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NORMAL FORMS AND BIFURCATIONS OF SOME EQUIVARIANT VECTOR FIELDS

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Many recent papers (e. g. [2], [8], [10], [20], [21]) and also some books (e. g. [9], [16], [19]) have been dealing with local bifurcations for equivariant vector fields and as the main tool the well-known Ljapunov-Schmidt reduction method has been used. Recently global bifurcations of periodic solutions of equivariant vector fields are very intensively studied (see, e. g., [7] and [18]) and besides the Ljapunov-Schmidt method also topological and homotopical methods are used there. However, there are only few remarks in the above mentioned papers and books about the existence of some further invariant sets and about dynamical properties of invariant sets including periodic trajectories. The Ljapunov-Schmidt method is very useful for detecting periodic trajectories but it can hardly be used for detecting other, more complicated invariant sests and their dynamical properties. A very effective method for a complex study of the local properties of differential equations is the normal forms method. Motivated by the paper of J. Guckenheimer [11] on a codimension two bifurcation with circular symmetry we study several bifurcation problems concerning equivariant vector fields whose normal forms possess some invariant subvarieties. The reductions of such vector fields to these subvarieties often represent bifurcation problems with known solutions. In the papers of A. Klíč [13], [14] the idea of looking for periodic solutions of symmetric equations on some invariant submanifolds defined by the symmetry of these equations is also used. Namely, he studies there the period doubling bifurcation for 1-parameter families of vector fields invariant with respect to an involutory mapping.

We use in this paper a normal form theorem published very recently in [6]. Normal form theorems of such kind (see also [4] and [18]) seem to be a very powerful tool especially for solving bifurcation problems of equivariant vector fields on higher dimensional spaces.

1. $O(2)$-equivariant vector fields on R^4

J. Guckenheimer studied in [11] vector fields on R^4 with non-zero nilpotent linear parts equivariant with respect to the diagonal action of $O(2)$, the group of orthogonal 2×2 matrices, on R^4. He derived a normal form of order 3 for such vector fields and studied their $0(2)$-equivariant unfoldings which have the form

$$\begin{aligned}
\dot{x}_1 &= y_1 + (x_1^2 + x_2^2)(b_{11}x_1 + b_{12}y_1) \\
\dot{x}_2 &= y_2 + (x_1^2 + x_2^2)(b_{11}x_2 + b_{12}y_2) \\
\dot{y}_1 &= \lambda_1 x_1 + \lambda_2 y_1 + (x_1^2 + x_2^2)(b_{21}x_1 + b_{22}y_1) \\
\dot{y}_2 &= \lambda_1 x_2 + \lambda_2 y_2 + (x_1^2 + x_2^2)(b_{21}x_2 + b_{22}y_2),
\end{aligned}$$

where $b_{ij} \in R$ and λ_1, λ_2 are real parameters.

The subvariety $D := \{(x, y) \in R^4 : x_1 y_2 - x_2 y_1 = 0\}$ is an invariant set of the family (1). The functions $\alpha = x_1^2 + x_2^2$, $\beta = y_1^2 + y_2^2$ and $\gamma = x_1 y_1 + x_2 y_2$ are invariant with respect to the diagonal action of $0(2)$ and if $(x_1(t), x_2(t), y_1(t), y_2(t))$ is a solution of (1) lying on D and $u(t) := \frac{1}{3} b_{12}(x_1^2(t) + x_2^2(t))^{3/2}$, $v(t) := (y_1^2(t) + y_2^2(t))^{1/2} + b_{11}(x_1^2(t) + x_2^2(t))^{3/2}$, then $(u(t), v(t))$ is a solution of the system

$$\begin{aligned}
\dot{u} &= v \\
\dot{v} &= \lambda_1 u + \lambda_2 v + \left(b_{21} + \frac{1}{2} b_{12}\lambda_1 - b_{11}\lambda_2\right) u^3 + 3(b_{11} + b_{22}) u^2 v + 0(3)
\end{aligned} \tag{2}$$

(cf. [11]). If $b_{21} \neq 0$, $b_{11} + b_{22} \neq 0$, then the bifurcations of (2) are well known (see e.g. [3], [12], [17]). It is also well known that under these generic assumptions the family (2) is structurally stable in the space of equivariant families of plane vector fields of the form $\dot{Z} = f(\lambda, Z)$, $\lambda = (\lambda_1, \lambda_2) \in R^2$, possessing the symmetry property $f(\lambda, -Z) = -f(\lambda, Z)$.

Using the normal form theorem from [6] we derive a normal form of order k $(3 \leqq k \leqq \infty)$ for an $0(2)$-equivariant vector field on R^4 with a non-zero nilpotent linear part. If we truncate terms of higher order than 3 in this vector field, we obtain a normal form of order 3 which differs from this studied by J. Guckenheimer in [11], however it is again $0(2)$-equivariant and the set D defined as above is its invariant set. The same is also true not only for this normal form of order 3 but for the normal form of arbitrarily large order.

