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INTEGRATION WITH RESPECT TO ORTHOGONALLY SCATTERED MEASURES

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ABSTRACT. We compare the Bochner and the monotone integrals for scalar measurable functions with respect to vector measures ranging on a Hilbert space.

1. Introduction

In Stochastic Integration, when it is necessary to integrate with respect to two-summable stochastic processes with independent increments, orthogonally scattered measures arise in a natural way ([9]).

In 1981 Chatterji [2] showed that every finitely additive measure $m\colon \Sigma \to H$ which ranges on a Hilbert space can be looked as a projection of a finitely additive orthogonally scattered measure $\widetilde{m}\colon \Sigma \to \widetilde{H}$, called an orthogonally scattered dilation of m.

A lot of authors have studied the problem of integration when the set function is a vector measure, but only in the last fifteen years it was possible to obtain meaningful developments in the integration on locally convex topological vector spaces.

The aim of this paper is to compare two classical definitions of integral with respect to a vector measure in a Hilbert space H. The two kinds of integrals considered are the Bochner integral, which is defined as a limit of integrals of a defining sequence of simple functions, and the De Giorgi-Letta integral which was defined for scalar integrands in [12] and [3], and further investigated in [4] and [6].

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Lavoro svolto nell' ambito dello G.N.A.F.A. del C.N.R.

The problem was already studied in [1] by Brooks–Martellotti for finitely additive measures ranging in Banach spaces and afterwards by Martellotti [8] for finitely additive measures on locally convex topological vector spaces.

Now, when H is a Hilbert space, the existence of an orthogonally scattered dilation allows to obtain a better comparison between the two integrals, when the functions are integrated either with respect to m or with respect to any orthogonally scattered dilation $\widetilde{m}$, and it yields sufficient conditions for the equivalence between them.

2. Preliminary remarks

2.1. Notation.

Let Ω be an arbitrary set, Σ a σ-algebra of subsets of Ω. H, $\widetilde{H}$ be separable Hilbert spaces, X a Banach space, X' its dual and $B_{X'}$ the unit ball of X'. Let $m\colon \Sigma \to X$ be a bounded finitely additive measure (f.a.m.). We denote by $\|m\|$ the semivariation of m and, if m is b.v., $|m|$ is its variation. We say that $\lambda\colon \Sigma \to \mathbb{R}_0^+$ is a control measure for m if λ is equivalent to $\|m\|$. λ is a Rybakov control for m if there exists $x' \in X'$ such that $\lambda(\cdot) = |\langle x' \mid m\rangle|(\cdot)$.

2.2. Some definitions.

A measure m is *s-bounded* if and only if for every $(A_n)_n$ in Σ with $A_n \cap A_m = \emptyset$ from $n \neq m$ it follows that $\lim\limits_{n\to\infty} m(A_n) = 0$.

If m is s-bounded we can define the $*$-semivariation of m as follows: for every $A \in \Sigma$

$$\|m\|^*(A) = \sup\big\{|\langle x' \mid m\rangle|(A) : \; x' \in B_{X'}\big\}. \tag{1}$$

We can observe that $\|m\|^*$ is equivalent to $\|m\|$ since for every $A \in \Sigma$ it is

$$\|m\|(A) \le \|m\|^*(A) \le 2\|m\|(A). \tag{2}$$

Let $m\colon \Sigma \to H$ be a finitely additive measure such that

$$\langle m(A), m(B)\rangle = 0 \qquad \text{if} \quad A \cap B = \emptyset.$$

m is said to be a *f.a.o.s. measure* (orthogonally scattered finitely additive).

Observe that if m is a f.a.o.s. measure then $\|m\|^2\colon \Sigma \to \mathbb{R}_0^+$ is a finitely additive measure. The f.a.m. thus obtained will be called the *finitely additive measure associated to* m, and will be denoted by μ_m.

2.3. The Stone space.

Let $(S, \mathcal{G})$ be the Stone space associated with (Ω, Σ), where $\mathcal{G}$ is the algebra of clopen sets of S, $h\colon \Sigma \to \mathcal{G}$ the Stone isomorphism and $\mathcal{G}_\sigma$ the σ-algebra on S generated by $\mathcal{G}$.

If m is a finitely additive s-bounded measure then we can define a measure $\overline{m}\colon \mathcal{G} \to X$ as follows: $\overline{m}(G) = m(h^{-1}(G))$ for every $G \in \mathcal{G}$. Since m is s-bounded $\overline{m}$ can be extended to $\mathcal{G}_\sigma$ in a countably additive way; the measure $\overline{m}$ will be called the *extended measure* of m.

For the reader's convenience we shall report some definitions and results that we shall largely use in the sequel. We refer to [11], [10], and [1] for the proofs.

DEFINITION 2.1. For every m-measurable function $f\colon \Omega \to \mathbb{R}$ we define the function $\overline{f}\colon S \to \overline{\mathbb{R}}$ as follows:

$$\overline{f}(s) = \sup\{a \in \mathbb{R} :\ s \notin h\big(f^{-1}(]-\infty, a[)\big)\}.$$

$\overline{f}$ satisfies the following properties:

- $\overline{f}$ is a continuous function;
- if f_1, f_2 are m-measurable and $c \in \mathbb{R}$ then:

(2.1.1) $\overline{cf_1} = c\overline{f}_1$;

(2.1.2) If $f_1 \le f_2$ then $\overline{f}_1 \le \overline{f}_2$;

(2.1.3) $\overline{f_1 + f_2} = \overline{f}_1 + \overline{f}_2$;

(2.1.4) For every $A \in \Sigma$, $\overline{f_1 \cdot 1_A} = \overline{f}_1 \cdot 1_{h(A)}$. Hence, if f is a simple function: $f = \sum_{i=1}^{n} x_i 1_{A_i}$ then $\overline{f} = \sum_{i=1}^{n} x_i 1_{h(A_i)}$;

(2.1.5) $|\overline{f}_1| = \overline{|f_1|}$.

PROPOSITION 2.2. *Let $\nu\colon \Sigma \to \mathbb{R}_0^+$ be a f.a.m. and $f\colon \Omega \to \mathbb{R}$ be a ν-measurable function. Then for every $t \in \mathbb{R}$ one has*

$$\overline{\nu}(\overline{f} > t) \le \nu(f \ge t) \le \overline{\nu}(\overline{f} \ge t). \tag{3}$$

Proof. Let $t \in \mathbb{R}$ be fixed. We set $H_t = \Big(h\big(f^{-1}(]-\infty, t[)\big)\Big)^c$. If $s \in H_t$ then $s \notin h\big(f^{-1}(]-\infty, t[)\big)$, and so $\overline{f}(s) = \sup\{a \in \mathbb{R} :\ s \notin h\big(f^{-1}(]-\infty, a[)\big)\} \ge t$; hence $H_t \subset \{\overline{f} \ge t\}$.

