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Mathematica Slovaca, Vol. 28 (1978), No. 4, 343--347

Persistent URL: http://dml.cz/dmlcz/129413

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CONGRUENCE FACTORIZATIONS ON DISTRIBUTIVE LATTICES

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G. A. Fraser and A. Horn [1] gave conditions for direct decompositions of congruences on a direct product of two algebras. The aim of this paper is to give some sufficient conditions for the direct factorization of congruences on distributive lattices, which are directly decomposable to an arbitrary number of chains.

Let A_i for $i \in I$ be lattices, θ_i be a congruence on A_i, $i_0 \in I$ and $J = I - \{i_0\}$. Denote by $A = \prod_{i \in I} A_i$ the direct product of A_i and let $\theta = \prod_{i \in I} \theta_i$ be a binary relation on A defined by:

$a\theta b$ if and only if $pr_i a \theta_i pr_i b$ for each $i \in I$, where pr_i denotes the projection of A onto A_i. Evidently, θ is a congruence on A. Further, denote by $a(i) = pr_i a$ and by $a(J)$ the projection of $a \in A$ onto $\prod_{i \in J} A_i$. If θ is a congruence on A and $\theta = \prod_{i \in I} \theta_i$ for some congruences θ_i on A_i $(i \in I)$, θ is called *directly factorizable.*

Let L be a conditionally complete lattice (see [2], p.64). Then there exist the supremum and infimum of every bounded family $\{a_\mu ; \mu \in M\}$ of elements $a_\mu \in L$; denote it by $\bigvee_{\mu \in M} a_\mu$ or $\bigwedge_{\mu \in M} a_\mu$, respectively. A congruence θ on L is said to be *conditionally complete,* whenever

$$a_\mu \theta b_\mu \quad \text{for} \quad \mu \in M \quad \text{imply} \quad \Big(\bigvee_{\mu \in M} a_\mu\Big) \,\theta\, \Big(\bigvee_{\mu \in M} b_\mu\Big)$$

and

$$\Big(\bigwedge_{\mu \in M} a_\mu\Big) \,\theta\, \Big(\bigwedge_{\mu \in M} b_\mu\Big)$$

for every two bounded families $\{a_\mu ; \mu \in M\}$ and $\{b_\mu ; \mu \in M\}$ of elements from L.

Theorem 1. *Let A_i be conditionally complete chains for $i \in I$ and θ be a conditionally complete congruence on $A = \prod_{i \in I} A_i$. Then there exist congruences θ_i on A_i such that $\theta = \prod_{i \in I} \theta_i$.*

Proof. Denote by $\{M_\gamma;\, \gamma \in \Gamma\}$ the set of all congruence classes of θ. The case $\operatorname{card}\Gamma = 1$ is trivial. Suppose $\operatorname{card}\Gamma \geq 2$ and denote $M_\gamma(i) = pr_i M_\gamma$ for $i \in I$.

a) First we prove that for each $i \in I$ the family $\{M_\gamma(i);\, \gamma \in \Gamma\}$ forms a partition of A_i. As $\bigcup_{\gamma \in \Gamma} M_\gamma = A$, clearly $\bigcup_{\gamma \in \Gamma} M_\gamma(i) = A_i$. It remains to prove that for all $\gamma', \gamma'' \in \Gamma$ and $i \in I$, $M_{\gamma'}(i) \cap M_{\gamma''}(i) = \emptyset$ implies $M_{\gamma'}(i) = M_{\gamma''}(i)$.

Let $i \in I$, $\gamma', \gamma'' \in \Gamma$, $\gamma' \neq \gamma''$ and $M_{\gamma'}(i) \cap M_{\gamma''}(i) \neq \emptyset$. If both $M_{\gamma'}(i)$ and $M_{\gamma''}(i)$ have no least and no greatest bound, then $M_{\gamma'}(i) = A_i = M_{\gamma''}(i)$ because A_i is a chain and the convexity of the congruence class M_γ on A implies the convexity of the set $M_\gamma(i)$ on A_i.

