Math. Slovaca, **46** (1996), No. 2–3, 279–284

Mathematica Slovaca

Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

ALMOST QUASICONTINUITY OF MULTIVALUED MAPS ON PRODUCT SPACES

JANINA EWERT

(*Communicated by Ľubica Holá*)

ABSTRACT. We define the almost quasicontinuity of multivalued maps and some stronger form of it. Further, we formulate sufficient conditions under which a multivalued map of two variables is upper (lower) almost quasicontinuous.

Let X, Z be topological spaces. For a set $A \subset X$ by $\operatorname{Cl} A$ and $\operatorname{Int} A$ are denoted the closure and the interior of A respectively. A function $f\colon X \to Z$ is said to be *almost quasicontinuous* if $f^{-1}(V) \subset \operatorname{Cl}\bigl(\operatorname{Int}\bigl(\operatorname{Cl} f^{-1}(V)\bigr)\bigr)$ for each open set $V \subset Z$ ([3], [11]). As it was shown in [7], separately almost quasicontinuous functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ need not be almost quasicontinuous. In the presented note, we define the upper and lower almost quasicontinuity of multivalued maps, and we formulate some sufficient conditions for almost quasicontinuity of maps on product spaces.

A set $A \subset X$ is said to be:

preopen, if $A \subset \operatorname{Int}(\operatorname{Cl} A)$;
semi-preopen, if $A \subset \operatorname{Cl}\bigl(\operatorname{Int}(\operatorname{Cl} A)\bigr)$.

These classes of sets are closed under any unions. The intersection of an open set and a preopen (semi-preopen) one is preopen (semi-preopen). For each $A \subset X$ the set $A \cap \operatorname{Int}(\operatorname{Cl} A)$ is preopen and $A \cap \operatorname{Cl}\bigl(\operatorname{Int}(\operatorname{Cl} A)\bigr)$ is semi-preopen ([1]). Following [9] we will denote

$$D(A) = \{y \in X :\ U \cap A \text{ is of the second category for each neighbourhood } U \text{ of } y\}.$$

AMS Subject Classification (1991): 54C60, 54C08, 26E25.
Key words: second category set, multivalued map, almost quasicontinuity.

Then any preopen set and each set A satisfying $A \subset D(A)$ is semi-preopen, but these three classes of sets can be different. For instance, in the space $\mathbb{R}$ of real numbers with the usual topology, let $\mathbb{Q}$ be the set of all rationals, $A = [0,1)$ and $B = \mathbb{Q} \cap [0,1]$. Thus $\mathbb{Q}$ is preopen, $D(\mathbb{Q}) = \emptyset$, $A \subset D(A)$, but A is not preopen. The set B is semi-preopen and not preopen, $B \not\subset D(B)$.

Now let $F\colon X \to Z$ be a multivalued map. For a set $W \subset Z$ we write $F^+(W) = \{x \in X : F(x) \subset W\}$, $F^-(W) = \{x \in X : F(x) \cap W \neq \emptyset\}$. A multivalued map $F\colon X \to Z$ is called *upper* (*lower*) *almost quasicontinuous at a point* $x \in X$ if for each open set $W \subset Z$ with $F(x) \subset W$ (resp. $F(x) \cap W \neq \emptyset$) it holds $x \in \mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}\,F^+(W))\big)$ (resp. $x \in \mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}\,F^-(W))\big)$). Equivalently, F is upper (lower) almost quasicontinuous at x if an only if there exists a semi-preopen set $A \subset X$ with $x \in A \subset F^+(W)$ (resp. $x \in A \subset F^-(W)$). A map F is called *upper* (*lower*) *almost quasicontinuous* if it has this property at each point.

The lower almost quasicontinuous maps under the name "lower demicontinuous" has been used in connection with the existence of densely defined continuous selections in [4], [5], [8].

By the standard arguments, we can prove the following (for functions it is [6; Theorem 1]):

PROPOSITION. *Let X, Z be topological spaces. For a multivalued map F: $X \to Z$ the following conditions are equivalent:*

(a) *F is upper almost quasicontinuous;*
(b) *$F^+(V) \subset \mathrm{Cl}\big(\mathrm{Int}(\mathrm{Cl}\,F^+(V))\big)$ for each open set $V \subset Z$;*
(c) *$\mathrm{Int}\big(\mathrm{Cl}(\mathrm{Int}\,F^-(A))\big) \subset F^-(A)$ for each closed set $A \subset Z$;*
(d) *the set $\mathrm{Cl}\,F^+(V)$ is regularly closed for each open set $V \subset Z$.*

Changing F^+ by F^- and conversely we obtain suitable equivalent conditions for lower almost quasicontinuity.