If $s \in H_t^c = h\big(f^{-1}(]-\infty, t[)\big)$ then $s \in h\big(f^{-1}(]-\infty, a[)\big)$ for every $a \ge t$ and so $\overline{f}(s) \le t$. Then $H_t^c \subset \{\overline{f} \le t\}$ i.e. $H_t \supset \{\overline{f} > t\}$. Since

$$\nu(f \ge t) = \nu\big((-\infty < f < t)^c\big) = \overline{\nu}\big(h\big(f^{-1}(]-\infty, t[)\big)^c\big) = \overline{\nu}(H_t)$$

we have the assertion by the monotonicity of $\overline{\nu}$. $\square$

COROLLARY 2.3. *If $m\colon \Sigma \to H$ is an s-bounded f.a.m. and $\overline{m}$ is the extended measure of m, then*

$$\|\overline{m}\|^*\big(\overline{f} > t\big) \leq \|m\|^*(f \geq t)\,, \tag{4}$$

$$\|m\|^*(f \geq t) \leq \|\overline{m}\|^*\big(\overline{f} \geq t\big)\,. \tag{5}$$

2.4. Integrals with respect to a vector finitely additive measure.

In [1] J. Brooks and A. Martellotti have introduced the following definition:

DEFINITION 2.4. A measurable function $f\colon \Omega \to \mathbb{R}$ is *m-integrable* if there exist a control measure $\lambda\colon \Sigma \to \mathbb{R}_0^+$ and a sequence of simple functions $(f_n)_n$ such that

(2.4.1) f_n λ-converges to f;

(2.4.2) $\big(\int_F f_n \, \mathrm{d}m\big)_n$ converges on H for every $F \in \Sigma$.

Then we set

$$\int_{\bullet} f \, \mathrm{d}m = \lim_{n\to\infty} \int_{\bullet} f_n \, \mathrm{d}m$$

and $L^1(m)$ denotes the set of m-integrable functions. The sequence $(f_n)_n$ will be said a *defining sequence* for f.

Let $m\colon \Sigma \to H$ be an s-bounded f.a.m.. Given an m-measurable function $f\colon \Omega \to [0,\infty[$, we introduce the following functions: $\varphi\colon [0,\infty[\to H$ and $\widehat{\varphi}, \overline{\varphi}\colon [0,\infty[\to [0,\infty[$ defined as follows:

$$\varphi(t) = m\big(\omega \in \Omega : f(\omega) > t\big)\,,$$

$$\widehat{\varphi}(t) = \|m\|\big(\omega \in \Omega : f(\omega) > t\big)\,,$$

$$\overline{\varphi}(t) = \|m\|^2\big(\omega \in \Omega : f(\omega) > t\big)\,.$$

In order to compare the m-integral with an extension of the De Giorgi-Letta integral we introduce the following definitions:

DEFINITION 2.5. Let $f\colon \Omega \to [0,\infty[$ be an m-measurable function. f is $(\widehat{\ })$*-integrable* with respect to m if and only if $\widehat{\varphi}$ is Lebesgue integrable; in this case in fact φ is Bochner integrable and we can set

$$\widehat{\int} f \, \mathrm{d}m = \int_0^\infty \varphi(t) \, \mathrm{d}t\,;$$

if f takes values in $\mathbb{R}$ we say that f is $(\widehat{\ })$*-integrable* if and only if f^+, f^- are $(\widehat{\ })$*-integrable.* We denote by $\widehat{L}^1(m)$ the set of $(\widehat{\ })$-integrable functions.

DEFINITION 2.6. Let m be a f.a.o.s. measure and $f\colon \Omega \to [0,\infty[$ be an m-measurable function. f is $(\tilde{\ })$-*integrable* with respect to m if $\overline{\varphi}$ is Lebesgue integrable. If f takes values in $\mathbb{R}$ we say that f is $(\tilde{\ })$-*integrable* if and only if f^+, f^- are $(\tilde{\ })$-integrable. We denote by $\widetilde{L}^1(m)$ the set of $(\tilde{\ })$-integrable functions.

3. Comparison between $L^1(m)$ and $\widehat{L}^1(m)$

We shall prove that under suitable conditions $L^1(m)$ and $\widehat{L}^1(m)$ are equivalent. In order to do this we begin with some propositions.

PROPOSITION 3.1. *Let $m\colon \Sigma \to H$ be an s-bounded f.a.m. and $\lambda = |\langle x_0 \mid m\rangle|$, with $\|x_0\| = 1$, be a Rybakov control for m. If $f\colon \Omega \to \mathbb{R}$ is m-integrable then $f \in L^1(\lambda)$.*

P r o o f . Since $f \in L^1(m)$, there exists a sequence of simple functions $(f_n)_n$ λ-converging to f. Moreover it is easy to check that

$$\begin{aligned}
\left|\int_F (f_k - f_n)\, \mathrm{d}\lambda\right| &\le \int_F |f_k - f_n|\, \mathrm{d}|\langle x_0 \mid m\rangle| \\
&= \operatorname{var}\left[\left\langle x_0 \,\middle|\, \int (f_k - f_n)\, \mathrm{d}m\right\rangle\right](F) \\
&\le 2 \sup_{G \in F\cap\Sigma} \left|\left\langle x_0 \,\middle|\, \int_G (f_k - f_n)\, \mathrm{d}m\right\rangle\right| \\
&\le 2\left\| \int_G (f_k - f_n)\, \mathrm{d}m \right\| = 2 \sup_{x^*} \left| x^* \int_G (f_k - f_n)\, \mathrm{d}m \right| \\
&\le 4 \sup_{G \in \Sigma\cap F} \left\| \int_G (f_k - f_n)\, \mathrm{d}m \right\|
\end{aligned}$$

and, as by [1; Remark 2.4], the convergence in (2.4.2) is uniform with respect to $F \in \Sigma$, and since $\left(\int_F f_n\, \mathrm{d}m\right)_n$ is Cauchy in H, it follows that $\left(\int_F f_n\, \mathrm{d}\lambda\right)_n$ is also Cauchy. □

Remark 3.2. Note that if $m\colon \Sigma \to H$ is a bounded countably additive measure (shortly a measure) and λ is a control for m by [5] there exists a function g Pettis integrable such that $g = \dfrac{\mathrm{d}m}{\mathrm{d}\lambda}$. If g is bounded, m is of bounded variation and $|m|(\cdot) \le k\lambda(\cdot)$ where k is such that $\|g(\omega)\| \le k$ for every $\omega \in \Omega$.

