Mathematica Slovaca

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Mathematica Slovaca, Vol. 39 (1989), No. 3, 289--295

Persistent URL: http://dml.cz/dmlcz/129691

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THE nTH ORDER DIFFERENTIAL EQUATION WITH DELAYED ARGUMENT

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In the paper an investigation of the nth order nonlinear differential equation with delayed argument of a form

$$L_n y(t) + H(t, y(g(t))) = b(t) \tag{1}$$

is made, where $L_n y$ is a differential operator of a form

$$L_n y(t) = a_n(t)(a_{n-1}(t)(\dots(a_1(t)(a_0(t)y(t))')'\dots)')',$$

the functions $a_0(t), a_1(t), \dots, a_n(t), b(t), g(t)$ are continuous on $[t_0, \infty)$ and $H(t, y)$ is continuous on $[t_0, \infty) \times R$. Further assume that $g(t) \leq t$, $g(t) \to \infty$ for $t \to \infty$ and that $a_i(t) > 0$ on $[t_0, \infty)$ for $i = 0, 1, \dots, n$.

We shall use the following notation:

$$L_0 y(t) = a_0(t) y(t), \ L_i y(t) = a_i(t)(L_{i-1} y(t))', \tag{2}$$

$$I_0(t, t_0) = 1, \ I_i(t, t_0, a_1, \dots, a_i) = \int_{t_0}^{t} \frac{1}{a_1(s)} I_{i-1}(s, t_0, a_2, \dots, a_i) \mathrm{d}s, \tag{3}$$

$$J_i(t, t_0) = \frac{1}{a_0(t)} I_i(t, t_0, a_1, \dots, a_i), \tag{4}$$

$$K_i(t, t_0) = \frac{1}{a_n(t)} I_i(t, t_0, a_{n-1}, \dots, a_{n-i}), \tag{5}$$

for $i = 1, 2, \dots, n$.

In paper [1] some asymptotic properties of solutions of the equation (1) were studied whereby the function $H(t, y)$ satisfied the assumption:

$$|H(t, y)| \leq f(t, |y|), \tag{6}$$

where $f(t, r)$ is a continuous function on $[t_0, \infty) \times R$, nondecreasing in r and such that $\frac{f(t, r)}{r}$ is nonincreasing in r, $r > 0$.

We shall consider the solutions of the equation (1) that exist on $[t_0, \infty)$ and satisfy condition $\sup\{|y(s)|, s \geq t\} > 0$ for every $t \geq t_0$. Let further

$$M = \{y(t);\ y(t) \text{ is an oscillatory solution of (1) such that } \lim_{t \to \infty} y(t) = 0\}.$$

Theorem 1. *Let* (6) *be valid and furthermore let*

$$\int_{t_0}^{\infty} \frac{|b(t)|}{a_n(t)} \mathrm{d}t < \infty \tag{7}$$

and

$$\int_{t_0}^{\infty} \frac{f(t, J_{n-1}(g(t), t_0))}{a_n(t)} \mathrm{d}t < \infty . \tag{8}$$

Then every solution of (1) *has a property*

$$y(t) = O(J_{n-1}(t, t_0)) \qquad \text{for } t \to \infty .$$

Proof. See the proof of theorem 1.1 in paper [1].

Theorem 2. *Let the conditions of theorem* 1 *be satisfied and let there exist*

$$\lim_{t \to \infty} \frac{\left| \int_t^{\infty} \frac{b(s)}{a_n(s)} \mathrm{d}s \right|}{\int_t^{\infty} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \mathrm{d}s} = \infty . \tag{9}$$

The every solution of (1) *is nonoscillatory.*

Proof. Let $y(t)$ be an oscillatory solution of (1). Then the functions $L_i y(t)$ are also oscillatory for $i = 0, 1, \ldots, n$ and then there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$ for $n \to \infty$ of zero points of the function $L_{n-1} y(t)$. From (1) we have

$$L_{n-1} y(t) = \int_{t_n}^{t} \frac{b(s)}{a_n(s)} \mathrm{d}s - \int_{t_n}^{t} \frac{H(s, y(g(s)))}{a_n(s)} \mathrm{d}s \tag{10}$$

for every $t \geq t_n$. Since the conditions of theorem 1 are satisfied there exists $c > 1$ such that

$$|y(g(t))| \leq c J_{n-1}(g(t), t_0) .$$

From the properties of (6) we have

$$f(t, |y(g(t))|) \leq f(t, cJ_{n-1}(g(t), t_0)) \leq cf(t, J_{n-1}(g(t), t_0)) \tag{11}$$

From the relation (10) by means of (6) and (11) we obtain

$$|L_{n-1}y(t)| = \left|\int_{t_n}^{t} \frac{b(s)}{a_n(s)}\,ds\right| + c\int_{t_n}^{t} \frac{f(s, J_{n-1}(g(s), t_0)}{a_n(s)}\,ds,$$

wherefrom with regard to (7), (8) and the fact that $t_n \to \infty$ it follows that there exists $\lim_{t\to\infty} L_{n-1}y(t) = 0$. From the relation (10) we have that

$$\int_{t_n}^{\infty} \frac{b(s)}{a_n(s)}\,ds = \int_{t_n}^{\infty} \frac{H(s, y(g(s)))}{a_n(s)}\,ds, \tag{12}$$

wherefrom

$$\left|\int_{t_n}^{\infty} \frac{b(s)}{a_n(s)}\,ds\right| \le c\int_{t_n}^{\infty} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)}\,ds,$$

where c is a constant $1 < c < \infty$.

