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STATISTICAL CONVERGENCE OF SEQUENCES OF FUZZY NUMBERS

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ABSTRACT. In this paper, the concepts of statistically convergent and statistically Cauchy sequences of fuzzy numbers have been introduced and discussed. Also $l(p)$-spaces of sequences of fuzzy numbers have been introduced.

1. Introduction and background

Let D denote the set of all closed bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line $\mathbb{R}$. For $A, B \in D$ define

$$A \le B \iff \underline{A} \le \underline{B} \text{ and } \overline{A} \le \overline{B},$$
$$d(A,B) = \max\bigl(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\bigr).$$

It is easy to see that d defines a metric on D and (D, d) is a complete metric space. Also $\le$ is a partial order in D. A fuzzy number is a fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support. In other words, if $X \in L(\mathbb{R})$, then, for any $\alpha \in [0,1]$, X^α is compact, where

$$X^\alpha = \begin{cases} t\colon X(t) \ge \alpha & \text{if } \alpha \in (0,1], \\ t\colon X(t) > 0 & \text{if } \alpha = 0. \end{cases}$$

Define $\overline{d}\colon L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$\overline{d}(X,Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \le Y$ if and only if $X^\alpha \le Y^\alpha$ for any $\alpha \in [0,1]$. We now recall the following definitions which were given in [3].

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DEFINITION 1.1. A *sequence* $X = \{X_k\}$ of fuzzy numbers is a function X from the set of all positive integers into $L(\mathbb{R})$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$ and is called the *kth term* of the sequence.

DEFINITION 1.2. A sequence $X = \{X_k\}$ of fuzzy numbers is said to be *convergent to the fuzzy number* X_0 if for every $\varepsilon > 0$ there exists a positive integer k_0 such that

$$\overline{d}(X_k, X_0) < \varepsilon \qquad \text{for} \quad k > k_0 .$$

Let $\mathfrak{c}$ denote the set of all convergent sequences of fuzzy numbers.

$X = \{X_k\}$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\overline{d}(X_k, X_m) < \varepsilon \qquad \text{for} \quad k, m > k_0 .$$

Let C denote the set of all Cauchy sequences of fuzzy numbers. It is easy to see that $\mathfrak{c} \subset C$.

It is known ([3]) that $L(\mathbb{R})$ is a complete metric space with the metric $\overline{d}$.

2. Statistical convergence

Recall ([5]) that the "natural density" of a set K of positive integers is defined by $\delta(K) = \lim_n 1/n\,|\{k \le n : \ k \in K\}|$, where $|\{k \le n : \ k \in K\}|$ denotes the number of elements of K not exceeding n. We shall be particularly concerned with integer sets having natural density zero. To facilitate this, we introduce the following notation:

If $X = \{X_k\}$ is a sequence that satisfies some property P for all k except a set of natural density zero, then we say that X_k satisfies P for "almost all k" and we abbreviate this by "a.a. k".

The concept of statistical convergence of real or complex sequences was introduced by F a s t [1] and studied by several authors including [7], [6], [2].

In this section, we shall introduce and discuss the concepts of statistically convergent and statistically Cauchy sequences of fuzzy numbers.

DEFINITION 2.1. A sequence $X = \{X_k\}$ of fuzzy numbers is said to be *statistically convergent to the fuzzy number* X_0, written as st-$\lim X_k = X_0$, if for every $\varepsilon > 0$,

$$\lim_n 1/n \left|\{k \le n : \ \overline{d}(X_k, X_0) \ge \varepsilon\}\right| = 0 ,$$

i.e.,

$$\overline{d}(X_k, X_0) < \varepsilon \qquad \text{a.a. } k .$$

It is clear that $\lim_k X_k = X_0$ implies st-$\lim X_k = X_0$.

DEFINITION 2.2. A sequence $X = \{X_k\}$ of fuzzy numbers is a *statistically Cauchy sequence* if for every $\varepsilon > 0$ there exists a number N $(= N(\varepsilon))$ such that

$$\lim_n 1/n \left|\{k \le n : \ \overline{d}(X_k, X_N) \ge \varepsilon\}\right| = 0,$$

i.e.,

$$\overline{d}(X_k, X_N) < \varepsilon \qquad \text{a.a. } k.$$

THEOREM 2.1. *A sequence $X = \{X_k\}$ of fuzzy numbers is statistically convergent if and only if $X = \{X_k\}$ is a statistically Cauchy sequence.*

P r o o f. Suppose that st-lim $X_k = X_0$ and $\varepsilon > 0$. Then

$$\overline{d}(X_k, X_0) < \varepsilon/2 \qquad \text{a.a } k,$$

and if N is chosen so that $\overline{d}(X_N, X_0) < \varepsilon/2$, then we have

$$\overline{d}(X_k, X_N) \le \overline{d}(X_k, X_0) + \overline{d}(X_N, X_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{a.a } k.$$

Hence, X is statistically Cauchy.