THEOREM 1. *A multivalued map $F\colon X \to Z$ is upper (lower) almost quasicontinuous at a point $x_0 \in X$ if and only if for each open set $W \subset Z$ with $F(x_0) \subset W$ (resp. $F(x_0) \cap W \neq \emptyset$) and for each neighbourhood U of x_0 there exists a nonempty preopen set $M \subset U \cap F^+(W)$ (resp. $M \subset U \cap F^-(W)$).*

Proof. Assume that F is upper almost quasicontinuous at $x_0 \in X$, $W \subset Z$ is an open set with $F(x_0) \subset W$, and U is a neighbourhood of x_0. There exists a semi-preopen set A such that $x_0 \in A \subset F^+(W)$. Then $U \cap \mathrm{Int}(\mathrm{Cl}\,A)$ is a nonempty set, so we have $\emptyset \neq U \cap \mathrm{Int}(\mathrm{Cl}\,A) \subset \mathrm{Cl}(U \cap \mathrm{Cl}\,A) = \mathrm{Cl}(U \cap A)$. Thus $(U \cap A) \cap \big(U \cap \mathrm{Int}(\mathrm{Cl}\,A)\big) \neq \emptyset$; therefore $M = U \cap A \cap \mathrm{Int}(\mathrm{Cl}\,A)$ is a nonempty preopen set and $M \subset U \cap A \subset U \cap F^+(W)$.

Conversely, let $W \subset Z$ be an open set with $F(x_0) \subset W$. Then for each neighbourhood U of x_0 there exists a nonempty preopen set $M_U \subset U \cap F^+(W)$. Let us put

$$M = \bigcup \{M_U : \ U \text{ is a neighbourhood of } x_0\}.$$

The set M is nonempty preopen and $M \subset F^+(W)$. Furthermore, $x_0 \in \operatorname{Cl} M = \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} M)) \subset \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} F^+(W)))$, which finishes the proof. For the lower almost quasicontinuity the proof is analogous. □

In a topological space X, a family P of nonempty open subsets is called *pseudo-base* if each nonempty open set in X contains some members of P. A topological space X is called *locally possessing a countable pseudo-base* if for each point $x \in X$ there exists a neighbourhood V of x which has a countable pseudo-base.

A multivalued map $F \colon X \to Z$ is said to be *upper* (*lower*) *D-continuous* if for each open set $W \subset Z$ it holds $F^+(W) \subset D(F^+(W))$ (resp. $F^-(W) \subset D(F^-(W))$). Evidently, the upper (lower) D-continuity implies the upper (lower) almost quasicontinuity.

Let $F \colon X \times Y \to Z$ be a multivalued map. For every $x \in X$, $y \in Y$, by F_x and F^y, we will denote multivalued maps $F_x \colon Y \to Z$ and $F^y \colon X \to Z$ assuming $F_x(y) = F(x,y) = F^y(x)$.

THEOREM 2. *Let X, Y, Z be topological spaces with Y locally possessing a countable pseudo-base and let $F \colon X \times Y \to Z$ be a multivalued map. If for all $x \in X$, $y \in Y$ the maps F_x are upper almost quasicontinuous and F^y are upper D-continuous, then F is upper almost quasicontinuous.*

Proof. Let us fix a point $(x_0, y_0) \in X \times Y$, a neighbourhood $U \times V$ of (x_0, y_0), and an open set $W \subset Z$ with $F(x_0, y_0) \subset W$. Without loss of generality, we can assume that V has a countable pseudo-base $\{V_n : \ n \geq 1\}$. Since F^{y_0} is upper D-continuous at x_0, the set $A = U \cap (F^{y_0})^+(W)$ is of the second category and $F(x, y_0) \subset W$ for each $x \in A$. Each of F_x is upper almost quasicontinuous at y_0, so – following Theorem 1 – for each $x \in A$ there exists a preopen set M_x such that $\emptyset \neq M_x \subset V \cap (F_x)^+(W)$. Let us put

$$A_n = \{x \in A : \ V_n \subset \operatorname{Int}(\operatorname{Cl} M_x)\}.$$

Then $A = \bigcup_{n=1}^{\infty} A_n$, and – since A is of the second category – for some $n \geq 1$ it holds $\operatorname{Int}(\operatorname{Cl} A_n) \neq \emptyset$. Thus $A_n \cap \operatorname{Int}(\operatorname{Cl} A_n)$ is a nonempty preopen set contained in A, therefore the set $(A_n \cap \operatorname{Int}(\operatorname{Cl} A_n)) \times V_n$ is nonempty preopen contained in $U \times V$. Now we denote

$$M = ((A_n \cap \operatorname{Int}(\operatorname{Cl} A_n)) \times V_n) \cap F^+(W).$$