Consider the system

$$\dot{y} = L_0 y + f(y), \tag{3}$$

where $y \in R^n$ or $y \in C^n$, $f \in C^r$, $2 \leqq r \leqq \infty$, $f(0) = 0$ and all eigenvalues of L_0 have zero real parts. A transformation of the form $y = x + \Phi(x)$, where Φ is a C^∞-map, transforms (3) into the form

$$\dot{x} = g(x) := L_0 x + F(x) + R(z), \tag{4}$$

where $R \in C^r$, $R(x) = 0(\|x\|^k)$, $1 \leqq k \leqq r \leqq \infty$ and F is a polynomial map of degree k. If F is as simple as possible (e. g. if it does not contain any resonant term; see [12]), then the system

$$\dot{x} = h(x) := L_0 x + F(x) \tag{5}$$

is called a normal form of the system (3) of order k. If $f \in C^\infty$, then a normal form of (3) can often be found in the form (5), where F is a smooth map (not a polynomial map in general) and the map R in (4) is a flat map, i. e. a smooth map with zero Taylor series at the origin. Then the system (5) is called a normal form of order ∞.

In our further considerations we assume $f \in C^\infty$.

Theorem 1. (see [6, Theorems 3, 4]). *Let the matrix L_0 of the linear part of (3) be in the Jordan form and all its eigenvalues have zero real parts. Then there is a smooth transformation of coordinates $y = x + \Phi(x)$ near the origin, transforming (3) into the following normal form of order ∞:*

$$\dot{x} = L_0 x + \sum_{j=1}^{n} \alpha_j(x) \mathscr{L}_j x, \tag{6}$$

where $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n$ are linear operators commuting with L_0^* (L_0^*— the adjoint of L_0) such that for almost all x the system $\{\mathscr{L}_j x : j = 1, 2, \ldots, n\}$ forms a basis for R^n (or C^n, if (3) is defined on C^n) and $\alpha_j(x), j = 1, 2, \ldots, n$ are first integrals of the system

$$\dot{x} = L_0^* x. \tag{7}$$

If there exists a linear unitary operator T which commutes with the vector field (3), then a normal form of the form (6) can be found which commutes with T as well as with $e^{L_0^* t}$, $t \in R$.

The assertions of Theorem 1 are formulated in [6] for normal forms of finite order only, however the Borel theorem (see, e. g., [1, Theorem 4. 9]) implies that the same assertions are valid also for normal forms of order ∞.

Now, let us consider a system of the form (3) on R^4 equivariant with respect to the diagonal action of $0(2)$. By [11] the matrix L_0 has the form

$$L_0 = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{11} & 0 & a_{13} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{31} & 0 & a_{33} \end{bmatrix} \tag{8}$$

and this matrix is nilpotent if and only if $a_{11} + a_{33} = 0$, $a_{11}a_{33} - a_{13}a_{31} = 0$. If the matrix (8) is non-zero and nilpotent, then, using an $0(2)$-equivariant linear change of coordinates, one can transform the system (3) into the same form with

$$L_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

which is also $O(2)$-equivariant. Therefore we assume L_0 to be in the form (9).

Let $M_4(O(2))$ be the set of all 4×4 matrices defining linear vector fields on R^4 eqivariant with respect to the diagonal action of $O(2)$ and let $N \subset M_4(O(2))$ consists of all non-zero nilpotent matrices. Since $a_{11} + a_{33} = 0$, $a_{11}a_{33} - a_{13}a_{31} = 0$, $L_0 \neq 0$ for $L_0 \in N$ and L_0 of the form (8), the implicit function theorem implies that N is a smooth submanifold of $M_4(O(2))$ of codimension 2. If $L: R^2 \to M_4(O(2))$,

$$L(\lambda) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 \end{bmatrix} \tag{10}$$

$\lambda = (\lambda_1, \lambda_2) \in \mathrm{R}^2$, *then* L *transvèrsally intersects* N.

We shall therefore study unfoldings of (3) with L_0 given by (9) which has its linear part defined by (10). First, let us outline how to derive a normal form for (3) with L_0 given by (9).

Let $x = (x_1, x_2, x_3, x_4)$, $X = x_1 + ix_2$, $Y = x_4 + ix_4$. Then one can write (3) in the folllowing complex form:

$$\begin{aligned} \dot{X} &= Y + F_1(X, \bar{X}, Y, \bar{Y}) \\ \dot{Y} &= F_2(X, \bar{X}, Y, \bar{Y}). \end{aligned} \tag{11}$$

If we denote $X = z$, $Y = w$, $u = \bar{X} = \bar{z}$, $v = \bar{Y} = \bar{w}$ and add to (11) its complex conjugate system, then we obtain a system of the form (3) on C^4 with

$$L_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Using Theorem 1 one can derive the following normal form of this system of order ∞:

$$\begin{aligned} \dot{z} &= w \\ \dot{w} &= wP_1(z, \bar{z}, w\bar{z} - z\bar{w}) + \bar{w}P_2(z, \bar{z}, w\bar{z} - z\bar{w}) + Q_1(z, \bar{z}) \\ \dot{\bar{z}} &= \bar{w} \end{aligned} \tag{13}$$

$$\dot{w} = \overline{\bar{w}P_1(z, \bar{z}, w\bar{z} - z\bar{w})} + \overline{wP_2(z, \bar{z}, w\bar{z} - z\bar{w})} + \overline{Q_1(z, \bar{z})}, \tag{14}$$

where P_1, P_2, Q_1 are complex valued functions. In [6] and also in [4] a normal form of systems on R^4 with the linear part defined by (12) is derived and it has formally the same form as (13), (14), where the real variables x_1, x_2, x_3, x_4 are instead of z, w, $\bar{z}$, $\bar{w}$, respectively, and some smooth real functions instead of P_1, P_2 and Q_1. Obviously, it is sufficient to consider the system (13) only. Since the original system (3) is $O(2)$-equivariant, the system (13) is invariant with respect to the transformations

$$X_1(z, w) = (e^{i\Theta}z, e^{i\Theta}w), \ \Theta \in R \tag{15}$$

$$X_2(z, w) = (\bar{z}, \bar{w}). \tag{16}$$

By Theorem 1 there is a normal form of the system (3) of the form (13), (14) which is also invariant with respect to these transformations. This invariance implies that

$$P_1(z, \bar{z}, w\bar{z} - z\bar{w}) = \psi_1(|z|^2, w\bar{z} - z\bar{w}), \ P_2(z, \bar{z}, w\bar{z} - z\bar{w}) \equiv 0,$$

