Ivan Kolář; Iveta Vadovičová

On the structure function of a G-structure

Mathematica Slovaca, Vol. 35 (1985), No. 3, 277--282

Persistent URL: http://dml.cz/dmlcz/129444

Math. Slovaca 35, 1985, No. 3, 277–282

ON THE STRUCTURE FUNCTION OF A G-STRUCTURE

IVAN KOLÁŘ—IVETA VADOVIČOVÁ

The first author deduced in [3] that the structure function of a generalized G-structure can be naturally defined in terms of the difference tensor of a semi-holonomic 2-jet. This approach leads to an original direct construction of the structure function of a classical (i.e. first order) G-structure. Since this problem is a matter of considerable interest to geometers, we now develop a complete version of the first order case including a detailed comparison with the classical constructions. Our approach underlines the fact that the structure function vanishes if and only if there exists the holonomic prolongation of the G-structure in question. (This result was recently derived in another way by A. Trautman, [6].) Then we deduce that every flat G-structure has the holonomic prolongation. The converse assertion is not true in general, which gives a clear interpretation of the fact that the vanishing of the structure function is only a necessary condition for flatness. Our consideration is in the category C^∞.

1. Semi-holonomic 2-jets

We first recall the basic facts from the theory of semi-holonomic 2-jets, [1]. Given two manifolds M, N, we denote by $J^1(M, N)$ the space of all first order jets of M into N and by α: $J^1(M, N)\to M$ or β: $J^1(M, N)\to N$ the source or target projection, respectively. Consider a local map φ of a neighbourhood of a point $x\in M$ into $J^1(M, N)$ satisfying $\alpha\circ\varphi=\mathrm{id}$ and

$$\varphi(x)=j^1_x(\beta\circ\varphi). \tag{1}$$

The 1-jet $A=j^1_x\varphi$ of such a map is said to be a semi-holonomic 2-jet of M into N with source $\alpha A=x$ and target $\beta A=\beta\varphi(x)$. The space of all these jets is denoted by $\bar J^2(M, N)$. The canonical coordinates u^i on $\boldsymbol{R}^m$ and v^p on $\boldsymbol{R}^n$ induce the additional coordinates $v^p_i=\partial v^p/\partial u^i$ on $J^1(\boldsymbol{R}^m, \boldsymbol{R}^n)$. If $v^p=f^p(u^1, \ldots, u^m)=f^p(u)$ and $v^p_i=f^p_i(u)$ is the coordinate expression of φ, then (1) implies $f^p_i(x)=(\partial f^p/\partial u^i)(x)$. Hence $v^p_{ij}=(\partial v^p_i/\partial u^j)$ are the additional coordinates on $\bar J^2(\boldsymbol{R}^m, \boldsymbol{R}^n)$. There is a canonical inclusion $J^2(M, N)\subset\bar J^2(M, N)$ of holonomic 2-jets into semi-holonomic ones defined by $j^2_x f \mapsto j^1_x(j^1_u f)$. In coordinates, a holonomic 2-jet is characterized by the property that its second order coordinates are symmetric in the subscripts.

The composition of semi-holonomic 2-jets is defined as follows. Consider $A \in \bar{J}^2(M, N)$, $A = j^1_x \varphi$ and $B \in \bar{J}^2(N, Q)$, $B = j^1_y \psi$ satisfying $\beta A = \alpha B$. Then the composition of first order jets $\psi(\beta\varphi(u)) \ \varphi(u)$ is a local map of M into $J^1(M, Q)$ of the type required in the definition of a semi-holonomic 2-jet and we set

$$(2) \qquad B \circ A := j^1_x[\psi(\beta\varphi(u)) \circ \varphi(u)] \in \bar{J}^2(M, Q).$$

If A and B are holonomic, one gets the usual composition of holonomic 2-jets. If in the coordinates $A = (y^p, v^p_i, v^p_{ij}, x^i)$ and $B = (z^a, w^a_p, w^a_{pq}, y^p)$, then (2) implies

$$(3) \qquad B \circ A = (z^a, w^a_p v^p_i, w^a_{pq} v^p_i v^q_j + w^a_p v^p_{ij}, x^i).$$

This composition is associative. Hence the set $\bar{L}^2_m$ of all invertible semi-holonomic 2-jets of $\boldsymbol{R}^m$ into $\boldsymbol{R}^m$ with source 0 and target 0 is a Lie group. By (3), its composition law in the coordinates a^i_j, a^i_{jk}, $\det a^i_j \neq 0$ is expressed by

$$(4) \qquad (b^i_j, b^i_{jk}) \circ (a^i_j, a^i_{jk}) = (b^i_k a^k_j, b^i_{lm} a^l_j a^m_k + b^i_l a^l_{jk}).$$

The subset $L^2_m \subset \bar{L}^2_m$ of all holonomic 2-jets is a Lie subgroup.