Let $(x,y) \in \big(A_n \cap \operatorname{Int}(\operatorname{Cl} A_n)\big) \times V_n$, and let $U' \times V'$ be a neighbourhood of this point with $V' \subset V_n$. The condition $x \in A_n$ implies

$$M_x \subset V \cap (F_x)^+(W) \quad \text{and} \quad V_n \subset \operatorname{Int}(\operatorname{Cl} M_x)\,.$$

Because M_x is preopen, we obtain

$$V' \subset V_n \subset \operatorname{Int}(\operatorname{Cl} M_x) \subset \operatorname{Int}\big(\operatorname{Cl}\big((F_x)^+(W)\big)\big)\,.$$

It means that $F(x,y') \subset W$ for some $y' \in V'$, i.e., $(x,y') \in M \cap (U' \times V')$. Thus we have shown

$$M \subset \big(A_n \cap \operatorname{Int}(\operatorname{Cl}(A_n)\big) \times V_n \subset \operatorname{Cl} M\,.$$

Since the set $\big(A_n \cap \operatorname{Int}(\operatorname{Cl} A_n)\big) \times V_n$ is preopen, it follows from the last inclusions that M is preopen. Moreover, $M \subset (U \times V) \cap F^+(W)$, so, according to Theorem 1, the map F is upper almost quasicontinuous. □

Remark 1. Looking carefully at the proof of Theorem 2, it can be easily seen that it suffices to assume F_x upper almost quasicontinuous for all x belonging to a residual subset of a Baire space X.

Using analogous arguments as above we can prove:

THEOREM 3. *Let X, Y, Z be topological spaces with Y locally possessing a countable pseudo-base, and let $F\colon X \times Y \to Z$ be a multivalued map. If all maps F_x are lower almost quasicontinuous and all F^y are lower D-continuous, then F is lower almost quasicontinuous.*

Any function f can be considered as a multivalued map F defined by $F(x) = \{f(x)\}$. Then the upper (lower) almost quasicontinuity of such multivalued map coincides with the almost quasicontinuity of f. Hence Theorem 2 and Remark 1 give Theorem 3 in [2]. Furthermore, Theorem 2 (Theorem 3) implies:

COROLLARY 1. (See [7; Theorem 1].) *Assume that X, Y are topological spaces with Y locally second countable, Z is metric one and $f\colon X \times Y \to Z$ is a function. If all functions f_x are almost quasicontinuous, and all f^y are D-continuous, then f is almost quasicontinuous.*

For any set $M \subset X \times Y$ we will write $M_x = \{y \in Y :\ (x,y) \in M\}$.

Now, we remind the well-known theorem:

KURATOWSKI-ULAM THEOREM. ([10], [9; p. 247]) *Let Y be a second countable space. If $M \subset X \times Y$ is of the first category, then there is a first category set $P \subset X$ such that for each $x \in X \setminus P$ the set M_x is of the first category.*

Remark 1. Looking at the proof of this theorem we observe that the assumption on Y can be weakened by taking a countable pseudo-base. In the sequel, we will apply this theorem in such more general form.

Remark 2. Kuratowski-Ulam Theorem need not be true if Y is locally second countable. For instance, let τ_1, τ_2 be the usual and the discrete topologies on $\mathbb{R}$ respectively, and let $M = \{(x,y) : x = y\}$. Then M is nowhere dense in $(\mathbb{R}^2, \tau_1 \times \tau_2)$, but for each $x \in \mathbb{R}$ the set M_x is of the second category (comp. [12]). The space $(\mathbb{R}, \tau_2)$ is locally second countable, but it has not a countable base.