$Q_1(z, \bar{z}) = z\Psi_2(|z|^2)$ for some complex valued functions Ψ_1, Ψ_2. We obtain a system of the form

$$\begin{aligned} \dot{z} &= w \\ \dot{w} &= w\Psi_1(|z|^2, w\bar{z} - z\bar{w}) + z\Psi_2(|z|^2). \end{aligned} \tag{17}$$

If $z = x_1 + ix_2$, $w = x_3 + ix_4$, then the system (17) written in real coordinates x_1, x_2, x_3, x_4 has the form

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= x_3P_1(x_1^2 + x_2^2, x_1x_4 - x_2x_3) - x_4P_2(x_1^2 + x_2^2, x_1x_4 - x_2x_3) + \\ &\quad + x_1R_1(x_1^2 + x_2^2) - x_2R_2(x_1^2 + x_2^2) \\ \dot{x}_4 &= x_3P_2(x_1^2 + x_2^2, x_1x_4 - x_2x_3) + x_4P_1(x_2^2 + x_2^2, x_1x_4 - x_2x_3) + \\ &\quad + x_1R_2(x_1^2 + x_2^2) + x_2R_1(x_1^2 + x_2^2) \end{aligned} \tag{18}$$

for some smooth real functions P_1, P_2, R_1, R_2. As a consequence of the invariance of the system (17) with respect to the map X_2 (see (16)) we have that Ψ_2 in (17) is a real function and this implies that in (18) we have $R_2 \equiv 0$. The invariance of (17) with respect to the map (16) implies the invariance of the system (18) with respect to the map (16) implies the invariance of the system (18) with respect to the map $X_3(x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, -x_4)$ and therefore $P_2 \equiv 0$. Thus we have obtained that the $O(2)$-equivariant system (3) on R^4 has the following normal form of order ∞:

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= x_3\tilde{P}(x_1^2 + x_2^2,\ x_1x_4 - x_2x_3) + x_1\tilde{Q}(x_1^2 + x_2^2) \\
\dot{x}_4 &= x_4\tilde{P}(x_1^2 + x_2^2,\ x_1x_4 - x_2x_3) + x_2\tilde{Q}(x_1^2 + x_2^2),
\end{aligned} \tag{19}$$

where $\tilde{P}$, $\tilde{Q} \in C^\infty$. Since our considerations are local, near the origin, we may assume without loss of generality that

$$|\tilde{P}(u, v)| \leqq K \quad \text{for all} \quad (u, v) \in R^2, \tag{20}$$

where K is a positive constant.

Lemma 1. *Let the condition* (20) *be satisfied*, $\varphi(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ *be a solution of the system* (19) *and* $\varphi(0) \in D := \{(x_1, x_2, x_3, x_4) \in R^4 : x_1x_4 - x_2x_3 = 0\}$. *Then* $\varphi(t) \in D$ *for all* $t \in R$.

Proof. If $\Psi(t) = x_1(t)x_4(t) - x_2(t)x_3(t)$, then

$$\frac{d\Psi(t)}{dt} = \tilde{P}(x_1^2(t) + x_2^2(t),\ \Psi(t))\,\Psi(t). \tag{21}$$

If $\varphi(0) \in D$, then $\Psi(0) = 0$ and therefore we have

$$\Psi(\mathrm{t}) - \Psi(0) = \int_0^t \tilde{P}(x_1^2(s) + x_2^2(s),\ \Psi(s))\,(\Psi(s) - \Psi(0))\,ds$$

and this yields the inequality

$$|\Psi(t) - \Psi(0)| \leqq K \int_0^t |\Psi(s) - \Psi(0)|\,ds.$$

The Gronwall lemma implies that $\Psi(t) = \Psi(0) = 0$ for all $t \in R$, i. e. $\Phi(t) \in D$ for all $t \in R$.

The reduction of the system (19) to the subvariety D has the form

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= x_3\hat{P}(x_1^2 + x_2^2) + x_1\hat{Q}(x_1^2 + x_2^2), \\
\dot{x}_4 &= x_4\hat{P}(x_1^2 + x_2^2) + x_2\hat{Q}(x_1^2 + x_2^2),
\end{aligned} \tag{22}$$

where $\hat{P}(u) := \tilde{P}(u, 0)$, $\hat{Q}(u) := \tilde{Q}(u, 0)$.

