_{t_1}^{t} \left(a_2(s) \int_{t_1}^{s} a_1(u)\, \mathrm{d}u\right) L_2 x(s)\, \mathrm{d}s \\ &\ge R[t, t_1] L_1 x(t) \qquad \text{for} \quad t \ge t_1 . \end{aligned}$$

There exists a $t_2 \geq t_1$ so that

$$x(t-g) \geq R[t-g,t_1] L_1 x(t-g) \qquad \text{for} \quad t \geq t_2 . \tag{6}$$

Using (6) in equation (E) and the fact that $L_1 x(t)$ is a decreasing function on $[t_1, \infty)$ and $h > g$, we obtain

$$L_n x(t) + \bigl(R[t-g,t_1]\bigr)^{m_1} \bigl(a_1(t-h)\bigr)^{m_2} \bigl(L_1 x(t-g)\bigr)^{m_1+m_2} \leq 0 \qquad \text{for} \quad t \geq t_2 .$$

Next, we set $v(t) = L_1 x(t)$, $t \geq t_2$; we get

$$M_m v(t) + \bigl(R[t-g,t_1]\bigr)^{m_1} \bigl(a_1(t-h)\bigr)^{m_2} \bigl(v(t-g)\bigr)^{m_1+m_2} \leq 0 .$$

The rest of the proof is similar to that of Theorem 1 (the case $N = 1$) and hence is omitted.

In the following theorem, we replace equation (E_1) by a set of first order equations

$$y'(t) + Q_i[t,T]\bigl(|y(t-g)|^{m_1+m_2}\bigr) \operatorname{sgn} y(t-g) = 0 , \qquad T \text{ is large} . \tag{$E_4; i$}$$

where $Q_i[t,T] = \bigl(a_1(t-h)\bigr)^{m_2} \bigl(A_{1,i}[t-g,T]\bigr)^{m_1} \bigl(A_{2,i}[t-h,T]\bigr)^{m_2}$, $i = 3, 5, \dots$ $\dots, n-1$, and obtain the following oscillation criterion for equation (E).

THEOREM 3. *Let conditions* (1) *and* (2) *hold. If for all large T with $t \geq T + g$, the equations* ($E_4; i$), $i = 3, 5, \dots, n-1$ *are oscillatory and all bounded solutions of equation* (E_3) (*or equation* (E_2), $c_2 > 0$) *are oscillatory, then equation* (E) *is oscillatory.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of equation (E), say $x(t) > 0$ and $x(t-g) > 0$ for $t \geq t_0$. As in the proof of Theorem 1, there exist a $t_1 \geq t_0$ and an integer $N \in \{1, 3, \dots, n-1\}$ so that (3) holds. We consider the two cases: $N = 1$ and $N > 1$. The proof of the case $N = 1$ is similar to that of Theorem 1 (or Theorem 2) and hence is omitted. Next, we consider the case $N > 1$. From Lemma 1 (ii), we get

$$L_N x(s) = \sum_{j=N}^{n-2} (-1)^{j-N} I_{j-N}(t,s; a_j, \dots, a_{N+1}) L_j x(t) \cdot$$

$$\cdot (-1)^{n-N-1} \int_s^t I_{n-N-2}(u,s; a_{n-2}, \dots, a_{N+1}) a_{n-1}(u) L_{n-1} x(u) \, \mathrm{d}u$$

$$\text{for} \quad t \geq s \geq t_1 .$$

Using (3) and the fact that $L_{n-1}x$ is decreasing function on $[t_1, \infty)$, we obtain

$$L_N x(s) \geq \left(\int_s^t T_{n-N-2}(u, s; a_{n-2}, \ldots, a_{N+1}) a_{n-1}(u)\, \mathrm{d}u \right) L_{n-1}x(t)$$

or

$$L_N x(s) \geq I_{n-N-1}(t, s; a_{n-1}, \ldots, a_{N+1}) L_{n-1}x(t)\,, \qquad t \geq s \geq t_1\,. \tag{7}$$