THEOREM 4. *Let X, Y, Z be topological spaces with Y locally possessing a countable pseudo-base, and let $F\colon X \times Y \to Z$ be a multivalued map. If all maps F_x, F^y for $x \in X$, $y \in Y$ are upper (lower) D-continuous, then F is upper (lower) D-continuous.*

Proof. Let $W \subset Z$ be an open set, and let $(x_0, y_0) \in F^+(W)$. We take neighbourhoods U, V of points x_0, y_0; without loss of generality, we can assume that V has a countable pseudo-base. Then the set $U \cap (F^{y_0})^+(W)$ is of the second category in X, and $F(x, y_0) \subset W$ for each $x \in U \cap (F^{y_0})^+(W)$. From the assumption on F_x, for each $x \in U \cap (F^{y_0})^+(W)$ the set $V \cap (F_x)^+(W)$ is of the second category in Y, so also in the subspace V. Now we put

$$M = \bigcup \Big\{ \{x\} \times \big(V \cap (F_x)^+(W)\big) : \ x \in U \cap (F^{y_0})^+(W) \Big\}.$$

We are going to show that M is of the second category in $X \times V$. Suppose not. Following Kuratowski-Ulam Theorem (comp. Remark 2), there is a first category set $P \subset X$ such that for each $x \in X \setminus P$ the set M_x is of the first category in V. Since $U \cap (F^{y_0})^+(W) \setminus P \neq \emptyset$, for some $x \in U \cap (F^{y_0})^+(W)$ the set $M_x = V \cap (F_x)^+(W)$ is of the first category. This contradiction means that M is of the second category in $X \times V$; consequently, also in $X \times Y$. For any point $(x,y) \in M$ we have $y \in (F_x)^+(W)$, thus $M \subset (U \times V) \cap F^+(W)$. This implies that $(U \times V) \cap F^+(W)$ is of the second category, which finishes the proof. □

For functions as a particular case of the last theorem we obtain Theorem 2 in [7]. Let us observe that, according to Remark 3, Theorem 4 can be true even for such spaces $X \times Y$ in which Kuratowski-Ulam Theorem does not hold.

If A is a subset of $\mathbb{R}$ (resp. $\mathbb{R}^2$), let $D_0(A)$ denote the set of all points $p \in \mathbb{R}$ (resp. $p \in \mathbb{R}^2$) such that for each neighbourhood U of p the Lebesgue measure of the set $U \cap A$ is positive. Then, by the same way as earlier, the D_0-continuity can be defined. Furthermore, in virtue of Fubini Theorem [12], for multivalued maps $F\colon \mathbb{R}^2 \to \mathbb{R}$ the result on D_0-continuity analogous to that in Theorem 4 can be formulated.

REFERENCES

[1] ANDRIJEVIĆ, D.: *Semi-preopen sets*, Mat. Vesnik **38** (1986), 24–32.

[2] BORSÍK, J.: *On almost quasicontinuous functions*, Math. Bohem. **118** (1993), 241–248.

[3] BORSÍK, J.—DOBOŠ, J.: *On decomposition of quasicontinuity*, Real Anal. Exchange **16** (1990-1991), 292–305.

[4] ČOBAN, M. M.—KENDEROV, P. S.—REVALSKI, J. P.: *Continuous selections of multivalued mappings*, C. R. Acad. Bulgare Sci. **44** (1991), 9–12.

[5] ČOBAN, M. M.—KENDEROV, P. S.—REVALSKI, J. P.: *Densely defined selections of multivalued mappings*, Trans. Amer. Math. Soc. **344** (1994), 533–552.

[6] EWERT, J.: *On almost quasicontinuity of functions*, Tatra Mt. Math. Publ. **2** (1993), 81–91.

[7] GRANDE, Z.: *A note about almost continuity*, Math. Slovaca **42** (1992), 265–268.

[8] KENDEROV, P. S.—REVALSKI, J. P.: *Residually defined selections for set-valued mappings*, Seminare d'Initation a l'Analyse (Seminare Choquet) **17** (1991), 1-7.

[9] KURATOWSKI, K.: *Topology. Vol.1*, Academic Press, Warszawa, 1966.

[10] KURATOWSKI, K.—ULAM, S.: *Quelques propriétés topologiques du produit combinatoire*, Fund. Math. **19** (1932), 247-251.

[11] NEUBRUNNOVÁ, A.—ŠALÁT, T.: *On almost quasicontinuity*, Math. Bohem. **117** (1992), 197–205.

[12] OXTOBY, J. C.: *Measure and Category*, Springer-Verlag, New York-Heidelberg-Berlin, 1971.

Received March 14, 1994
Revised November 6, 1995

Department of Mathematics
Pedagogical University
Arciszewskiego 22b
PL-76-200 Słupsk
POLAND