Now, let us consider an $0(2)$-equivariant unfolding of (19) which has the form

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4
\end{aligned} \tag{23}$$

$$\dot{x}_3 = \lambda_1 x_1 + \lambda_2 x_3 + x_3 G(x_1^2 + x_2^2, x_1x_4 - x_2x_3, \lambda) + x_1 H(x_1^2 + x_2^2, \lambda)$$
$$\dot{x}_4 = \lambda_1 x_2 + \lambda_2 x_4 + x_4 G(x_1^2 + x_2^2, x_1x_4 - x_2x_3, \lambda) + x_2 H(x_1^2 + x_2^2, \lambda),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in R^k$ is a parameter, $G, H \in C^\infty$, $G(u, v, 0) = \tilde{P}(u, v)$, $H(u, 0) = \tilde{Q}(u)$ for $(u, v) \in R^2$. We assume that

$$|G(u, v, \lambda)| \leqq L \quad \text{for all} \quad (u, v, \lambda) \in R^2 \times R^k, \tag{24}$$

where L is a positive constant. By the same procedure is in the proof of Lemma 1 one can show that if the condition (24) is satisfied, then the subvariety D from Lemma 1 is also an invariant set of the unfolding (23). The reduction of the system (23) to the set D has the form

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \lambda_1 x_1 + \lambda_2 x_3 + x_3 P(x_1^2 + x_2^2, \lambda) + x_1 Q(x_1^2 + x_2^2, \lambda) \\
\dot{x}_4 &= \lambda_1 x_2 + \lambda_2 x_4 + x_4 P(x_1^2 + x_2^2, \lambda) + x_2 Q(x_1^2 + x_2^2, \lambda),
\end{aligned} \tag{25}$$

where $P(u, 0) = \hat{P}(u)$, $Q(u, 0) = \hat{Q}(u)$ for $(\mathrm{u}, \mathrm{v}) \in \mathrm{R}^2$. The family (25) is an $0(2)$-equivariant unfolding of (22). One can check that if $x = (x_1, x_2, x_3, x_4)$ is a solution of the system (25) and $\alpha = x_1^2 + x_2^2$, $\beta = x_3^2 + x_4^2$, $\gamma = x_1x_4 + x_2x_3$, then (α, β, γ) is a solution of the system

$$\begin{aligned}
\dot{\alpha} &= 2\gamma \\
\dot{\beta} &= 2(\lambda_1 \gamma + \lambda_2 \beta + P(\alpha, \lambda)\beta + Q(\alpha, \lambda)\gamma) \\
\dot{\gamma} &= \beta + \lambda_1 \alpha + \lambda_2 \gamma + P(\alpha, \lambda)\gamma + Q(\alpha, \lambda)\alpha.
\end{aligned} \tag{26}$$

Since $\alpha\beta = \gamma^2 + \delta^2$, where $\delta = x_1x_4 - x_2x_3$, the equality $\delta = 0$ implies that $\alpha\beta = \gamma^2$. Using this equality one can show that if $X = \sqrt{\alpha}$, $Y = \sqrt{\beta}$, then (X, Y) is a solution of the system

$$\begin{aligned}
\dot{X} &= Y \\
\dot{Y} &= \lambda_1 X + \lambda_2 Y + Q(X^2, \lambda) X + P(X^2, \lambda) Y.
\end{aligned} \tag{27}$$

Theorem 2. *Let F be a smooth vector field on R^4 which is equivariant with respect to the diagonal acation of $0(2)$ on R^4. If the vector field F has the equilibrium point at the origin with linearization defined by a non-zero nilpotent matrix, then the following assertions hold:*

(1) *A normal form of order ∞ of the vector field F has the form* (19) *and it is $0(2)$-equivariant.*
(2) *If the condition* (20) *is satisfied, then the subvariety $D := \{(x_1, x_2, x_3, x_4) \in R^4 : x_1x_4 - x_2x_3 = 0\}$ is an invariant set of the system* (19).
(3) *If $\tilde{F}$ is the vector field defined by* (19), *then the vector field $\tilde{F}/D$ is represented by the system* (22) *which is $0(2)$-equivariant.*

(4) *The family* (23) *is an* $O(2)$-*equivariant unfolding of* (19), *the set* D *is its invariant set and the reduction of* (23) *to this set has the form* (25) *which is an* $O(2)$-*equivariant system.*
(5) *If* $x = (x_1, x_2, x_3, x_4)$ *is a solution of the system* (25) *and* $\alpha = x_1^2 + x_2^2$, $\beta = x_3^2 + x_4^2$, $\gamma = x_1x_3 + x_2x_4$, *then* (α, β, γ) *is a solution of the system* (26). *Moreover, if* $x \in D$ *and* $X = \sqrt{\alpha}$, $Y = \sqrt{\beta}$, *then* (X, Y) *is a solution of the system* (27)
(6) *If* $\dfrac{\partial P(0, 0)}{\partial x} \neq 0$, $\dfrac{\partial Q(0, 0)}{\partial x} \neq 0$ *and* $\lambda \in R^2$, *then* (27) *is the structurally stable family in the space of all 2-parameter families of plane vector fields equivariant with respect to the rotation by* π *in the plane (for the bifurcations of* (27) *see, e.g.,* [3], [12], [17]).

Proof. The assertions (1)—(5) follow from Lemma 1 and the considerations before Theorem 2. The assertion (6) is a consequence of the results published by J. Carr in [3].

If $\dfrac{\partial P(0, 0)}{\partial x} \cdot \dfrac{\partial Q(0, 0)}{\partial x} = 0$, then the family (27) is an unfolding of a vector field possessing a singularity of codimension greater than 2. Bifurcations of vector fields of the form (27) near a codimension 3 singularity are studied in [5] (see also [15] for the nonsymmetric case).