We first prove the equivalence in the countably additive case. In the sequel without loss of generality we shall always assume, when considering real $f \in \widehat{L}^1(m)$ or $f \in \widetilde{L}^1(m)$, that f is non-negative.

PROPOSITION 3.3. *Let $m\colon \Sigma \to H$ be a bounded measure, λ a Rybakov control for m. If $g = \dfrac{\mathrm{d}m}{\mathrm{d}\lambda}$ is bounded then $f \in L^1(m)$ if and only if $f \in \widehat{L}^1(m)$.*

Proof. By [1; Theorem 3.9] the inclusion $\widehat{L}^1(m) \subset L^1(m)$ holds. Conversely if $f \in L^1(m)$ then, by Proposition 3.1 we have that $f \in L^1(\lambda)$ which is equal to $\widehat{L}^1(\lambda)$ because λ is a scalar measure ([1; Theorem 3.6]). Then, by Remark 3.2, if k is such that $\|g(\omega)\| \leq k$ for every $\omega \in \Omega$,

$$\int_0^\infty \|m\|(f > t)\ \mathrm{d}t \leq \int_0^\infty |m|(f > t)\ \mathrm{d}t \leq \int_0^\infty k\lambda(f > t)\ \mathrm{d}t < +\infty$$

and so $f \in \widehat{L}^1(m)$. □

Now we want to extend Proposition 3.3 to the finitely additive case. In order to do this we need some propositions and we will introduce an "extended" function when f ranges on H.

We suppose now that m is an s-bounded f.a.m. and we introduce some preliminary propositions concerning the extended function $\overline{f}$ already introduced for real-valued function f.

PROPOSITION 3.4. *Let $m\colon \Sigma \to H$ be an s-bounded f.a.m. and $\overline{m}$ is the extended measure of m. If $f \in L^1(m)$ then $\overline{f} \in L^1(\overline{m})$ and for every $E \in \Sigma$*

$$\int_E f\ \mathrm{d}m = \int_{h(E)} \overline{f}\ \mathrm{d}\overline{m}.$$

Proof. If $f \in L^1(m)$ there exists a sequence $(f_n)_n$ of simple functions such that f_n $\|m\|$-converges to f and $\left(\int f_n\ \mathrm{d}m\right)_n$ is Cauchy. $\overline{f_n}$ $\|\overline{m}\|^*$-converges to $\overline{f}$. In fact, by (2.1.3), (2.1.5), (4) and (2), applied to $|\overline{f_n - f}|$, for every $\alpha > 0$ we have $\|\overline{m}\|^*(|\overline{f_n - f}| > \alpha) \leq 2\|m\|(|f_n - f| > \alpha)$.

Moreover, by (2.1.4), for every $G \in \mathcal{G}$ if $F = h^{-1}(G)$ we have

$$\int_G \overline{f_n}\ \mathrm{d}\overline{m} = \int_F f_n\ \mathrm{d}m.$$

Thus $\left(\int_G \overline{f_n}\ \mathrm{d}\overline{m}\right)_n$ is Cauchy in H for every $G \in \mathcal{G}$. This implies that $\overline{f} \in L^1(\overline{m})$ and that

$$\int_G \overline{f}\ \mathrm{d}\overline{m} = \lim_{n\to\infty} \int_G \overline{f_n}\ \mathrm{d}\overline{m} = \lim_{n\to\infty} \int_{h(F)} f_n\ \mathrm{d}m = \int_{h(F)} f\ \mathrm{d}m.$$

□

PROPOSITION 3.5. *If $m\colon \Sigma \to H$ is an s-bounded f.a.m. and $\overline{m}$ is the extended measure of m then $f \in \widehat{L}^1(m)$ if and only if $\overline{f} \in \widehat{L}^1(\overline{m})$.*

P r o o f. By (2) and (4) we have

$$\int_0^\infty \|\overline{m}\|(\overline{f} > t)\ dt \le \int_0^\infty \|\overline{m}\|^*(\overline{f} > t)\ dt \le \int_0^\infty \|m\|^*(f \ge t)\ dt$$
$$\le 2\int_0^\infty \|m\|(f \ge t)\ dt < +\infty\,.$$

Conversely, by (2) and (5)

$$\int_0^\infty \|m\|(f \ge t)\ dt \le \int_0^\infty \|m\|^*(f \ge t)\ dt \le \int_0^\infty \|\overline{m}\|^*(\overline{f} \ge t)\ \ dt$$
$$\le 2\int_0^\infty \|\overline{m}\|(\overline{f} \ge t)\ dt\,.$$

□

We remember that a function $\psi\colon \Omega \to H$ is *totally λ-measurable* if and only if there exists a sequence of simple functions $(\psi_n)_n$ which λ-converges to ψ.

We are now going to define a Stone extended function $\overline{f}$ for *vector-valued* functions. If $f\colon \Omega \to H$ is simple, say $f = \sum_{i=1}^n x_i 1_{A_i}$, we set

$$\overline{f} = \sum_{i=1}^n x_i 1_{h(A_i)}\,.$$

Let $f\colon \Omega \to H$ be a totally λ-measurable function; then there exists a sequence of simple functions $(f_n)_n$ such that f_n λ-converges to f. Since $(f_n)_n$ is Cauchy in λ measure, by (3) $\left(\overline{f}_n\right)_n$ is also Cauchy in $\overline{\lambda}$ measure and so there exists a $\mathcal{G}_\sigma$-measurable function ψ such that $\overline{f}_n$ $\overline{\lambda}$-converges to ψ. It is obvious that if $\|f\|$ is bounded then $\|\psi\|$ is bounded.