For every $A \in \bar{J}^2(M, N)$, the first author [2] introduced the difference tensor $\triangle(A) \in T_y N \otimes \wedge^2 T^*_x M$, $x = \alpha A$, $y = \beta A$, see also [4]. If v^p_{ij} are the second order coordinates of A, then the coordinates of $\triangle(A)$ are $v^p_{[ij]}$, where the square bracket denotes antisymmetrization. Hence A is holonomic if and only if $\triangle(A) = 0$.

The space $\bar{H}^2 M$ of all invertible semi-holonomic 2-jets of $\boldsymbol{R}^m$ into M with source 0 is a principal fibre bundle over M with the structure group $\bar{L}^2_m$, the action of $\bar{L}^2_m$ on $\bar{H}^2 M$ being defined by the composition of jets, $m = \dim M$. The coordinates on $\bar{H}^2 \boldsymbol{R}^m$ are u^i, u^i_j, u^i_{jk}, $\det u^i_j \neq 0$. The classical second order frame bundle $H^2 M$ of M (i.e. the subspace $H^2 M \subset \bar{H}^2 M$ of all holonomic 2-jets) is a reduction of $\bar{H}^2 M$ to $L^2_m \subset \bar{L}^2_m$.

Consider further the first jet prolongation $J^1 H^1 M$ of the fibred manifold $H^1 M \to M$ of all first order frames on M. We introduce a map i: $J^1 H^1 M \to \bar{H}^2 M$ as follows. Denote by t_u: $\boldsymbol{R}^m \to \boldsymbol{R}^m$ the translation $x \mapsto x + u$. Having $X \in J^1 H^1 M$, $X = j^1_x s$, $s(x) = j^1_0 \varphi$, φ: $\boldsymbol{R}^m \to M$, we construct the composition $s(\varphi(u)) \ j^1_u(t^{-1}_u) \in J^1(\boldsymbol{R}^m, M)$ and set

$$(5) \qquad i(X) := j^1_0[s(\varphi(u)) \circ j^1_u(t^{-1}_u)] \in \bar{H}^2 M.$$

The coordinates on $H^1 \boldsymbol{R}^m$ being u^i, u^i_j, $\det u^i_j \neq 0$, we introduce the additional coordinates on $J^1 H^1 \boldsymbol{R}^m$ by $\bar{u}^i_{jk} = \partial u^i_j / \partial u^k$. If $X = (u^i, u^i_j, \bar{u}^i_{jk}) \in J^1 H^1 \boldsymbol{R}^m$, then we deduce from (5) that the coordinates of $i(X) \in \bar{H}^2 \boldsymbol{R}^m$ are u^i, u^i_j, and

$$(6) \qquad u^i_{jk} = \bar{u}^i_{jl} u^l_k.$$

As u^i_j is a regular matrix, we have proved

Proposition 1. *i: $J^1H^1M \to \bar{H}^2M$ is a fibred manifold isomorphism over H^1M.*

2. Prolongations of groups

By (4) the restriction of the group composition to the kernel of the jet projection $\beta_1: \bar{L}^2_m \to L^1_m$ is the vector addition, so that we have an exact sequence of groups

(7) $$0 \to \boldsymbol{R}^m \otimes \boldsymbol{R}^{m*} \otimes \boldsymbol{R}^{m*} \to \bar{L}^2_m \xrightarrow{\beta_1} L^1_m \to 0.$$

There is a splitting λ: $L^1_m \to L^2_m \subset \bar{L}^2_m$, $a^i_j \mapsto (a^i_j, 0)$. (Geometrically, every $a = (a^i_j) \in L^1_m$ determines a linear transformation $\text{lin}\, a$: $\bar{u}^i = a^i_j u^j$ and we set $\lambda(a) = j^2_0(\text{lin}\, a)$.) Hence $\bar{L}^2_m$ can be expressed as a semi-direct product of L^1_m and $\text{Ker}\,\beta_1$. For every $A \in \bar{L}^2_m$, we have $a = \beta_1 A \in L^1_m$ and $A_1 := \lambda(a)^{-1} \circ A \in \text{Ker}\,\beta_1$. We shall write $A = (a, A_1)$, which determines a decomposition $\bar{L}^2_m = L^1_m \times (\boldsymbol{R}^m \otimes \boldsymbol{R}^{m*} \otimes \boldsymbol{R}^{m*})$. If $a = (a^i_j)$ and $A_1 = (\delta^i_j, a^i_{jk})$, then