2. On vector fields with a pair of pure imaginary eigenvalues of multiplicity 2

Let us consider a smooth vector field on R^4, represented by the system (3) with

$$\begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix} \tag{28}$$

The analysis of bifurcations of unfoldings of such a vector field is still an open problem. For a normal form and quotations of papers concerning this case see, e.g., [12].

The system (3) with L_0 given by (28) can be written in the following complex form:

$$\begin{aligned} \dot{z}_1 &= i\omega z_1 + z_2 + F_1(z_1, z_2, \bar{z}_1, \bar{z}_2) \\ \dot{z}_2 &= i\omega z_2 + F_2(z_1, z_2, \bar{z}_1, \bar{z}_2). \end{aligned} \tag{29}$$

Assume that the following conditions are satisfied:

$$\overline{F_i(\bar{z}_1, \bar{z}_2, z_1, z_2)} = F_i(z_1, z_2, \bar{z}_1, \bar{z}_2) \quad \text{for} \quad i = 1, 2 \quad \text{and all} \quad z_1, z_2 \in C, \tag{30}$$

$$\begin{gathered} e^{i\Theta} F_j(z_1, z_2, \bar{z}_1, \bar{z}_2) = F_j(e^{i\Theta} z_1, e^{i\Theta} z_2, e^{-\Theta} \bar{z}_1, e^{-i\Theta} \bar{z}_2) \\ \text{for} \quad j = 1, 2 \quad \text{and all} \quad \Theta \in R, \; z_1, z_2 \in C. \end{gathered} \tag{31}$$

After introducing new variables $w_1 = e^{-i\omega t} z_1$, $w_2 = e^{-i\omega t} z_2$ the system (29) becomes

$$\begin{aligned} \dot{w}_1 &= w_2 + F_1(w_1, w_2, \bar{w}_1, \bar{w}_2) \\ \dot{w}_2 &= F_2(w_1, w_2, \bar{w}_1, \bar{w}_2). \end{aligned} \tag{32}$$

Since we assume that the functions F_1, F_2 satisfy the conditions (30), (31), the system (32) is invariant with respect to the transformations X_1, X_2 (see (15), (16)), i. e. the corresponding real system is $O(2)$-equivariant. By Theorem 1 (see also [6]) the system (32) has the following normal form of order ∞:

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1 \varphi_1(|y_1|^2, y_1\bar{y}_2 - \bar{y}_1 y_2) + y_2 \varphi_2(|y_1|^2, y_1 \bar{y}_2 - \bar{y}_1 y_2), \end{aligned} \tag{33}$$

and this system is also invariant with respect to the transformations X_1 and X_2. Using this invariance property one can check that if $y_1 = x_1 + ix_2$, $y_2 = x_3 + ix_4$, then the system (33) written in the real variables x_1, x_2, x_3, x_4 has the form

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= x_3 \bar{P}(x_1^2 + x_2^2, x_1 x_4 - x_2 x_3) + x_1 \bar{Q}(x_1^2 + x_2^2, x_1 x_4 - x_2 x_3) \\ \dot{x}_4 &= x_4 \bar{P}(x_1^2 + x_2^2, x_1 x_4 - x_2 x_3) + x_2 \bar{Q}(x_1^2 + x_2^2, x_1 x_4 - x_2 x_3). \end{aligned} \tag{34}$$

Since our considerations are local, near the origin, we may assume without loss of generality that

$$|\bar{P}(u, v)| \leqq K \quad \text{for all} \quad (u, v) \in R^2. \tag{35}$$

Similarly as in the proof of Lemma 1 one can show that the subvariety $D := \{(x_1, x_2, x_3, x_4) \in R^4 : x_1 x_4 - x_2 x_3 = 0\}$ is an invariant set of the system (34). The reduction of (34) to the subvariety D has the form (22), where $\hat{P}(u) = \bar{P}(u, 0)$, $\hat{Q}(u) = \bar{Q}(u, 0)$.

Now, consider the following $O(2)$-equivariant unfolding of the system (34):

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= \lambda_1 x_1 + \lambda_2 x_3 + x_3 \Phi(x_1^2 + x_2^2, x_1 x_4 - x_2 x_3, \lambda) + \\ &\quad + x_1 \Psi(x_1^2 + x_2^2, x_1 x_4 - x_2 x_3, \lambda) \end{aligned} \tag{36}$$

$$\dot{x}_4 = \lambda_1 x_2 + \lambda_2 x_4 + x_4 \Phi(x_1^2 + x_2^2, x_1x_4 - x_2x_3, \lambda) + \\ + x_2 \Psi(x_1^2 + x_2^2, x_1x_4 - x_2x_3, \lambda),$$

where Φ, Ψ are smooth functions, $\Phi(u, v, 0) = \bar{P}(u, v)$, $\Psi(u, v) = \bar{Q}(u, v)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in R^k$.

We assume that the following condition is satisfied:

$$|\Phi(u, v, \lambda)| \leqq L \quad \text{for all} \quad (u, v, \lambda) \in R^2 \times R^k, \tag{37}$$

where L is a positive constant. Then the set D is an invariant set of the system (36) (see Lemma 1) and the reduction of this system to the set D has the form (25), where $P(u, \lambda) = \Phi(u, 0, \lambda)$, $Q(u, \lambda) = \Psi(u, 0, \lambda)$. We have proved the following theorem.