Suppose $a_0 \in M_{\gamma'}(i) \cap M_{\gamma''}(i)$ and $M_{\gamma'}(i)$ has an upper bound. As θ and A_i are conditionally complete, we have $j' = \sup M_{\gamma'}(i) \in M_{\gamma'}(i)$. Suppose the existence of $b \in M_{\gamma''}$ with $b(i) = b_0 > j'$ and put $J = I - \{i\}$. Since $a_0 \leq j' < b_0$, there exists $h \in M_{\gamma''}$ with $h(i) = j'$ (with respect to the convexity of $M_{\gamma''}(i)$). Let $x \in M_{\gamma'}$ with $x(i) = j'$ and choose $y \in A$ with $y(i) = b_0$, $y(J) = x(J)$. Then $y \in M_\gamma$ for some $\gamma \in \Gamma$ and $\gamma \neq \gamma'$ because of $y(i) > j'$.

Since b, $h \in M_{\gamma''}$, there exists $\gamma_1 \in \Gamma$ with $p = x \vee h \in M_{\gamma_1}$, $q = x \vee b \in M_{\gamma_1}$. Then there exists $\gamma_2 \in \Gamma$ with $y \wedge p \in M_{\gamma_2}$, $y \wedge q \in M_{\gamma_2}$. However,

$$
\begin{gathered}
y(i) \wedge p(i) = b_0 \wedge (x(i) \vee h(i)) = b_0 \wedge j' = j' \\
y(J) \wedge p(J) = y(J) \wedge (x(J) \vee h(J)) = x(J) \wedge (x(J) \vee h(J)) = x(J),
\end{gathered}
$$

i.e. $y \wedge p = x \in M_{\gamma'}$, and

$$
\begin{gathered}
y(i) \wedge q(i) = b_0 \wedge (j' \vee b_0) = b_0 \\
y(J) \wedge q(J) = x(J), \\
\text{i.e.} \quad y \wedge q = y \in M_\gamma .
\end{gathered}
$$

Further, $\gamma \neq \gamma'$ implies $M_\gamma \cap M_{\gamma'} = \emptyset$, which contradicts the existence of γ_2.

Hence $b(i) > j'$ for no element b of $M_{\gamma''}(i)$, thus j' is also an upper bound of $M_{\gamma''}(i)$. Denote $j'' = \sup M_{\gamma''}(i) \in M_{\gamma''}(i)$. By the foregoing part of the proof, we have $j'' \leq j'$. The converse inequality can be proved analogously, thus $j'' = j'$.

Provided $M_{\gamma'}(i)$ has a lower bound, then also $M_{\gamma''}(i)$ has a lower bound and $\inf M_{\gamma'}(i) = \inf M_{\gamma''}(i)$; it can be proved by dualization. Hence $M_{\gamma'}(i) = M_{\gamma''}(i)$ with respect to the convexity of the sets $M_\gamma(i)$.

b) Prove $M_\gamma = \prod_{i \in I} M_\gamma(i)$ for each $\gamma \in \Gamma$. If $\gamma_0 \in \Gamma$ and M_{γ_0} has not this property, then there exists $a \in A$ with $a(i) \in M_{\gamma_0}(i)$ for each $i \in I$ and $a \notin M_{\gamma_0}$. Then $a \in M_{\gamma'}$ for some $\gamma' \in \Gamma$, $\gamma' \neq \gamma_0$. Hence $a(i) \in M_{\gamma'}(i)$ for each $i \in I$, i.e. $a(i) \in M_{\gamma_0}(i) \cap M_{\gamma'}(i)$, thus, by the foregoing part of the proof, $M_{\gamma_0}(i) = M_{\gamma'}(i)$ for all $i \in I$. As M_{γ_0}, $M_{\gamma'}$ are congruence classes, we obtain easily $M_{\gamma_0} \cap M_{\gamma'} \neq \emptyset$, which is a contradiction. Accordingly, $M_\gamma = \prod_{i \in I} M_\gamma(i)$ for each $\gamma \in \Gamma$, thus $\theta = \prod_{i \in I} \theta_i$, where θ_i is an equivalence on A_i induced by the partition $\{M_\gamma(i);\ \gamma\in\Gamma\}$; evidently, every θ_i is a congruence on A_i.

Q.E.D.