Hence $\dfrac{\left|\int_{t_n}^{\infty} \frac{b(s)}{a_n(s)}\,ds\right|}{\int_{t_n}^{\infty} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)}\,ds} \le c$, which contradicts assumption (9). This completes the proof of the theorem.

Example 1. We shall consider

$$(t^3y''(t))' + \frac{60}{(t^2+1)^{\frac{1}{3}}t^{\frac{8}{3}}}\,y^{\frac{1}{3}}(t) = -6t^{-2}, \qquad t > 0. \tag{13}$$

The conditions of theorem 2 are satisfied and thus every solution of (13) is nonoscillatory. The solution of the equation is, e.g. $y(t) = t^{-2} + t^{-4}$.

Theorem 3. *Let* (7) *be satisfied and let* $H(t, y) = a(t)h(y)$, *whereby* $\lim_{y\to 0} h(y) = 0.$ (14)

and

$$\int_{t_0}^{\infty} \frac{|a(t)|}{a_n(t)}\,dt < \infty \tag{15}$$

If

$$\liminf_{t\to\infty} \frac{\left|\int_{t}^{\infty} \frac{b(s)}{a_n(s)}\,ds\right|}{\int_{t}^{\infty} \frac{a(s)}{a_n(s)}\,ds} = \beta > 0, \tag{16}$$

then the set M *is empty.*

Proof. Let there exist the oscillatory solution $y(t)$ of (1) such that $\lim_{t \to \infty} y(t) = 0$. From the condition (14) it follows that to any arbitrary positive number γ there exists T such that $|h(y(g(t)))| < \gamma$ for every $t > T$. Let $\gamma < \beta$. From the relation (10) we have that

$$|L_{n-1} y(t)| = \left| \int_{t_n}^{t} \frac{b(s)}{a_n(s)} \, ds \right| + \gamma \int_{t_n}^{t} \frac{|a(s)|}{a_n(s)} \, ds ,$$

from what with respect to (7), (15) and the fact that $t_n \to \infty$ for $n \to \infty$ it follows that $\lim_{t \to \infty} L_{n-1} y(t) = 0$. Then from the relation (12) we have that

$$\left| \int_{t_n}^{\infty} \frac{b(s)}{a_n(s)} \, ds \right| = \gamma \int_{t_n}^{\infty} \frac{|a(s)|}{a_n(s)} \, ds \qquad \text{for } t_n > T .$$

But from the last relation it results that

$$\liminf_{n \to \infty} \frac{\left| \int_{t_n}^{\infty} \frac{b(s)}{a_n(s)} \, ds \right|}{\int_{t_n}^{\infty} \frac{|a(s)|}{a_n(s)} \, ds} \le \gamma < \beta ,$$

which contradicts assumption (16), hence the set M is empty. This completes the proof of the theorem.

Theorem 4. *Let the conditions of theorem* 1 *be satisfied and if furthermore*

$$\lim_{t \to \infty} \left| \int_{t_0}^{t} \frac{J_{n-1}(g(t), s)\, b(s)}{a_n(s)} \, ds \right| < \infty \tag{17}$$

and

$$\lim_{t \to \infty} \int_{t_0}^{t} \frac{J_{n-1}(g(t), s) f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds < \infty . \tag{18}$$

then every oscillatory solution is bounded.

Proof. Let $y(t)$ be an oscillatory solution of (1). Then there exist T_i $i = 1, \ldots, n$ such that $L_{n-i} y(T_i) = 0$, $T \le T_1 \le \ldots \le T_n$. We integrate (1) n-times successively from T_i to $t > T_n$, multiplying the result of the ith integration by the function $\dfrac{1}{a_{n-i+1}(t)}$. We obtain

$$a_0(t) y(t) = \int_{T_n}^{t} \frac{1}{a_1(s_1)} \int_{T_{n-1}}^{s_1} \frac{1}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-1}} \frac{b(s) - H(s, y(g(s)))}{a_n(s)} \, ds \, ds_{n-1} \ldots ds_1 ,$$

wherefrom

$$|y(t)| \leq \frac{1}{a_0(t)} \int_{T_1}^{t} \frac{1}{a_1(s_1)} \int_{T_1}^{s_1} \frac{1}{a_2(s_2)} \cdots \int_{T_1}^{s_{n-1}} \frac{|b(s)| + |H(s, y(g(s)))|}{a_n(s)} \, ds \, ds_{n-1} \ldots ds_1 . \tag{19}$$