(8) $$A = (a^i_j, 0) \circ (\delta^i_j, a^i_{jk}) = (a^i_j, a^i_l a^l_{jk}).$$

For any $a = (a^i_j) \in L^1_m$ and $A = (a^i_{jk}) \in \boldsymbol{R}^m \otimes \boldsymbol{R}^{m*} \otimes \boldsymbol{R}^{m*}$, we set

(9) $$\text{ad}(a)(A) = (a^i_l a^l_{mn} \tilde{a}^m_j \tilde{a}^n_k),$$

where $\tilde{a}^i_j$ means the inverse matrix to a^i_j.

Lemma 1. *In the decomposition (8) the multiplication in $\bar{L}^2_m$ is expressed by*

(10) $$(b, B_1) \circ (a, A_1) = (ba, \text{ad}(a^{-1})B_1 + A_1).$$

Proof. According to (4), $(b^i_j, b^i_l b^l_{jk}) \circ (a^i_j, a^i_l a^l_{jk}) = (b^i_k a^k_j, b^i_l b^l_{mn} a^m_j a^n_k + b^i_m a^m_l a^l_{jk}) = (b^i_k a^k_j, b^i_p a^p_l (\tilde{a}^l_q b^q_{mn} a^m_j a^n_k + a^l_{jk}))$,

QED.

Let G be any Lie group, whose multiplication will be denoted by a dot. Then the space $T^1_m G$ of all 1-jets of $\boldsymbol{R}^m$ into G with source 0 is also a Lie group with the composition law

(11) $$(j^1_0 \varphi(u)) \cdot (j^1_0 \psi(u)) := j^1_0(\varphi(u) \cdot \psi(u)).$$

The target projection β: $T^1_m G \to G$ is a group homomorphism. Let $\mathbf{g}$ be the Lie algebra of G.

Lemma 2. *We have an exact sequence of groups*

(12) $$0 \to \mathbf{g} \otimes \boldsymbol{R}^{m*} \to T^1_m G \xrightarrow{\beta} G \to 0.$$

Proof. The kernel of β is the set of all 1-jets of $\boldsymbol{R}^m$ into G with source 0 and the target at the unit of G. As a set, this is equal to $\text{Hom}(\boldsymbol{R}^m, \mathbf{g}) = \mathbf{g} \otimes \boldsymbol{R}^{m*}$. Using the basic facts on the Lie groups, one finds easily that the group composition in $\mathbf{g}\otimes \boldsymbol{R}^{m*}$ coincides with the vector addition, QED.

In particular, if $G=L^1_m$ and the additional coordinates on $T^1_m L^1_m$ are $a^i_{jk}=(\partial a^i_j/\partial u^k)(0)$, then the multiplication in $T^1_m L^1_m$ is given by

$$(b^i_j,\ b^i_{jk})\cdot(a^i_j,\ a^i_{jk})=(b^i_k a^k_j,\ b^i_{lk}a^l_j+b^i_l a^l_{jk}). \tag{13}$$

On the other hand, we introduce a map ν: $T^1_m L^1_m \to \bar{L}^2_m$ as follows. Having $A\in T^1_m L^1_m$, $A=j^1_0\gamma(u)$, $\gamma(0)=j^1_0\psi(u)$, we set

$$\nu(A):=j^1_0[j^1_0(t_{\psi(u)})\circ\gamma(u)\circ j^1_u(t^{-1}_u)]. \tag{14}$$

If A has some coordinates a^i_j, a^i_{jk} in $T^1_m L^1_m$, then $\nu(A)$ has the same coordinates in $\bar{L}^2_m$. Comparing (4) and (13), we find that ν is not a group homomorphism. Nevertheless, (14) and (2) imply

Lemma 3. *If G is a subgroup in L^1_m, then $\nu(T^1_m G)$ is a subgroup in $\bar{L}^2_m$.*

The latter group will be denoted by $\bar{G}$ and called the semi-holonomic prolongation of G.

Lemma 4. *We have an exact sequence of groups*

$$0\to\mathbf{g}\otimes\boldsymbol{R}^{m*}\to\bar{G}\xrightarrow{\beta_1}G\to 0 \tag{15}$$

Proof. By (4) and (13) the composition laws in $T^1_m L^1_m$ and $\bar{L}^2_m$ coincide on $\operatorname{Ker}\beta$ and $\operatorname{Ker}\beta_1$, so that (15) is a consequence of (12).