On the other hand, from Lemma 1 (i), we have

$$x(t) = \sum_{j=0}^{N-1} I_j(t, t_1; a_1, \ldots, a_j) L_j x(t_1) + \int_{t_1}^t I_{N-1}(t, s; a_1, \ldots, a_{N-1}) a_N(s) L_N x(s)\, \mathrm{d}s$$

or

$$x(t) \geq \int_{t_1}^t I_{N-1}(t, s; a_1, \ldots, a_{N-1}) a_N(s) L_N x(s)\, \mathrm{d}s\,, \qquad t \geq t_1\,. \tag{8}$$

Combining (7) and (8) we get

$$x(t) \geq A_{1,N}[t, t_1] L_{n-1}x(t) \qquad \text{for} \quad t \geq t_1\,.$$

Also, from Lemma 1 (i), we have

$$x'(t) = \left[\sum_{j=1}^{N-1} I_{j-1}(t, t_1; a_2, \ldots, a_j) L_j x(t_1) + \int_{t_1}^t I_{N-2}(t, s; a_2, \ldots, a_{N-1}) a_N(s) L_N x(s)\, \mathrm{d}s \right] a_1(t)$$

or

$$x'(t) \geq a_1(t) \int_{t_1}^t I_{N-2}(t, s; a_2, \ldots, a_{N-1}) a_N(s) L_N x(s)\, \mathrm{d}s\,. \tag{9}$$

Combining (7) and (9) we obtain

$$\overset{\prime}{x}(t) \geq a_1(t)A_{2,N}[t,t_1]L_{n-1}x(t) \qquad \text{for} \quad t \geq t_1 .$$

There exists a $t_2 \geq t_1$ so that

$$x(t-g) \geq A_{1,N}[t-g,t_1]L_{n-1}x(t-g) \qquad \text{for} \quad t \geq t_2 \tag{10}$$

and

$$\overset{\prime}{x}(t-h) \geq a_1(t-h)A_{2,N}[t-h,t_1]L_{n-1}x(t-h) \qquad \text{for} \quad t \geq t_2 .$$

Using the fact that $L_{n-1}x$ is a decreasing function on $[t_1,\infty)$ and $h > g$, we have

$$\overset{\prime}{x}(t-h) \geq a_1(t-h)A_{2,N}[t-h,t_1]L_{n-1}x(t-g) \qquad \text{for} \quad t \geq t_2 . \tag{11}$$

Now, using (10) and (11) in equation (E), we get

$$\begin{aligned} L_n x(t) &= -f\big(t, x(t-g), \overset{\prime}{x}(t-h)\big) \\ &\leq -q(t)\big(x(t-g)\big)^{m_1}\big(\overset{\prime}{x}(t-h)\big)^{m_2} \\ &\leq -q(t)\big(A_{1,N}[t-g,t_1]\big)^{m_1}\big(a_1(t-h)\big)^{m_2} \cdot \\ &\qquad \cdot \big(A_{2,N}[t-h,t_1]\big)^{m_2}\big(L_{n-1}x(t-g)\big)^{m_1+m_2} \qquad \text{for} \quad t \geq t_2 . \end{aligned}$$

Setting $y(t) = L_{n-1}x(t)$ yields

$$\begin{aligned} \overset{\prime}{y}(t) + q(t)\big(a_1(t-h)\big)^{m_2}\big(A_{1,N}[t-g,t_1]\big)^{m_1}\big(A_{2,N}[t-h,t_1]\big)^{m_2} \cdot \\ \cdot \big(y(t-g)\big)^{m_1+m_2} \leq 0 \qquad \text{for} \quad t \geq t_2 . \end{aligned}$$

But in view of [11; Corollary 1], each of the equations $(\mathrm{E}_4; N)$, $N = 3, 5, \dots, n-1$, has an eventually positive and decreasing solution, which is a contradiction. This completes the proof.

The following results are immediate consequences of Theorem 3. The Corollaries below follow readily from results in [1], [7] and [9].