From the last relation we have by using (6) and notation (3), (4)

$$|y(t)| \leq \int_{T_1}^{t} \frac{J_{n-1}(t,s)|b(s)|}{a_n(s)} \, ds + \int_{T_1}^{t} \frac{J_{n-1}(t,s) f(s, |y(g(s))|)}{a_n(s)} \, ds$$

Since $g(t) \leq t$, $g(t) \to \infty$ for $t \to \infty$ from the last relation is

$$|y(g(t))| \leq \int_{T_1}^{g(t)} \frac{J_{n-1}(g(t),s)|b(s)|}{a_n(s)} \, ds + \\ + c \int_{T_1}^{g(t)} \frac{J_{n-1}(g(t),s) f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds \tag{20}$$

for every $t \geq T^*$ such that $g(t) \geq T_1$. With regard to (17) and (18) we shall obtain the assertion of the theorem.

Theorem 5. *Let the conditions of theorem* 4 *be fulfilled and if furthermore*

$$\lim_{t \to \infty} J_{n-1}(g(t), t_0) < \infty , \tag{21}$$

then for every oscillatory solution $y(t)$ *of* (1) $\lim_{t \to \infty} y(t) = 0$.

Proof. Since $J_{n-1}t, s)$ is a nonincreasing function in s from relation (20) we have

$$|y(g(t))| \leq J_{n-1}(g(t), T_1)\left(\int_{T_1}^{g(t)} \frac{|b(s)|}{a_n(s)} \, ds + \right. \\ \left. + c \int_{T_1}^{g(t)} \frac{f(s, J_{n-1}(g(s), t_0))}{a_n(s)} \, ds \right),$$

which, with respect to (21), (7), (8) and the fact that T_1 may be arbitrary large, leads to the assertion of the theorem.

Theorem 6. *Let the conditions of theorem* 1, (14), (16) *and* (21) *be fulfilled. Then every solution of equation* (1) *is nonoscillatory.*

Proof. Let there exist an oscillatory solution $y(t)$ of (1). From Theorem 5 it results that then there exists $\lim_{t \to \infty} y(t) = 0$ i.e. $y(t) \in M$. Since the assumptions of theorem 3 be fulfilled we have a contradiction.
This completes the proof of the theorem.

Theorem 7. *Let the conditions of theorem* 1 *be satisfied and let furthermore*

$$\int_{t_0}^{\infty} K_{n-1}(t, t_0)|b(t)| \, \mathrm{d}t < \infty , \tag{22}$$

$$\int_{t_0}^{\infty} K_{n-1}(t, t_0) f(t, J_{n-1}(g(t), t_0)) \, \mathrm{d}t < \infty , \tag{23}$$

$$\liminf_{t \to \infty} a_0(t) > 0 . \tag{24}$$

Then for every oscillatory solution $y(t)$ *of* (1) $\lim_{t \to \infty} y(t) = 0$.

Proof. See the proof of theorem 1.2 in paper [1].

Theorem 8. *Let the assumptions of theorem* 7 *and* (14), (16) *be satisfied. Then every solution of* (1) *is nonoscillatory.*

Proof. It follows from theorems 7 and 3.

Note. Sufficient conditions for the nonoscillation of equation (1) presented in theorems 2, 6 and 8 are not equivalent, which results from the next examples.

Example 2. Consider an equation

$$(t^3 y''(t))' + \frac{1}{t^3} y^{\frac{1}{2}}(t) = t^{-\frac{5}{2}}, \qquad t > 0 . \tag{25}$$

The conditions of theorems 2 and 6 are not satisfied, but the conditions of theorem 8 are satisfied and thus every solution of (25) is nonoscillatory. The equation has a nonoscillatory solution, e.g. $y(t) = t$.

Example 3. Consider an equation

$$(t^3 y'(t))'' + \frac{6}{t^2 (t^{3/2} + 1)^{1/3}} y^{\frac{1}{3}}(t) = \frac{3}{8} t^{-\frac{3}{2}}, \qquad t > 0 . \tag{26}$$

In the equation the conditions of theorem 8 are not satisfied but the conditions of theorem 6, resp. 2 are fulfilled and thus every solution of (26) is nonoscillatory. The equation has nonoscillatory solutions, e.g.

$$y(t) = t^{-3} + t^{-\frac{3}{2}} .$$

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Received September 24, 1987

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АСИМПТОТИЧЕСКИЕ СВОЙСТВА РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ n-ОГО ПОРЯДКА С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

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Резюме

В работе исследуются асимптотические свойства решений дифференциального уравнения n-ого порядка в форме

$$L_n y(t) + H(t, y(g(t))) = b(t), \qquad \text{для } n \geq 2,$$

где $L_n y(t) = a_n(t)(a_{n-1}(t)(\ldots(a_1(t)(a_0(t)y(t))')' \ldots)')'$.

Для каждого уравнения приводятся достаточные условия, при которых каждое решение является неколеблющимся.