Theorem 3. *Let F be a smooth vector field on R^4 defined by the equation* (3) *with L_0 given by* (28), *and let the system* (29) *be its complexification. Then the following assertions hold*:

(1) *The transformations $w_1 = E^{-i\omega t} z_1$, $w_2 = e^{-i\omega t} z_2$ transform the system* (29) *into the form* (32) *which is invariant with respect to the transformations* (15), (16).
(2) *The real system corresponding to the system* (32) *has the form* (34) *and this system is $0(2)$- equivariant.*
(3) *If the condition* (37) *is satisfied, then the set $D := \{(x_1, x_2, x_3, x_4) \in R^4 : x_1x_4 - x_2x_3\}$ is an invariant set of the system* (34) *and its reduction to the set D has the form* (22), *where $\hat{P}(u) = \bar{P}(u, 0)$, $\hat{Q}(u) = \bar{Q}(u, 0)$.*
(4) *The family* (36) *is an $0(2)$-equivariant unfolding of the system* (34) *and if the condition* (37) *is satisfied, then the set D is an invariant set of this family. The reduction of the system* (36) *to the set D has the form* (25) *and for this family the assertions* (5) *and* (6) *of Theorem* 2 *are valid.*

3. $0(2) \times S^1$-equivariant vector fields on C^2

M. Golubitsky an M. Roberts studied in their paper [8] the Hopf bifurcation for vector fields on C^2 equivariant with respect to the following transformations:

$$\begin{aligned} \Theta(z_1, z_2) &= (e^{i\Theta} z_1, e^{-i\Theta} z_2), \ \Theta \in R \\ K(z_1, z_2) &= (z_2, z_1) \\ \varphi(z_1, z_2) &= (e^{i\varphi} z_1, e^{i\varphi} z_2), \ \varphi \in R \end{aligned} \tag{38}$$

($0(2) \times S^1$-equivariant vector fields; see [8]). By [8, Proposition 2.1] such equivariant vector fields have the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = (p + iq) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (r + is)\,\delta \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{39}$$

where $z_1, z_2 \in C$, p, q, r, s are smooth functions of (N, Δ) with real values, $N = |z_1|^2 + |z_2|^2$, $\Delta = \delta^2$, $\delta = |z_2|^2 - |z_1|^2$. Following [8] we write $z_1 = xe^{i\Psi_1}$, $z_2 = ye^{i\Psi_2}$ and from (39) we obtain the system

$$\begin{aligned} \dot{x} &= (p + r\delta)\,x \\ \dot{y} &= (p - r\delta)\,y \end{aligned} \qquad \text{amplitude equations} \tag{40}$$

$$\begin{aligned} \dot{\Psi}_1 &= q + s\delta \\ \dot{\Psi}_2 &= q - s\delta \end{aligned} \qquad \text{phase equations.} \tag{41}$$

The equations (40) are invariant with respect to the transformations $I:(x, y) \to (x, -y)$ and $J:(x, y) \to (y, x)$, i.e. these equations are invariant with respect to the action of the dihedral group D_4. By [2] this is the general form of D_4-equivariant vector fields on R^2.

Now assume that the functions p, q, r, s depend also on some parameter $\lambda \in R^k$. Then we have the family

$$\begin{aligned} \dot{x} &= p(N, \Delta, \lambda)\,x + r(N, \Delta, \lambda)\,\delta x \\ \dot{y} &= p(N, \Delta, \lambda)\,y - r(N, \Delta, \lambda)\,\delta y, \end{aligned} \tag{42}$$

where $N = x^2 + y^2$, $\delta = y^2 - x^2$, $\Delta = \delta^2$ and p, r are smooth functions. The phase equations have the form

$$\begin{aligned} \dot{\Psi}_1 &= q(N, \Delta, \lambda) + s(N, \Delta, \lambda)\,\delta \\ \dot{\Psi}_2 &= q((N, \Delta, \lambda) - s(N, \Delta, \lambda)\,\delta, \end{aligned} \tag{43}$$

where q, s are smooth functions. Since we are interested in bifurcations near the origin, it is convenient to study a modification of the system (42). If U, V are neighbourhoods of the origin in R^{k+2} such that $\bar{U} \subset V$ and $\bar{V}$ is a compact set, then there are smooth functions $\tilde{r}: R^{k+2} \to R$, $\tilde{p}: R^{k+2} \to R$ such that $\tilde{r} = r$, $\tilde{p} = p$ on U and $\tilde{r} = 0$, $\tilde{p} = 0$ outside V. By the modification of the system (42) we mean the system of the same form, where the functions $\tilde{p}$ and $\tilde{r}$ are instead of p and r, respectively. Then the functions $\tilde{p}$ and $\tilde{r}N$ are obviously bounded and so we may assume without loss of generality that for the original system (42) the following condition is satisfied:

$$|r(u, v, \lambda)\,u| \leqq K, \quad |p(u, v, \lambda)| \leqq K \tag{44}$$

for all $(u, v, \lambda) \in R^2 \times R^k$, where K is a positive constant. Using the system (42) it is easy to prove the following lemma.