Conversely, suppose that X is a statistically Cauchy sequence. Then $\overline{d}(X_k, X_N) < \varepsilon/2$ a.a. k. Chose N such that $\overline{d}(X_N, X_0) < \varepsilon/2$, then for every $\varepsilon > 0$, we have

$$\overline{d}(X_k, X_0) \le \overline{d}(X_k, X_N) + \overline{d}(X_N, X_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \text{a.a. } k.$$

Hence st-lim $X_k = X_0$. □

THEOREM 2.2. *If $X = \{X_k\}$ is a sequence of fuzzy numbers for which there is a convergent sequence $Y = \{Y_k\}$ such that $X_k = Y_k$ a.a. k, then X is statistically convergent.*

P r o o f. Let $X = \{X_k\}$ be a sequence of fuzzy numbers such that $\lim_k Y_k = X_0$. Suppose $\varepsilon > 0$. Then for each n

$$\{k \le n : \ \overline{d}(X_k, X_0) \ge \varepsilon\} \subseteqq \{k \le n : \ X_k \ne Y_k\} \cup \{k \le n : \ \overline{d}(Y_k, X_0) > \varepsilon\}$$

since $\lim_k Y_k = X_0$, the latter set contains a fixed number of integers, say $s = s(\varepsilon)$. Then

$$\lim_n 1/n \left|\{k \le n : \ \overline{d}(X_k, X_0) \ge \varepsilon\}\right| \le \lim_n 1/n \left|\{k \le n : \ X_k \ne Y_k\}\right| + \lim_n s/n = 0$$

because $X_k = Y_k$ a.a. k. Hence $\overline{d}(X_k, X_0) < \varepsilon$ a.a. k, so X is statistically convergent to X_0. □

3. $l(p)$-spaces of sequences of fuzzy numbers

In [4], N a n d a introduced and discussed l_p-spaces of sequences of fuzzy numbers as follows:

$$l_p = \Big\{ X = \{X_k\} : \ \sum_k \big[\,\overline{d}(X_k, 0)\big]^p < \infty\,, \ \ l \le p < \infty \Big\}.$$

In this section, we shall introduce $l(p)$-spaces of sequences of fuzzy numbers.

We have

$$l(p) = \Big\{ X = \{X_k\} : \ \sum_k \big[\,\overline{d}(X_k, 0)\big]^{p_k} < \infty \Big\},$$

where (p_k) is a bounded sequence of strictly positive real numbers. If $p_k = p$ for all k, then $l(p) = l_p$, which was introduced by N a n d a .

We have the following result.

THEOREM 3.1. *$l(p)$ is a complete metric space with the metric ϱ defined by*

$$\varrho(X, Y) = \Big(\sum_k \big[\,\overline{d}(X_k, Y_k)\big]^{p_k} \Big)^{1/M},$$

where $X = \{X_k\}$ and $Y = \{Y_k\}$ are sequences of fuzzy numbers which are in $l(p)$ and $M = \max\Big(1, \sup_k p_k\Big)$.

P r o o f . It is straightforward to see that ϱ is a metric on $l(p)$. To show that $l(p)$ is complete in this metric, let $\{X^j\}$ be a Cauchy sequence in $l(p)$. Then for each fixed k,

$$\overline{d}(X_k^i, X_k^j) \le \Big(\sum_k [d(X_k^i, Y_k^j)]^{p_k} \Big)^{1/M} = \varrho(X^i, X^j)\,,$$

it follows that $\{X_k^i\}$ is a Cauchy sequence in $L(\mathbb{R})$. But $\big(L(\mathbb{R}), \overline{d}\,\big)$ is complete. Hence $\lim_i X_k^i = X_k$ for each k. Put $X = \{X_k\}$. We shall show that $\lim_i X^i = X$ and $X \in l(p)$.

We know that $\varrho(X^i, 0)$ is bounded, say, $\varrho(X^i, 0) \le L$. Now, for any t,

$$\Big(\sum_{k=1}^{t} \big[\,\overline{d}(X_k, 0)\big]^{p_k} \Big)^{1/M} \le \varrho(X^i, 0) \le L\,.$$

Letting $i \to \infty$, and then $t \to \infty$, we obtain

$$\Big(\sum_{k=1}^{t} \big[\,\overline{d}(X_k, 0)\big]^{p_k} \Big)^{1/M} \le L\,,$$

and this shows that $X \in l(p)$. It remains to show $\varrho(X^i, X) \to 0$. Let $\varepsilon > 0$ be given. Then there is a integer N such that $\varrho(X^i, X^j) < \varepsilon$ for $i, j \geq N$. Therefore for any t,

$$\left(\sum_{k=1}^{t}\left[\overline{d}(X_k^i, X_k^j)\right]^{p_k}\right)^{1/M} \leq \varrho(X^i, X^j) \leq \varepsilon \qquad \text{for} \quad i, j \geq N .$$

Letting $j \to \infty$ we obtain

$$\left(\sum_{k=1}^{t}\left[\overline{d}(X_k^i, X_k)\right]^{p_k}\right)^{1/M} \leq \varepsilon \qquad \text{for} \quad i \geq N .$$

Since t is arbitrary, we let $t \to \infty$ and obtain $\varrho(X^i, X) < \varepsilon$ for $i \geq N$. This completes the proof. □

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