For all large $T \geq t_0$ with $t \geq T + g$, we put

$$Q_1[t,T] = \big(a_1(t-h)\big)^{m_2}\big(R[t-g,T]\big)^{m_1} q(t) .$$

COROLLARY 1. *Let conditions* (1) *and* (2) *hold and* $m_1+m_2<1$. *Moreover, suppose that for all large* T *with* $t>T+g$

$$\int^{\infty} Q_N[s,T]\,\mathrm{d}s=\infty\,,\qquad \text{for}\quad N=3,5,\dots,n-1\,, \tag{12; N}$$

and

$$\liminf_{t\to\infty}\int_{t-g}^{t} I_{n-2}(s,s-g;a_{n-1},\dots,a_2)Q_1[s,T]\,\mathrm{d}s>0\,. \tag{13}$$

Then equation (E) *is oscillatory.*

COROLLARY 2. *Let conditions* (1) *and* (2) *hold and* $m_1+m_2=1$. *In addition, we assume that for all large* T *with* $t>T+g$

$$\liminf_{t\to\infty}\int_{t-g}^{t} Q_N[s,T]\,\mathrm{d}s>\frac{1}{\mathrm{e}}\qquad \text{for}\quad N=3,5,\dots,n-1\,, \tag{14; N}$$

and for some $i=0,1,\dots,n-2$

$$\limsup_{t\to\infty}\int_{t-g}^{t} I_{n-i-2}(s,t-g;a_{n-1},\dots,a_{i+2})I_i(t-g,s-g;a_{i+1},\dots,a_2)\cdot \cdot Q_1[s,T]\,\mathrm{d}s>1\,. \tag{15}$$

Then equation (E) *is oscillatory.*

Remark 1. From the known oscillation criteria for undamped equations of type (E) (i.e., equation (E) with $m_2=0$) in [1] and [7] and the references cited therein, we see that Theorem 1 applies to equation (E) with $m_1>0$ and $0\le m_2\le 1$ while Theorems 2 and 3 are applicable to (E) with $0<m_1+m_2\le 1$.

The following example is illustrative:

Example 1. Consider the fourth order differential equation

$$\left(\frac{1}{t}\left(\frac{1}{t}\left(\frac{1}{t}x'(t)\right)'\right)'\right)'+\frac{231}{16}t^{-13/2}\left(2(t-h)^{1/2}\right)^{m_4}(t-g)^{-m_3/2}\cdot \cdot\left(|x(t-g)|^{m_1}\right)\left(|x'(t-h)|^{m_2}\right)\operatorname{sgn}x(t-h)=0\,,\qquad t>g\,, \tag{E$_5$}$$

where g, h, m_j, $j = 1,2,3,4$ are real constants, $m_1 > 0$, $m_2 \geq 0$ and $g \geq h \geq 0$. It is easy to check the following:

(i) when $m_2 = 0$, $m_4 - m_3 > 3$ and $m_1 \geq 1$, equation (E_5) is oscillatory by [1; Theorems 2 and 4] and [7; Theorems 3.2 and 5.1];

(ii) when $m_2 > 0$ and $\frac{1}{4}(13 - m_4 + m_3 + 2m_2) \leq m_1 + m_2 \leq 1$, equation (E_5) is oscillatory by Theorems 2 and 3;

(iii) when $m_2 = m_4 \geq 0$ and $m_1 = m_3 > 0$, equation (E_5) has a nonoscillatory solution $x(t) = t^{1/2}$.

Thus, we conclude that the damping term which appeared in equation (E_5) (i.e., equation (E_5) with $m_2 \neq 0$) plays important role in preserving or disrupting the oscillatory character of undamped equation (E_5) (i.e., equation (E_5) with $m_2 = 0$).

Theorems 1–3 applied to the special equation

$$\left(\frac{1}{a_1(t)}\, x'(t)\right)' + f\big(t, x(t-g), x'(t-h)\big) = 0 \tag{E_6}$$

(i.e., equation (E) with $n = 2$) yields the following corollary.