Lemma 2. *If* $(x(t),\ y(t))$ *is a solution of the system* (42) *and* $N(t) = x^2(t) + y^2(t)$, $\delta(t) = y^2(t) - x^2(t)$, *then* $(N(t),\ \delta(t))$ *is a solution of the system*

$$\begin{aligned} \dot{N} &= 2(p(N,\ \delta^2,\ \lambda)\, N + r(N,\ \delta^2,\ \lambda)\, \delta^2) \\ \dot{\delta} &= 2(p(N,\ \delta^2,\ \lambda)\, \delta - r(N,\ \delta^2,\ \lambda)\, N\delta) \end{aligned} \tag{45}$$

Lemma 3. *Let* $\varphi(t) = (x(t),\ y(t))$ *be a solution of the system* (42) *and* $\varphi(0) \in E := \{(x,\ y) \in R^2 : \delta := y^2 - x^2 = 0\}$. *Assume that the condition* (44) *is satisfied. Then* $\varphi(t) \in E$ *for all* $t \in R$.

Proof. From the condition (44) we obtain that if $\varphi(0) \in E$ and $\Psi(t) = y^2(t) - x^2(t)$, then

$$|\Psi(t) - \Psi(0)| \leqq 4K \int_0^t |\Psi(s) - \Psi(0)|\, ds$$

and the Gronwall lemma implies that $\Psi(t) = \Psi(0) = 0$ for all $t \in R$, i.e. $\varphi(t) \in E$ for all $t \in R$.

The reduction of the system (42), (43) to the invariant set E is

$$\begin{aligned} \dot{x} &= P(2x^2,\ \lambda)\, x \\ \dot{y} &= P(2y^2,\ \lambda)\, y \end{aligned} \tag{46}$$

$$\begin{aligned} \dot{\Psi}_1 &= Q(2x^2,\ \lambda) \\ \dot{\Psi}_2 &= Q(2x^2,\ \lambda). \end{aligned} \tag{47}$$

Theorem 4. *Let*

$$\dot{z} = f(z,\ \lambda),\quad z \in C^2,\quad \lambda \in R^k \tag{48}$$

be a smooth k-parameter family of vector fields on C^2 *equivariant with respect to the transformations* (38) *and assume that* $f(0,\ 0) = 0$. *Then the following assertions hold*:

(1) *The family* (48) *has the form* (39).

(2) *If* $z = (z_1,\ z_2)$, $z_1 = xe^{i\Psi_1}$, $z_2 = ye^{i\Psi_2}$, *then the equations for* x, y, Ψ_1, Ψ_2 *have the form* (42), (43).

(3) *If the condition* (44) *is satisfied, then the set* $\tilde{E} := \{(x,\ y,\ \Psi_1,\ \Psi_2) \in R^4 : y^2 - x^2 = 0\}$ *is an invariant set of the family* (42), (43).

(4) *The reduction of the family* (42), (43) *to the invariant set* $\tilde{E}$ *has the form* (46), (47).

(5) *If* $(x_0,\ y_0,\ \Psi_1^0,\ \Psi_2^0) := \gamma_0 \in \tilde{E}$ *and* $\gamma(t) = (x(t), y(t),\ \Psi_1(t),\ \Psi_2(t))$ *is a solution of the system* (46), (47) *satisfying the initial condition* $\gamma(0) = \gamma_0$, *then* $z(t) = (z_1(t),\ z_2(t)) = (x(t)\, e^{i\Psi_1(t)},\ y(t)\, e^{i\Psi_2(t)})$ *is a solution of the system* (48), *i.e.* (39).

(6) *If* (x_0, y_0) *is an equilibrium of the system* (46), *then the set* $T(x_0, y_0) := \{(z_1, z_2) \in C^2 : z_1 = x_0 e^{i\Psi_1}, z_2 = y_0 e^{i\Psi_2}, \Psi_1, \Psi_2 \in R\}$ *is an invariant set of the system*

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = (P(|z_1|^2 + |z_2|^2, \lambda) + iQ(|z_1|^2 + |z_2|^2, \lambda)) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{49}$$

where $P(u, \lambda) = p(u, 0, \lambda)$, $Q(u, \lambda) = q(u, 0, \lambda)$, *i. e. this is the system* (39) *with* $\delta = 0$, $T(0, 0) = \{(0, 0)\}$ *and if* $x_0 \neq 0$, $y_0 = \pm x_0$, *then* $T(x_0, y_0)$ *is an invariant 2-tori* T^2 *of* (48).

(7) *An equilibrium solution* (x_0, y_0) *of the amplitude equations* (46) *is stable if and only if the corresponding equilibrium point* $T(0, 0)$ (*if* $x_0 = 0$) *or invariant 2-torus* $T^2 = T(x_0, y_0)$ (*if* $x_0 \neq 0$, $y_0 = \pm x_0$) *is a stable invariant set of the system* (48).

Proof. The assertions (1)—(5) follow from Lemma 2, Lemma 3 and the considerations before Theorem 4. The assertions (6), (7) follow from the relations $z_1 = xe^{i\Psi_1}$, $z_2 = ye^{i\Psi_2}$, where (z_1, z_2) is a solution of (49) and (x, y, Ψ_1, Ψ_2) is a solution of the system (46), (47).