PROPOSITION 3.6. *If $f\colon \Omega \to H$ is totally λ-measurable then $\|\psi\| = \overline{\|f\|}$ $\overline{\lambda}$-a.e..*

P r o o f . The equality is obvious if f is simple. We then suppose that f is totally λ-measurable. By definition there exists a sequence $(f_n)_n$ such that f_n λ-converges to f; then it follows that $\overline{f}_n$ $\overline{\lambda}$-converges to ψ, $\|\overline{f}_n\|$ $\overline{\lambda}$-converges to $\|\psi\|$ and $\|f_n\|$ λ-converges to $\|f\|$. On the other hand

$$\overline{\lambda}\Big\{\big|\overline{\|f_n\|} - \overline{\|f\|}\big| > \alpha\Big\} = \overline{\lambda}\Big\{\overline{\big|\|f_n\| - \|f\|\big|} > \alpha\Big\} \leq \lambda\Big\{\big|\|f_n\| - \|f\|\big| > \alpha\Big\}$$

and so $\|\overline{f}_n\| = \overline{\|f_n\|}$ $\overline{\lambda}$-converges to $\|\psi\|$ and $\overline{\|f\|}$. This implies that $\|\psi\| = \overline{\|f\|}$ $\overline{\lambda}$-a.e.. □

PROPOSITION 3.7. *If $f\colon \Omega \to H$ is totally λ-measurable then for every $x \in B_H$ one has $\langle x \mid \psi\rangle = \overline{\langle x \mid f\rangle}$ $\overline{\lambda}$-a.e..*

P r o o f . If f is a simple function then it is easy to prove the equality. Then we suppose that f is a totally λ-measurable function. Let f_n be a sequence of simple functions λ-converging to f. For every $x \in B_H$ $\langle x \mid f_n\rangle$ λ-converges to $\langle x \mid f\rangle$ and so applying (2.1.3), (2.1.5) and (3), we have

$$\begin{aligned}\overline{\lambda}\Big\{\big|\overline{\langle x \mid f_n\rangle} - \overline{\langle x \mid f\rangle}\big| > \alpha\Big\} &= \overline{\lambda}\big\{\overline{|\langle x \mid f_n\rangle - \langle x \mid f\rangle|} > \alpha\big\} \\ &\leq \lambda\big\{|\langle x \mid f_n\rangle - \langle x \mid f\rangle| \geq \alpha\big\}.\end{aligned}$$

This proves that $\overline{\langle x \mid f_n\rangle} = \langle x \mid \overline{f}_n\rangle$ $\overline{\lambda}$-converges to $\overline{\langle x \mid f\rangle}$. On the other hand by the Schwartz's inequality and Proposition 3.6

$$\begin{aligned}\overline{\lambda}\{|\langle x \mid \overline{f}_n\rangle - \langle x \mid \psi\rangle| > \alpha\} \leq \overline{\lambda}\{\|\overline{f}_n - \psi\| > \alpha\} &= \overline{\lambda}\{\overline{\|f_n - f\|} > \alpha\} \\ &\leq \lambda\{\|f_n - f\| \geq \alpha\}.\end{aligned}$$

We set $\overline{f} = \psi$. Observe that, via Proposition 3.7, $\overline{f}$ is well-defined. Indeed, if $(f_n)_n$ and $(g_n)_n$ both λ-converge to f, then $\overline{\lambda}$-a.e.

$$\Big\langle x \mid \lim_{n\to\infty} \overline{f}_n\Big\rangle = \Big\langle x \mid \lim_{n\to\infty} \overline{g}_n\Big\rangle$$

for every $x \in B_H$, namely $\overline{\lambda}$-a.e. $\lim\limits_{n\to\infty} \overline{f}_n = \lim\limits_{n\to\infty} \overline{g}_n$ scalarly. □

PROPOSITION 3.8. *If $f\colon \Omega \to H$ is bounded, totally λ-measurable and λ-integrable then $\overline{f}$ is $\overline{\lambda}$-integrable and for every $F \in \Sigma$*

$$\int_F f \,\mathrm{d}\lambda = \int_{h(F)} \overline{f} \,\mathrm{d}\overline{\lambda}.$$

Proof. By [1; Theorem 3.6] $\|f\|$ is $(\widehat{\ })$-integrable with respect to λ. Since $\overline{\lambda}\{\|\overline{f}\| > \alpha\} = \overline{\lambda}\{\overline{\|f\|} > \alpha\} \leq \lambda\{\|f\| \geq \alpha\}$, $\|\overline{f}\|$ is $(\widehat{\ })$-integrable with respect to $\overline{\lambda}$ and, by [1; Theorem 3.9], $\|\overline{f}\|$ is $\overline{\lambda}$-integrable. By [7; Theorem III.2.22] it follows that $\overline{f}$ is $\overline{\lambda}$-integrable.

Moreover, if $(f_n)_n$ is any defining sequence for f, for every $F \in \Sigma$

$$\int_F f \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_F f_n \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_{h(F)} \overline{f}_n \,\mathrm{d}\overline{\lambda} = \int_{h(F)} \overline{f} \,\mathrm{d}\overline{\lambda}.$$

□

THEOREM 3.9. *Let $m\colon \Sigma \to H$ be an s-bounded finitely additive measure. If there exists $y \in H$ such that*

1) *$|\langle y \mid \overline{m}\rangle|$ is a Rybakov control for $\overline{m}$,*
2) *$\dfrac{\mathrm{d}m}{\mathrm{d}|\langle y \mid m\rangle|}$ is bounded,*

then $f \in L^1(m)$ if and only if $f \in \widehat{L}^1(m)$.

Proof. The implication $f \in \widehat{L}^1(m) \implies f \in L^1(m)$ is proven in [1; Theorem 3.9]. We now prove the converse implication. We first observe that if $\lambda = |\langle y \mid m\rangle|$, then $\overline{\lambda} = |\langle y \mid \overline{m}\rangle|$. We denote by $g = \dfrac{\mathrm{d}m}{\mathrm{d}\lambda}$ and we want to prove that $\overline{g} = \dfrac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}}$. Since g is bounded and λ-integrable, by Proposition 3.8, $\overline{g}$ is bounded also and $\overline{\lambda}$-integrable. Moreover for every $G \in \mathcal{G}$

$$\int_G \overline{g} \,\mathrm{d}\overline{\lambda} = \int_{h^{-1}(G)} g \,\mathrm{d}\lambda = m\bigl(h^{-1}(G)\bigr) = \overline{m}(G).$$

We now prove that the last equality holds for every $G \in \mathcal{G}_\sigma$, too. Let $G \in \mathcal{G}_\sigma$, $\varepsilon > 0$ be fixed and let $\delta > 0$ be that of the absolute continuity of $\int_{\bullet} \|\overline{g}\| \,\mathrm{d}\overline{\lambda}$ with respect to $\overline{\lambda}$. Let $\sigma(\delta) > 0$ be such that if $\overline{\lambda}(E) < \delta$ then $\|\overline{m}\|(E) < \sigma$. There exists $A \in \mathcal{G}$ such that $\overline{\lambda}(G \Delta A) < \delta$.