Let L be a lattice and $\mathfrak{M}$ a cardinal number. L is said to be *join-$\mathfrak{M}$-complete* provided the supremum $\bigvee_{\mu\in M} a_\mu$ exists for every family $\{a_\mu;\ \mu\in M\}$ of elements $a_\mu\in L$ with $\operatorname{card} M\leqslant\mathfrak{M}$. A congruence θ on a join-$\mathfrak{M}$-complete lattice L is called *join-$\mathfrak{M}$-complete* provided $a_\mu\theta b_\mu$ for $\mu\in M$, $\operatorname{card} M\leqslant\mathfrak{M}$ imply $\left(\bigvee_{\mu\in M} a_\mu\right)\theta\left(\bigvee_{\mu\in M} b_\mu\right)$ for every $a_\mu,\ b_\mu\in L$. A homomorphism h of a join-$\mathfrak{M}$-complete lattice L into a lattice L_1 is called *join-$\mathfrak{M}$-complete,* provided $\bigvee_{\mu\in M} h(a_\mu)$ exists for every family $\{a_\mu;\ \mu\in M,\ a_\mu\in L,\ \operatorname{card} M\leqslant\mathfrak{M}\}$ and $h\left(\bigvee_{\mu\in M} a_\mu\right) = \bigvee_{\mu\in M} h(a_\mu)$.

Theorem 2. *Let A_i, B_j $(i\in I,\ j\in J)$ be at least two-element chains with least elements,* $\operatorname{card} I=\mathfrak{M}$ *and* $\prod_{i\in I} A_i$, $\prod_{j\in J} B_j$ *be join-$\mathfrak{M}$-complete lattices. If θ is a join-$\mathfrak{M}$-complete congruence on* $\prod_{i\in I} A_i$ *and* $\left(\prod_{i\in I} A_i\right)\Big/\theta\cong\prod_{j\in J} B_j$, *then*

a) $\operatorname{card} J\leqslant\operatorname{card} I$

b) $\theta=\prod_{i\in I}\theta_i$ *for some congruences θ_i on A_i $(i\in I)$*

c) *there exists an injective mapping $\pi: J\to I$ with $A_{\pi(j)}/\theta_{\pi(j)}\cong B_j$ for each $j\in J$.*

Proof. Since $\left(\prod_{i\in I} A_i\right)\Big/\theta\cong\prod_{j\in J} B_j$, there exists a homomorphism h of $A=\prod_{i\in I} A_i$ onto $B=\prod_{j\in J} B_j$. As θ is join-$\mathfrak{M}$-complete, also h has this property. Let $i_0\in I$, $a\in A_{i_0}$. Denote by $\bar a$ the element of A with $\bar a(i_0)=a$, $\bar a(i)=0_i$ for $i\neq i_0$, where 0_i is the least element of A_i. Introduce $\bar A_i=\{\bar a_i;\ a_i\in A_i\}$. Clearly $\bar A_i$ is a chain which is a sublattice of A isomorphic with A_i. Denote by 0_A the element of A with $0_A(i)=0_i$ for all $i\in I$. Clearly, 0_A is the least element of A. The concepts of $\bar B_j$ and 0_B are introduced analogously.

1°. Let $j\in J$ and $b\in\bar B_j$, $b\neq 0_B$. Prove the existence of $i_0\in I$ and $\overline{a(i_0)}\in\bar A_{i_0}$ such that $h(\overline{a(i_0)})=b$. Let $a\in A$ and $h(a)=b$.

Since $\overline{a(i)}\leqslant a$ for each $i\in I$, we have $h(\overline{a(i)})\leqslant h(a)=b$. As $\bar B_j$ is an ideal of B, $h(\overline{a(i)})\in\bar B_j$. Suppose the existence of $i,\ i'\in I,\ i\neq i'$ with $pr_j h(\overline{a(i)})\neq 0_j\neq pr_j h(\overline{a(i')})$. Then $0_A=\overline{a(i)}\wedge\overline{a(i')}$, $0_B=h(0_A) = h(\overline{a(i)})\wedge h(\overline{a(i')})$, hence

$0_j = pr_j h(\overline{a(i)}) \wedge pr_j h(\overline{a(i')})$, which is a contradiction. Let $i_0 \in I$ and $pr_j h(\overline{a(i_0)}) \neq 0_j$. As $h(a) = b \neq 0_B$, such i_0 exists. Since $b = h(a) = \bigvee_{i \in I} h(\overline{a(i)})$ and $pr_j h(\overline{a(i)}) = 0_j$ for $i \neq i_0$, we obtain $b = \overline{pr_j b} = \overline{\bigvee_{i \in I} pr_j h(a(i))} = h(\overline{a(i_0)})$.