Remark. Since we have to do with the reduction to the invariant set E defined by the equation $x^2 = y^2$, this reduction does not possess the invariant circle $T(x_0, 0)$ or $T(0, y_0)$ for $x_0 \neq 0$, $y_0 \neq 0$. Of course, the original general system (39) may possess such invariant circles (see [8]). However, our reduction (46), (47) is very simple and its equilibria are relatively simply computable. The form of these equations enables us to solve also higher codimensional bifurcation problems. Using the Malgrange-Weierstrass preparation theorem (see, e. g., [1, Theorem 6.3]) we are able to simplify the system (46), (47) near the origin as follows. By this theorem the following holds:
If

$$P(0, 0) = \frac{\partial^k P(0, 0)}{\partial x^k} = 0, \qquad k = 1, 2, \ldots, n-1, \frac{\partial^n P(0, 0)}{\partial x^n} \neq 0,$$

$$Q(0, 0) = \frac{\partial^j Q(0, 0)}{\partial x^j} = 0, \qquad j = 1, 2, \ldots, m-1, \frac{\partial^m Q(0, 0)}{\partial x^m} \neq 0,$$

then there is a neighbourhood $W_1 \times W_2 \subset R \times R^k$ of the origin and smooth functions $\Theta(x, \lambda)$, $\varphi_k(\lambda)$, $k = 1, 2, \ldots, n$, $\eta(x, \lambda)$, $\omega_j(\lambda)$, $j = 1, 2, \ldots, m$ defined for all $x \in W_1$, $\lambda \in W_2$ such that $\Theta(x, \lambda) \neq 0$, $\eta(x, \lambda) \neq 0$ for all $(x, \lambda) \in W_1 \times W_2$, $\varphi_k(0) = 0$, $\omega_j(0) = 0$ for all k, j and $P(2x^2, \lambda) = \Theta(x, \lambda)((x^2)^n + \varphi_n(\lambda)(x^2)^{n-1} + \ldots + \varphi_1(\lambda))$, $Q(2x^2, \lambda) = \eta(x, \lambda)((x^2)^m + \omega_m(\lambda)(x^2)^{m-1} + \ldots + \omega_1(\lambda))$ for all $(x, \lambda) \in W_1 \times W_2$. Putting these expressions into the system (46), (47) and dividing it by the function $\Theta(x, \lambda)$ we obtain the system

$$\dot{x} = x^{2n} + \varphi_n(\lambda) x^{2n-2} + \ldots + \varphi_1(\lambda)$$

$$\dot{y} = y^{2n} + \varphi_n(\lambda) y^{2n-2} + \ldots + \varphi_1(\lambda) \tag{50}$$

$$\begin{aligned} \dot{\Psi}_1 &= R(x, \lambda)\, (x^{2m} + \omega_m(\lambda) x^{2m-2} + \ldots + \omega_1(\lambda)) \\ \dot{\Psi}_2 &= R(x, \lambda)\, (x^{2m} + \omega_m(\lambda) x^{2m-2} + \ldots + \omega_1(\lambda)), \end{aligned} \tag{51}$$

where $R(x, \lambda) = \eta(x, \lambda)\, (\Theta(x, \lambda))^{-1}$.

If $k = \dim \lambda = n$, $\Phi := (\varphi_1, \varphi_2, \ldots, \varphi_n) : R^n \to R^n$ and if we assume that the derivative $d\Phi(0) : R^n \to R^n$ is an isomorphism, then we may introduce new coordinates $\varepsilon_i = \varphi_i(\lambda)$, $i = 1, 2, \ldots, n$ in a sufficiently small neighbourhood of the origin in the parameter space. In these new coordinates the family (50), (51) has the form

$$\begin{aligned} \dot{x} &= x^{2n} + \varepsilon_n x^{2n-2} + \ldots + \varepsilon_1 \\ \dot{y} &= y^{2n} + \varepsilon_n y^{2n-2} + \ldots + \varepsilon_1 \end{aligned} \tag{52}$$

$$\begin{aligned} \dot{\Psi}_1 &= \tilde{R}(x, \varepsilon)\, (x^{2m} + \tilde{\omega}_m(\varepsilon) x^{2m-2} + \ldots + \tilde{\omega}_1(\varepsilon)) \\ \dot{\Psi}_2 &= \tilde{R}(x, \varepsilon)\, (x^{2m} + \tilde{\omega}_m(\varepsilon) x^{2m-2} + \ldots + \tilde{\omega}_1(\varepsilon)) \end{aligned} \tag{53}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in R^n$ and the functions $\tilde{R}$, $\tilde{\omega}_1, \ldots, \tilde{\omega}_m$ have the same properties as R and $\omega_1, \omega_2, \ldots, \omega_m$, respectively.

Obviously, if (x_0, y_0) is an equilibrium of (52) and $x_0^2 = y_0^2$, then we have three equilibria (x_0, x_0), $(x_0, -x_0)$, $(-x_0, x_0)$ generated by the real root of the equation

$$x^{2n} + \varepsilon_n x^{2n-2} + \ldots + \varepsilon_1 = 0 \tag{54}$$

and the following proposition holds.

Proposition. *If c is the numbear of real roots of the equation* (54), *then the number of equilibria of* (52) *lying in the invariant set $E := \{(x, y) \in R^2 : x^2 - y^2 = 0\}$ is $3c$.*

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НОРМАЛЬНЫЕ ФОРМЫ И БИФУРКАЦИИ НЕКОТОРЫХ ЭКВИВАРИАНТНЫХ ВЕКТОРНЫХ ПОЛЕЙ

Milan Medveď

Резюме

В этой статье рассматриваются бифуркации гладких $O(2)$-эквивариантных векторных полей на R^4 и тоже $O(2) \times S^1$-эквивариантных векторных полей на C^2. Показано что некоторые нормальные формы бесконечного порядка таких ве кторных полей имеют специальные инвариантные многообразия и приведены некоторые теоремы касающееся бифуркации редукции таких нормальных форм на эти многообразия.