$$\begin{aligned}
&\Bigl\| \overline{m}(G) - \int_G \overline{g} \,\mathrm{d}\overline{\lambda} \Bigr\| \\
&\leq \|\overline{m}(G) - \overline{m}(A)\| + \Bigl\| \overline{m}(A) - \int_A \overline{g} \,\mathrm{d}\overline{\lambda} \Bigr\| + \Bigl\| \int_A \overline{g} \,\mathrm{d}\overline{\lambda} - \int_G \overline{g} \,\mathrm{d}\overline{\lambda} \Bigr\| \\
&\leq 2\|\overline{m}\|(G \Delta A) + \int_{A \Delta G} \|\overline{g}\| \,\mathrm{d}\overline{\lambda} \leq 2\sigma + \varepsilon.
\end{aligned}$$

By the arbitrariness of ε we obtain that $\overline{g} = \dfrac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}}$ $\overline{\lambda}$-a.e.. By Proposition 3.3 it follows that $L^1(\overline{m}) = \widehat{L}^1(\overline{m})$. Hence if $f \in L^1(m)$, by Proposition 3.4, $\overline{f} \in L^1(\overline{m}) = \widehat{L}^1(\overline{m})$, and, by Proposition 3.5, $f \in \widehat{L}^1(m)$. □

4. Comparison between $L^1(\widetilde{m})$ and $\widetilde{L}^1(\widetilde{m})$

Let $m\colon \Sigma \to H$, $\widetilde{m}\colon \Sigma \to \widetilde{H}$ be f.a. measures. According to [2], we shall say that $\widetilde{m}$ is a dilation of m if for every $A \in \Sigma$, $m(A) = VP\widetilde{m}(A)$ where P is a projection of $\widetilde{H}$ onto a linear manifold M and $V\colon M \to H$ is a unitary isomorphism. In [2] Chatterji has obtained the following result:

THEOREM 4.1. *If $m\colon \Sigma \to H$ is a bounded f.a.m., then there exists a dilation of m, $\widetilde{m}\colon \Sigma \to \widetilde{H}$, which is orthogonally scattered. Moreover if m is countably additive $\widetilde{m}$ can be chosen countably additive also.*

PROPOSITION 4.2. *Let $m\colon \Sigma \to H$ be an s-bounded finitely additive measure. If $\widetilde{m}$ is any dilation of m and $f \in L^1(\widetilde{m})$, then $f \in L^1(m)$ and $\int_\bullet f \,\mathrm{d}\widetilde{m}$ is an orthogonally scattered dilation of $\int_\bullet f \,\mathrm{d}m$.*

Proof. If f is simple, the implication is trivial. Let now $f \in L^1(\widetilde{m})$. Then if λ is a control for $\widetilde{m}$ there exists a defining sequence $(f_n)_n$ for f. Let ν be a control for m. Since λ is equivalent to $\|\widetilde{m}\|$ and

$$\|m\|(\cdot) = \|VP\widetilde{m}\|(\cdot) \le \|P\|\,\|\widetilde{m}\|(\cdot) \le \|\widetilde{m}\|(\cdot),$$

f_n ν-converges to f. It only remains to prove that for every $E \in \Sigma$ the sequence $\left(\int_E f_n \,\mathrm{d}m\right)_n$ is Cauchy in H. By [2; Lemma 2] there exists a finitely additive measure $\mu\colon \Sigma \to \mathbb{R}_0^+$ such that for every $E \in \Sigma$ and $n \in \mathbb{N}$

$$\left\| \int_E f_n \,\mathrm{d}m \right\|^2 \le \int_E |f_n|^2 \,\mathrm{d}\mu$$

and, by [2; Lemma 3] and [9; Definition 1.4], we can choose $\mu = \|\widetilde{m}\|^2$. So the assertion follows by

$$\left\| \int_E f_n \,\mathrm{d}m - \int_E f_k \,\mathrm{d}m \right\|^2 \le \int_E (f_n - f_k)^2 \,\mathrm{d}\mu = \left\| \int_E f_n \,\mathrm{d}\widetilde{m} - \int_E f_k \,\mathrm{d}\widetilde{m} \right\|^2 .$$

Moreover, by the continuity of V and P, for every $E \in \Sigma$

$$VP \int_E f \, \mathrm{d}\widetilde{m} = VP \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\widetilde{m} = VP \lim_{n\to\infty} \widetilde{\int_E f_n \, \mathrm{d}m}$$

$$= \lim_{n\to\infty} VP \widetilde{\int_E f_n \, \mathrm{d}m} = \lim_{n\to\infty} \int_E f_n \, \mathrm{d}m = \int_E f \, \mathrm{d}m \,.$$

□

The converse inclusion $L^1(\widetilde{m}) \supset L^1(m)$ is not true in general. In order to exhibit a suitable counter-example, we shall need a preliminary result:

PROPOSITION 4.3. *If $\widetilde{m}\colon \Sigma \to \widetilde{H}$ is a c.a.o.s. measure then $f \in L^1(\widetilde{m})$ implies that $f \in L^2(\mu_{\widetilde{m}})$.*

P r o o f . Let $f \in L^1(\widetilde{m})$ be fixed. Then if λ is a control for $\widetilde{m}$ there exists a defining sequence $(f_n)_n$ for f (with respect to λ). Then f_n $\mu_{\widetilde{m}}$-converges to f, and for every $E \in \Sigma$, $\varepsilon > 0$ fixed there exists $k(\sqrt{\varepsilon}) \in \mathbb{N}$ such that for every $n, p > k$

$$\left\| \int_E f_n \, \mathrm{d}\widetilde{m} - \int_E f_p \, \mathrm{d}\widetilde{m} \right\|^2 < \varepsilon \,.$$

Since f_n, f_p are simple

$$\int_E (f_n - f_p)^2 \, \mathrm{d}\mu_{\widetilde{m}} = \left\| \int_E f_n \, \mathrm{d}\widetilde{m} - \int_E f_p \, \mathrm{d}\widetilde{m} \right\|^2 < \varepsilon \,.$$

So $(f_n)_n$ is Cauchy in $L^2(\mu_{\widetilde{m}})$ which is complete; hence there exists $\varphi \in L^2(\mu_{\widetilde{m}})$ such that for every $E \in \Sigma$

$$\lim_{n\to\infty} \int_E (f_n - \varphi)^2 \, \mathrm{d}\mu_{\widetilde{m}} = 0 \,.$$