COROLLARY 3. *Let conditions* (1) *and* (2) *hold. If for all large T every bounded solution of the equation*

$$y'(t) + Q_1[t,T]\big(|y(t-g)|^{m_1+m_2}\big)\operatorname{sgn} y(t-g) = 0 \tag{E_7}$$

is oscillatory, or for all large T and every $c > 0$, all bounded solutions of the equation

$$v'(t) + c\big(a_1(t-h)\big)^{m_2} q(t)\big(|v(t-h)|^{m_2}\big)\operatorname{sgn} v(t-h) = 0, \tag{E_8}$$

are oscillatory, then equation (E_6) *is oscillatory.*

R e m a r k 2. In view of Corollaries 1 and 2, one can easily see that Corollary 3 is an extension of our results in [4] and some of the results in [12].

R e m a r k 3. From the proof of Theorem 3, we see that Theorem 3 remains valid when the constant m_2 in condition (2) is identically zero, i.e., f satisfies

$$f(t, x_1, x_2)\operatorname{sgn} x_1 \geq q(t)|x_1|^{m_1}, \qquad m_1 > 0 \quad \text{and} \quad x_1 \neq 0, \tag{16}$$

where q is defined as in condition (2).

In this case, we establish a criterion for the oscillation of equation (E) which improves our earlier result in [5].

Now we state this result by noting that for $t \geq T$

[illegible]

THEOREM 4. *Let conditions* (1) *and* (16) *hold. If for all large* T *and* $N = 1,3,\ldots,n-1$, *the equations*

$$y'(t) + C_i[t,T]\big(|y(t-g)|^{m_1}\big)\operatorname{sgn} y(t-g) = 0 \tag{17; N}$$

are oscillatory, then equation (E) *is oscillatory.*

P r o o f. It follows from the proof of Theorem 3, and hence is omitted.

Theorems 1–3 seems to be new even when specialized to the equation

$$x^{(n)}(t) + f\big(t, x(t-g), x'(t-h)\big) = 0\,, \qquad n \quad \text{is even}\,, \tag{E_9}$$

for which condition (2) is satisfied. So, we state them below as corollaries by noting that in this case for $t \geq s$

$$I_{n-1}(t,s;a_1,\ldots,a_{n-1}) = I_{n-1}(t,s;a_{n-1},\ldots,a_1) = \frac{(t-s)^{n-1}}{(n-1)!}\,.$$

Next, for all large T, p_i, $0 < p_i < 1$, $i = 1,3,\ldots,n-1$ such that

$$Q_1[t,p_1] = p_1 t^{m_1} q(t)\,,$$

$$Q_i[t,p_i] = p_i K_i t^{B} q(t)\,,$$

where

$$B = (n-1)m_1 + (n-2)m_2\,,$$

and

$$K_i = \frac{1}{(n-1)^{m_1}(n-2)^{m_2}\big((i-1)!\big)^{m_1}\big((i-2)!\big)^{m_2}\big((n-i-1)!\big)^{m_1+m_2}}\,.$$

COROLLARY 4. *Suppose that condition* (2) *holds. If for every* p_i, $0 < p_i < 1$, $i = 1,3,\ldots,n-1$, *the equations*

$$y'(t) + Q_i[t,p_i]\big(|y(t-g)|^{m_1+m_2}\big)\operatorname{sgn} y(t-g) = 0\,, \qquad \textit{for} \quad i = 3,5,\ldots,n-1\,, \tag{18; N}$$

are oscillatory and every bounded solution of either

$$w^{(n-1)} + Q_1[t,p_1]\big(|w(t-g)|^{m_1+m_2}\big)\operatorname{sgn} w(t-g) = 0 \tag{19}$$

or

$$v^{(n-1)}(t) + cq(t)\big(|v(t-h)|^{m_2}\big)\operatorname{sgn} v(t-h) = 0\,, \qquad \textit{for every} \quad c > 0\,, \tag{20}$$

is oscillatory, then equation (E_9) *is oscillatory.*

R e m a r k 1. One can draw more corollaries from Theorems 1–4, similar to those given above. Here we omit the details.

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