2°. Prove that this index $i_0 \in I$ is the same for all non-zero elements from $\bar{B}_j$. Let $b_1, b_2 \in \bar{B}_j$, $b_1 \neq 0_B \neq b_2$. By 1°, there exist $i_1, i_2 \in I$ and $a_{i_1} \in A_{i_1}$, $a_{i_2} \in A_{i_2}$ with $h(\bar{a}_{i_1}) = b_1$, $h(\bar{a}_{i_2}) = b_2$. Clearly $\bar{a}_{i_1} \neq 0_A \neq \bar{a}_{i_2}$. Suppose $i_1 \neq i_2$. As $\bar{B}_j$ is a chain, we have

$$0_B = h(0_A) = h(\bar{a}_{i_1} \wedge \bar{a}_{i_2}) = h(\bar{a}_{i_1}) \wedge h(\bar{a}_{i_2}) = b_1 \wedge b_2 \neq 0_B\,,$$

a contradiction. Thus $i_1 = i_2$. If $b = 0_B$, then $h(0_A) = 0_B$ and $0_A \in \bar{A}_{i_0}$. Summarizing, $\bar{B}_j \subseteq h(\bar{A}_{i_0})$. As j was chosen arbitrarily, we obtain: for each $j \in J$ there exists $i_j \in I$ with $h(\bar{A}_{i_j}) \supseteq \bar{B}_j$; putting $b_1 = b_2$ we obtain the proof of unicity of such i_j.

3°. Suppose $\bar{B}_{j_1} \subseteq h(A_i)$, $\bar{B}_{j_2} \subseteq h(A_i)$ for $i \in I$. As $\bar{A}_i$ is a chain, also $h(\bar{A}_i)$ is a chain; however, $\bar{B}_{j_1} \cup \bar{B}_{j_2}$ is not a chain for $j_1 \neq j_2$, thus $\bar{B}_{j_1} \cup \bar{B}_{j_2} \subseteq h(\bar{A}_i)$ is impossible. Hence $j \mapsto i_j$ is an injective mapping of J into I.

4°. Prove $h(\bar{A}_{i_j}) = \bar{B}_j$ for each $j \in J$. Suppose $c \in h(\bar{A}_{i_j}) - \bar{B}_j$. Then $c(j') \neq 0_{j'}$ for some $j' \neq j$, $j' \in J$. Choose $d \in \bar{A}_{i_j}$ with $h(d) = c$. Since $\overline{c(j')} \in \bar{B}_{j'}$, there exists $d_1 \in \bar{A}_{i_{j'}}$ with $h(d_1) = \overline{c(j')}$. As $j \mapsto i_j$ is injective, we have $i_j \neq i_{j'}$. Hence $d \wedge d_1 = 0_A$ and

$$pr_{j'} h(d \wedge d_1) = pr_{j'}(c \wedge \overline{c(j')}) = c(j') \wedge c(j') = c(j') \neq 0_{j'}\,,$$

a contradiction.

5°. Denote by π the mapping $j \mapsto i_j$ of J into I with $h(\bar{A}_{\pi(j)}) = \bar{B}_j$. Put $\bar{\theta}_i = \theta \cap (\bar{A}_i \times \bar{A}_i)$. Then $\bar{A}_{\pi(j)}/\bar{\theta}_{\pi(j)} \cong \bar{B}_j$. The congruence $\bar{\theta}_i$ on $\bar{A}_i$ induces a congruence θ_i on A_i by the rule:

$$a_i \theta_i b_i \quad \text{if and only if} \quad \bar{a}_i \bar{\theta}_i \bar{b}_i\,.$$

Then, evidently, $A_{\pi(j)}/\theta_{\pi(j)} \cong B_j$.

If $i \in I - \pi(J)$, put $\theta_i = A_i \times A_i$. Let $p, q \in A$ and $p\theta q$. Then $pr_j h(p) = pr_j h(q)$ for each $j \in J$, hence $p(\pi(j))\theta_{\pi(j)} q(\pi(j))$ for each $j \in J$, thus $p(i)\theta_i q(i)$ for all $i \in I$. The converse implication is clear, thus $\theta = \prod_{i \in I} \theta_i$.

Q.E.D.

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Received February 9, 1976

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ПРЯМАЯ РАЗЛОЖИМОСТЬ КОНГРУЭНЦИЙ ДИСТРИБУТИВНЫХ РЕШЕТОК

Иван Хайда

Резюме

Дж. А. Фрейзер и А. Хорн определили условия для прямой разложимости конгруэнций на прямых произведениях двух алгебр. В этой работе даны достаточные условия для прямой разложимости конгруэнций дистрибутавной решетки, которая прямо разлагается в произвольное число цепей.