Since $(f_n)_n$ $L^2(\mu_{\widetilde{m}})$-converges to φ it is possible to obtain a subsequence $(f_{n_k})_k$ which converges to φ $\mu_{\widetilde{m}}$-almost everywhere. As $(f_{n_k})_k$ $\|\widetilde{m}\|$-converges to f there exists $(f_{n_{k_p}})_p$ which converges to f $\|\widetilde{m}\|$-almost everywhere. As $\mu_{\widetilde{m}}(\cdot) = 0$ if and only if $\|\widetilde{m}\|(\cdot) = 0$, $\varphi = f$ $\mu_{\widetilde{m}}$-almost everywhere. □

EXAMPLE 4.4. Let $\Omega = [0,1]$, $\mathcal{B}$ be the σ-algebra of Borel and m the Lebesgue measure. From the construction of $\widetilde{m}$ due to Chatterji [2], it follows that there exists an orthogonally scattered dilation $\widetilde{m}$ such that $\mu_{\widetilde{m}} = m$. Consider $f \colon \Omega \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & x \in]0,1]\,, \\ 0 & x = 0\,. \end{cases}$$

Then $f \in L^1(m)$ but, by Proposition 4.3, $f \notin L^1(\widetilde{m})$, as $f \notin L^2(\mu_{\widetilde{m}})$.

PROPOSITION 4.5. *Let $\widetilde{m} \colon \Sigma \to \widetilde{H}$ be a bounded c.a.o.s. measure. If $f \in L^1(\widetilde{m})$ then $f \in \widetilde{L}^1(\widetilde{m})$.*

Proof. If $f \in L^1(\widetilde{m})$, by Proposition 4.3, $f \in L^2(\|\widetilde{m}\|^2)$. Since $\|\widetilde{m}\|^2$ is bounded $f \in L^1(\|\widetilde{m}\|^2)$ and so, by [1; Theorem 3.4]

$$\int_0^\infty \|\widetilde{m}\|^2(f > t)\ \mathrm{d}t = \widehat{\int_\Omega} f\ \mathrm{d}\|\widetilde{m}\|^2 = \int_\Omega f\ \mathrm{d}\|\widetilde{m}\|^2 < +\infty\,,$$

namely $f \in \widetilde{L}^1(\widetilde{m})$. □

We now want to obtain the analogous results in the finitely additive case.

COROLLARY 4.6. *Let $\widetilde{m} \colon \Sigma \to \widetilde{H}$ be an s-bounded f.a.o.s. measure and $\mu = \|\overline{\widetilde{m}}\|^2$, if $\overline{f} \in L^1(\overline{\widetilde{m}})$ then $\overline{f} \in \widetilde{L}^1(\overline{\widetilde{m}})$.*

Proof. By [9; Theorem 2.3] $\overline{\|\widetilde{m}\|^2} = \|\overline{\widetilde{m}}\|^2$ and by Proposition 4.3 $\overline{f} \in L^2(\mu)$. The remaining of the proof is identical with that of Proposition 4.5. □

COROLLARY 4.7. *Under the same assumptions of* Corollary 4.6 *if $\overline{f} \in \widetilde{L}^1(\overline{\widetilde{m}})$ then $f \in \widetilde{L}^1(\widetilde{m})$.*

Proof. We prove the assertion when $f \geq 0$. If f ranges on $\mathbb{R}$ it suffices to consider f^+, f^-. Observe that if $\psi(t) = \|\widetilde{m}\|^2(f \geq t)$ and $\chi(t) = \|\widetilde{m}\|^2(f > t)$, as $\|\widetilde{m}\|^2$ is σ-finite the set $H = \{t \in [0, +\infty] :\ \psi(t) \neq \chi(t)\}$ is at most countable. Moreover, since $\|\overline{\widetilde{m}}\|^2 = \overline{\|\widetilde{m}\|^2}$, by [1; Theorem 3.6] and (3) we have

$$\begin{aligned} \int_\Omega f\ \mathrm{d}\|\widetilde{m}\|^2 &= \int_0^\infty \chi(t)\ \mathrm{d}t = \int_0^\infty \psi(t)\ \mathrm{d}t \\ &\leq \int_0^\infty \overline{\|\widetilde{m}\|^2}(\overline{f} \geq t)\ \mathrm{d}t = \int_0^\infty \|\overline{\widetilde{m}}\|^2(\overline{f} \geq t)\ \mathrm{d}t = \int_0^\infty \|\overline{\widetilde{m}}\|^2(\overline{f} > t)\ \mathrm{d}t\,. \end{aligned}$$

□

COROLLARY 4.8. *Under the same assumptions of* Corollary 4.6 *if* $f \in L^1(\widetilde{m})$ *then* $f \in \widetilde{L}^1(\widetilde{m})$.

Proof. It is a consequence of Proposition 3.4, Proposition 4.5 and Corollary 4.7. □

5. Comparison between $\widehat{L}^1(m)$ and $\widetilde{L}^1(\widetilde{m})$

Let $m\colon \Sigma \to H$ be a bounded measure. Then according to [2; Theorem 1] we can define $\widetilde{H} = H \oplus L^2(\lambda)$, since a multiple of λ satisfies [2; Lemma 2]; define $\widetilde{m}^*\colon \Sigma \to \widetilde{H}$ by the law

$$\widetilde{m}^*(A) = \left[T(1_A), (I - T^*T)^{\frac{1}{2}}(A)\right]$$

where $T\colon L^2(\lambda) \to H$ is defined by $T(h) = \int h \,\mathrm{d}m$ and $T^*\colon H \to L^2(\lambda)$ is such that for every $h \in L^2(\lambda)$ and for every $x \in H$

$$\langle T(h) \mid x\rangle = \langle h \mid T^*(x)\rangle .$$

Note that $(I - T^*T)$ is a positive Hermitian operator from $L^2(\lambda)$ to $L^2(\lambda)$ since T has been supposed to be a contraction; hence the positive square root $(I-T^*T)^{\frac{1}{2}}$ is a well-defined (positive, Hermitian) operator from $L^2(\lambda)$ to $L^2(\lambda)$. Then

$$\widetilde{m}(A) = \left[m(A), \sigma(A)\right] = \left[m(A), 1_A - \langle m(A), g\rangle^{\frac{1}{2}}\right].$$

THEOREM 5.1. *Let* $m\colon \Sigma \to H$ *be a bounded measure; assume that there exists a control measure* λ *such that* $g = \dfrac{\mathrm{d}m}{\mathrm{d}\lambda}$ *is bounded. If* $f \in \widehat{L}^1(m)$ *then* $f \in \widetilde{L}^1(\widetilde{m})$.