Since the coordinates in $T^1_m L^1_m$ coincide with those in $\bar{L}^2_m$, there holds $\lambda(G)\subset\bar{G}$. Then Lemma 1 gives

Proposition 2. *We have $\bar{G}=G\times(\mathbf{g}\otimes\boldsymbol{R}^{m*})$ with composition law (10).*

The intersection $G':=\bar{G}\cap L^2_m$ will be called the (holonomic) prolongation of G. Obviously, $(a^i_j, 0)\in G'$ for each $a^i_j\in G$, so that β_1: $G'\to G$ is surjective. Let $p(\mathbf{g})=(\mathbf{g}\otimes\boldsymbol{R}^{m*})\cap(\boldsymbol{R}^m\otimes\boldsymbol{R}^{m*}\odot\boldsymbol{R}^{m*})$ be the Spencer prolongation of $\mathbf{g}$. By Proposition 2 we obtain immediately

Proposition 3. *We have $G'=G\times p(\mathbf{g})$ with composition law (10).*

3. The structure function

Consider a G-structure $P\subset H^1M$. Hence $J^1P\subset J^1H^1M$. If $X\in J^1P$ and $A\in T^1_m G$ are as in (5) and (14), then

$$i(X)\ \nu(A)=j^1_0[s(\varphi(\psi(u)))\ \ \gamma(u)\ \ j^1_u(t^{-1}_u)]\in i(J^1P).$$

Conversely, for any other $\bar{X}\in J^1P$, $\bar{X}=j^1_x\bar{s}$, $\bar{s}(x)=j^1_0\bar{\varphi}$, there exists exactly one

$A \in T^1_m G$ satisfying $i(X)\ v(A) = i(\bar{X})$. Indeed, the equation $s \circ \mu = \bar{s}$ determines a local map of M into G and $A = j^1_0 \mu(\bar{\varphi}(u))$. Thus, we have proved

Proposition 4. *$i(J^1P)$ is a reduction of $\bar{H}^2M$ to $\bar{G} \subset \bar{L}^2_m$.*

For every $B \in \bar{H}^2M$ we can construct its difference tensor $\triangle(B) \in T_xM \otimes \wedge^2 \boldsymbol{R}^{m*}$, $x = \beta B$, further, $b = \beta_1 b \in H^1M$ can be interpreted as a linear map $b\colon \boldsymbol{R}^m \to T_xM$. Then $\tilde{\triangle}(B) := b^{-1} \triangle(B) \in \boldsymbol{R}^m \otimes \wedge^2 \boldsymbol{R}^{m*}$. If in coordinates $B = (x^i, b^i_j, b^i_{jk})$, then $\tilde{\triangle}(B) = \tilde{b}^i_l b^l_{[jk]}$.

Definition 1. *The structure function $\tau(b)$ of a G-structure P at $b \in P$ is the set $\tilde{\triangle}(i(X))$ for all $X \in J^1P$, $\beta X = b$.*

Since $\mathbf{g} \subset \boldsymbol{R}^m \otimes \boldsymbol{R}^{m*}$, there is $\mathbf{g} \otimes \boldsymbol{R}^{m*} \subset \boldsymbol{R}^m \otimes \boldsymbol{R}^{m*} \otimes \boldsymbol{R}^{m*}$ and $\mathfrak{A}(\mathbf{g} \otimes \boldsymbol{R}^{m*}) \subset \boldsymbol{R}^m \otimes \wedge^2 \boldsymbol{R}^{m*}$, where $\mathfrak{A}$ means the antisymetrization with respect to $\boldsymbol{R}^{m*} \otimes \boldsymbol{R}^{m*}$. The space $H^{0,2}(\mathbf{g}) = \boldsymbol{R}^m \otimes \wedge^2 \boldsymbol{R}^{m*} / \mathfrak{A}(\mathbf{g} \otimes \boldsymbol{R}^{m*})$ is the Spencer cohomology class of bidegree (0,2) of $\mathbf{g}$.

Proposition 5. *$\tau(b)$ belongs to $H^{0,2}(\mathbf{g})$ for every $b \in P$.*

Proof. By Proposition 4 any other $i(Y) \in i(J^1P)$, $\beta Y = b$, is of the form $i(X) \circ A$, $A \in \mathbf{g} \otimes \boldsymbol{R}^{m*}$. If $A = (\delta^i_j, a^i_{jk})$ and $i(X) = (x^i, u^i_j, u^i_{jk})$, then $i(X) \circ A = (x^i, u^i_j, u^i_{jk} + u^i_l a^l_{jk})$ and $\tilde{\triangle}(i(X) \circ A) = \tilde{u}^i_l u^l_{[jk]} + a^i_{[jk]}$, QED.