Proof. We want to obtain an estimate for $\|\widetilde{m}\|^2$. For every $A \in \Sigma$

$$\|\widetilde{m}^*\|^2_{\widetilde{H}}(A) = \langle (m \oplus \sigma)(A) \mid (m \oplus \sigma)(A)\rangle = \|m\|^2_H(A) + \|\sigma\|^2_2(A).$$

Since

$$\begin{aligned}
\|\sigma\|^2_2(A) &= \int |1_A - \langle m(A) \mid g\rangle| \,\mathrm{d}\lambda \\
&\le \lambda(A) + \int |\langle m(A) \mid g\rangle| \,\mathrm{d}\lambda \\
&= \lambda(A) + \operatorname{var}\left[\int \left\langle m(A) \;\middle|\; \frac{\mathrm{d}m}{\mathrm{d}\lambda}\right\rangle \mathrm{d}\lambda\right](\Omega) \\
&= \lambda(A) + \operatorname{var}\left[\left\langle m(A) \;\middle|\; \int \frac{\mathrm{d}m}{\mathrm{d}\lambda}\right\rangle \mathrm{d}\lambda\right](\Omega) \\
&\le \lambda(A) + 2\|m\|(A)\,\|m\|(\Omega),
\end{aligned}$$

hence

$$\|\widetilde{m}^*\|^2(f > t) \leq \|m\|^2(f > t) + \lambda(f > t) + 2\|m\|(\Omega)\,\|m\|(f > t)\,. \qquad (6)$$

Since $f \in \widehat{L}^1(m)$, $\int_0^\infty \|m\|(f > t)\ \mathrm{d}t < \infty$. From [1; Theorem 3.9] $f \in L^1(m)$, from Proposition 3.1 $f \in L^1(\lambda)$ and finally, applying [1; Theorem 3.6], $f \in \widehat{L}^1(\lambda)$; so $\int_0^\infty \lambda(f > t)\ \mathrm{d}t < \infty$. For what concerns the first summand of the r.h.s. of (6) we can observe that the function $\widehat{\varphi}(t)$ is non increasing and $\lim_{t\to\infty} \widehat{\varphi}(t) = 0$, so there exists a $\bar{t}$ such that for every $t > \bar{t}$ $\|m\|^2(f > t) = \widehat{\varphi}^2(t) < \widehat{\varphi}(t) < 1$. Then

$$\int_0^\infty \|m\|^2(f > t)\ \mathrm{d}t = \int_0^{\bar{t}} \|m\|^2(f > t)\ \mathrm{d}t + \int_{\bar{t}}^\infty \|m\|^2(f > t)\ \mathrm{d}t$$
$$\leq \bar{t}\|m\|^2(\Omega) + \int_{\bar{t}}^\infty \|m\|(f > t)\ \mathrm{d}t\,,$$

and so the assertion follows. □

COROLLARY 5.2. *Let* $m\colon \Sigma \to H$ *be a bounded measure satisfying the same assumptions of* Theorem 5.1 *for a Rybakov control* $|\langle y \mid m\rangle|$. *Then the following chain of implications holds:*

$$f \in L^1(\widetilde{m}^*) \implies f \in L^1(m) \iff f \in \widehat{L}^1(m) \implies f \in \widetilde{L}^1(\widetilde{m}^*)\,.$$

P r o o f . It follows immediately from Proposition 4.2, Proposition 3.3 and Theorem 5.1. □

Now we prove that there holds:

PROPOSITION 5.3. *If* $m\colon \Sigma \to H$ *is an s-bounded finitely additive measure then* $\overline{\widetilde{m}}$ *is an orthogonally scattered dilation of* $\overline{m}$.

P r o o f . It is easy to show that $\overline{\widetilde{m}}$ is orthogonally scattered. We prove now that $\overline{\widetilde{m}}$ is a dilation of $\overline{m}$, namely for every $B \in \mathcal{G}_\sigma$

$$VP\overline{\widetilde{m}}(B) = \overline{m}(B)\,.$$

Since $\widetilde{m}$ is an orthogonally scattered dilation of m for every $G \in \mathcal{G}$

$$VP\overline{\widetilde{m}}(G) = VP\widetilde{m}\big(h^{-1}(G)\big) = m(h^{-1}(G)) = \overline{m}(G)\,.$$

Let now $\varepsilon > 0$ and $B \in \mathcal{G}_\sigma$ be fixed. We denote by ν_1, ν_2 two controls for $\overline{\widetilde{m}}$ and $\overline{m}$ respectively and let $\lambda = \nu_1 + \nu_2$. Since ν_1 and ν_2 are controls there exist $\delta_1, \delta_2 > 0$ such that if $\nu_1(A) < \delta_1$ then $\|\overline{\overline{m}}\|(A) < \varepsilon$; if $\nu_2(A) < \delta_2$ then $\|\overline{m}\|(A) < \varepsilon$. We set $\delta(\varepsilon) = \min\{\delta_1(\frac{\varepsilon}{4\|P\|}), \delta_2(\frac{\varepsilon}{4})\}$. Since $B \in \mathcal{G}_\sigma$ there exists $G \in \mathcal{G}$ such that $\lambda(B\Delta G) < \delta$ and hence

$$\|\overline{\overline{m}}\|(B\Delta G) < \frac{\varepsilon}{4\|P\|}, \qquad \|\overline{m}\|(B\Delta G) < \frac{\varepsilon}{4}.$$

So it follows that

$$\left\|\overline{\overline{m}}(B) - \overline{\overline{m}}(G)\right\| = \left\|\overline{\overline{m}}(B-G) - \overline{\overline{m}}(G-B)\right\| \leq 2\|\overline{\overline{m}}\|(B\Delta G) < \frac{\varepsilon}{2\|P\|},$$

$$\|\overline{m}(B) - \overline{m}(G)\| \leq 2\|\overline{m}\|(B\Delta G) < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned} &\|\overline{m}(B) - VP\overline{\overline{m}}(B)\| \\ &\leq \|\overline{m}(B) - \overline{m}(G)\| + \|\overline{m}(G) - VP\overline{\overline{m}}(G)\| + \|VP\overline{\overline{m}}(G) - VP\overline{\overline{m}}(B)\| \\ &\leq \frac{\varepsilon}{2} + \|P\|\left\|\overline{\overline{m}}(B) - \overline{\overline{m}}(G)\right\| \leq \varepsilon. \end{aligned}$$

The result follows by the arbitrariness of ε. $\square$

The following proposition is straightforward.

PROPOSITION 5.4. *Let $m\colon \Sigma \to H$ be an s-bounded f.a.m. and let h denote the Stone isomorphism. Then $\widetilde{\overline{m}}\circ h\colon \Sigma \to \widetilde{H}$ is an orthogonally scattered dilation of m where $\widetilde{H} = H \oplus L^2(\overline{\lambda})$ and $\overline{\lambda} = |\langle y \mid \overline{m}\rangle|$ is a control for $\overline{m}$.*

By Theorem 5.1 the following Corollary is true.

COROLLARY 5.5. *Let $m\colon \Sigma \to H$ be an s-bounded f.a.m.. If there exists $y \in H$ such that*

(5.5.1) $|\langle y \mid \overline{m}\rangle|$ *is a control for* $\overline{m}$,

(5.5.2) $\dfrac{\mathrm{d}m}{\mathrm{d}|\langle y \mid m\rangle|}$ *is bounded,*

then $\overline{f} \in \widehat{L}^1(\overline{m})$ implies that $\overline{f} \in \widetilde{L}^1(\widetilde{\overline{m}}^*)$.

THEOREM 5.6. *Let $m\colon \Sigma \to H$ be a bounded f.a.m.. If there exists $y \in H$ such that*

(5.6.1) $|\langle y \mid \overline{m}\rangle|$ *is a control for* $\overline{m}$,

(5.6.2) $\dfrac{\mathrm{d}m}{\mathrm{d}|\langle y \mid m\rangle|}$ *is bounded,*

then the following implications hold:

$$f \in L^1(\widetilde{m}^*) \implies f \in L^1(m) \iff f \in \widehat{L}^1(m) \implies f \in \widetilde{L}^1(\widetilde{m}^*).$$

P r o o f . From Proposition 4.2, Theorem 3.9, Proposition 3.5 and Theorem 5.1 we have the chain of implications

$$f \in L^1(\widetilde{m}^*) \Longrightarrow f \in L^1(m) \Longleftrightarrow f \in \widehat{L}^1(m) \Longleftrightarrow \overline{f} \in \widehat{L}^1(\overline{m}) \\ \Longrightarrow f \in \widetilde{L}^1\big(\widetilde{\overline{m}}^*\big).$$

On the other hand, from Corollary 4.7, we know that $\overline{f} \in \widetilde{L}^1\big(\widetilde{\overline{m}}\big) \Longrightarrow f \in \widetilde{L}^1(\widetilde{m})$, for any orthogonally scattered dilation $\widetilde{m}$ of m.

We want to show that $\overline{\widetilde{m}^*} = \widetilde{\overline{m}}^*$. This will imply that $\widetilde{L}^1\big(\widetilde{\overline{m}}^*\big) = \widetilde{L}^1\big(\overline{\widetilde{m}^*}\big)$ and thus it will conclude the proof.

$\widetilde{\overline{m}}^* = \overline{m} + \tau$, where $\tau\colon \mathcal{G}_\sigma \to L^2(\overline{\lambda})$ is defined by

$$\tau(G) = 1_G - \left\langle \overline{m}(G) \mid \frac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}} \right\rangle,$$

while $\overline{\widetilde{m}^*} = \overline{m + \sigma} = \overline{m} + \overline{\sigma}$ where $\sigma\colon \Sigma \to L^2(\lambda)$ is defined by

$$\sigma(A) = 1_A - \left\langle m(A) \mid \frac{\mathrm{d}m}{\mathrm{d}\lambda} \right\rangle.$$

Hence it is enough to show that $\tau = \overline{\sigma}$. Let $A \in \Sigma$, $G = h(A) \in \mathcal{G}$. Since

$$\tau(G) = 1_G - \left\langle \overline{m}(G) \mid \frac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}} \right\rangle = 1_G - \left\langle m(A) \mid \frac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}} \right\rangle$$

and

$$\overline{\sigma}(G) = \overline{1_A - \left\langle m(A) \mid \frac{\mathrm{d}m}{\mathrm{d}\lambda} \right\rangle} = 1_{h(A)} - \overline{\left\langle m(A) \mid \frac{\mathrm{d}m}{\mathrm{d}\lambda} \right\rangle} = 1_G - \overline{\left\langle m(A) \mid \frac{\mathrm{d}m}{\mathrm{d}\lambda} \right\rangle}$$

again it suffices to show that

$$\left\langle \overline{m}(G) \mid \frac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}} \right\rangle = \overline{\left\langle m(A) \mid \frac{\mathrm{d}m}{\mathrm{d}\lambda} \right\rangle} \quad \overline{\lambda}\text{-a.e.}. \tag{7}$$

In general, given $\mu\colon \Sigma \to H$, $\lambda\colon \Sigma \to \mathbb{R}_0^+$, $\mu \ll \lambda$ such that $\dfrac{\mathrm{d}\mu}{\mathrm{d}\lambda}\colon \Omega \to H$ exists, for every $y \in H$, the scalar f.a.m. $\langle y \mid \mu \rangle$ admits a density with respect to λ, as

$$\frac{\mathrm{d}\langle y \mid \mu \rangle}{\mathrm{d}\lambda} = \left\langle y \mid \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \right\rangle.$$

Applying this fact to the left hand side of (7) we find

$$\left\langle \overline{m}(G) \mid \frac{\mathrm{d}\overline{m}}{\mathrm{d}\overline{\lambda}} \right\rangle = \frac{\mathrm{d}}{\mathrm{d}\overline{\lambda}} \langle \overline{m}(G) \mid \overline{m} \rangle = \frac{\mathrm{d}}{\mathrm{d}\overline{\lambda}} \langle m(A) \mid \overline{m} \rangle.$$

On the other hand

$$\left\langle m(A) \mid \frac{\mathrm{d}m}{\mathrm{d}\lambda} \right\rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda} \langle m(A) \mid m \rangle.$$

Since $\overline{\langle m(A) \mid m \rangle} = \langle m(A) \mid \overline{m} \rangle$, from the same argument used to prove Theorem 3.9, we get

$$\frac{\mathrm{d}}{\mathrm{d}\overline{\lambda}} \langle m(A) \mid \overline{m} \rangle = \overline{\frac{\mathrm{d}}{\mathrm{d}\lambda} \langle m(A) \mid m \rangle} \quad \overline{\lambda}\text{-a.e.}.$$

This shows that τ and $\overline{\sigma}$ coincide on $\mathcal{G}$, and hence they coincide on $\mathcal{G}_\sigma$. The proof is now complete. □

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