In coordinates, one verifies easily that our structure function coincides with the classical one, see, e.g., [5]. We remark that our method leads to a simple derivation of the classical transformation law of the structure function. The space $\mathfrak{A}(\mathbf{g} \otimes \boldsymbol{R}^{m*})$ being invariant with respect to the action (9) of G, [5], we have an induced action ϱ of G on the factor space $H^{0,2}(\mathbf{g})$.

Proposition 6. *There holds $\varrho(g^{-1})\tau(b) = \tau(bg)$ for all $g \in G$ and $b \in P$.*

Proof. By (6), if (u^i, u^i_j, u^i_{jk}) are coordinates of $i(X)$, then the coordinates of X are $(u^i, u^i_j, u^i_{jl} \tilde{u}^l_k)$. Take an element $a^i_j \in G$ and construct the image X' of X by the right translation determined by a^i_j. Then the coordinates of X' are $(u^i, u^i_k a^k_j, u^i_{ml} \tilde{u}^l_k a^m_j)$ and the second order coordinates of $i(X')$ are $u^i_{lm} a^l_j a^m_k$. Hence $\tilde{\triangle}(i(X')) = \tilde{a}^i_p \tilde{u}^p_l u^l_{[mn]} a^m_j a^n_k$, which proves our assertion.

4. Prolongability and flatness

Definition 2. *A G-structure P is called prolongable if the intersection of $i(J^1P)$ and H^2M is non-empty over every $b \in P$.*

If P is prolongable, then the intersection $P' := i(J^1P) \cap H^2M$ is said to be the (holonomic) prolongation of P.

Proposition 7. *If P is prolongable, then P' is a reduction of H^2M to $G' \subset L^2_m$.*

Proof. This follows from Proposition 4 and from the fact that the composition of two holonomic 2-jets is holonomic.

Proposition 8. *A G-structure P is prolongable if and only if its structure function vanishes.*

Proof. By definition, P is prolongable if and only if for every $b \in P$ there exists an $X \in J^1 P$, $\beta X = b$, such that $i(X) \in H^2 M$. This is equivalent to $\triangle(i(X)) = 0$, which is the same as $\tau(b) = 0 \in H^{0,2}(\mathbf{g})$.

We recall that a G-structure on M is said to be flat if it is locally isomorphic to the standard flat G-structure $\boldsymbol{R}^m \times G \subset H^1 \boldsymbol{R}^m$. The well-known fact that the structure function of a flat G-structure vanishes can be rededuced as follows. If we take a constant section s: $u^i \mapsto (u^i, a^i_j)$ of $\boldsymbol{R}^m \times G$, we have $j^1_x s = (x^i, a^i_j, 0)$ and $i(j^1_x s) \in H^2 M$. This implies

Proposition 9. *Every flat G-structure is prolongable.*

The converse assertion is not true in general. This clarifies in a conceptual way the relation between the vanishing of the structure function and the flatness of a G-structure.

REFERENCES

[1] EHRESMANN, C.: Extension du calcul des jets aux jets non-holonomes, C.R. Acad. Sci. Paris, 239, 1954, 1762—1764.

[2] KOLÁŘ, I.: Higher order torsions of spaces with Cartan connection, Cahiers Topologie Géom. Différentielle, 12, 1971, 137—146.

[3] KOLÁŘ, I.: Generalized G-structures and G-structures of higher order, Boll. Un. Mat. Ital. (4) 12, Suppl. fasc. 3, 1975, 245—256.

[4] PRADINES, J.: Fibrés vectoriels doubles symétriques et jets holonomes d'ordre 2, C.R. Acad. Sci. Paris, Sér. A, 278, 1974, 1557—1560.

[5] STERNBERG, S.: Lectures on differential geometry, Prentice Hall, Inc., Englewood Cliffs, N.J., 1964.

[6] TRAUTMAN, A.: A classification of space-time structures, Rep. Mathematical Phys., 10, 1976, 297—310.

Received March 31, 1983

Matematický ústav ČSAV
Mendelovo nám. 1
603 00 Brno

Katedra matematiky
Vysoká škola dopravy a spojov
Marxa-Engelsa 25
010 88 Žilina

СТРУКТУРНАЯ ФУНКЦИЯ G-СТРУКТУРЫ

Ivan Kolář—Iveta Vadovičová

Резюме

Работа посвящена построению структурной функции G-структуры с помощью разностного тензора полуголомонного 2-джета и исследованию некоторых ей свойств. Показано, что структурная функция обращается в нуль тогда и только тогда, когда существует голономное продолжение изучаемой G-структуры.