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ON HIGHER ORDER POINT SINGULARITIES OF SOME GEOMETRIC OBJECT FIELDS

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Let M be a differentiable manifold, $n=\dim M$. Let $T_k^r M$ be the manifold of all k^r-velocities, i.e. the space of all r-jets of mappings $R^k \to M$ with the source $O \in R^k$. It will be said that a geometric object field $\sigma: M \to T_k^r M$, i.e. a crosssection σ of the fibre bundle β: $T_k^r M \to M$, has the singularity of order s (shortly s-singularity) at $x \in M$ if $j_x^s \sigma = j_x^s O_k^r$, where O_k^r: $M \to T_k^r M$ is such a geometric object field that for any $x \in M$ $O_k^r(x) = j_0^r \gamma$, $\gamma(y) = x$ for any $y \in R^k$. In this paper we study higher order singularities of some geometric object fields. Our considerations are in the category C^∞.

1. Let (x^i) be a local chart on M. Let $x = a^i(x)\partial/\partial x^i$ be a vector field on M. Then X: $M \to TM$ has r-singularity at $(x_0) \in M$ if and only if

$$a^i(x_0)=0, \quad \partial_j a^i(x_0)=0, \ldots, \partial_{j_1 \ldots j_r} a^i(x_0)=0,$$

where $\partial_{j_1 \ldots j_p}$ denotes $\partial/\partial x_{j_1 \ldots j_p}$.

Let $(x^i, x_\lambda^i, \ldots, x_{\lambda_1 \ldots \lambda_r}^i)$ be a local chart on $T_k^r M$. Let $X^{(r)}$ be the r-prolongation of X on $T_k^r M$. Locally

$$X^{(r)} = a^i \partial/\partial x^i + \partial_j(a^i)\, x_\lambda^j \partial/\partial x_\lambda^i + \ldots +$$

$$+ (\partial_{j_1 \ldots j_r}(a^i) x_{\lambda_1}^{j_1} \ldots x_{\lambda_r}^{j_r} + \ldots + \partial_j(a^i) x_{\lambda_1 \ldots \lambda_r}^j) \partial/\partial x_{\lambda_1 \ldots \lambda_r}^i.$$

It immediately gives

Proposition 1. *The field X has at $x \in M$ the r-singularity if and only if $X^{(r)}(h)=0$ for any $h \in (T_k^r M)_x$.*

Let π: $E \to M$ be a fibre manifold. Let (x^i, y^α) be a local chart on E. Let $X = a^i(x)\, \partial/\partial x^i + b^\alpha(x, y)\, \partial/\partial y^\alpha$ be a projectable tangent vector field on E. Let $X^{(r)}$ be the r-prolongation of X on $J^r E$. Locally, for example,

$$X^{(1)} = a^i \partial/\partial x^i + b^\alpha \partial/\partial y^\alpha + (\partial_\beta(b^\alpha) y_i^\beta - y_k^\alpha \partial_i(a^k) + \partial_i(b^\alpha))\partial/\partial y_i^\alpha, \tag{1}$$

where $(x^i, y^\alpha, y_i^\alpha)$ is a local chart on $J^1 E$. It is obvious that if X has at $(x_0, y_0) = e \in E$ the r-singularity, then $X^{(r)}(h)=0$ for any $h \in (J^r E)_u$, $\beta h = u$. Conversely this is not true.

Let $\Gamma: E \to J^1E$ be a generalized connection on E (see for example [4]). Denote by Γ_u the horizontal tangent subspace determined by Γ at $u \in E$, $T_uE = T_uE_x \oplus \Gamma_u$. Let $X \in T_uE$. Then $X = vX + hX$, $vX \in T_uE_x$, $hX \in \Gamma_u$, $\pi u = x$. Let φ, or ψ, be the morphism of Γ, or the curvature morphism of Γ, i.e. $\varphi(X) = vX$, $\psi(X, Y) = \varphi([\tilde{X}, \tilde{Y}])$, where $\tilde{X}, \tilde{Y}$ are such horizontal tangent vector fields that $\tilde{X}(u) = h(X)$, $\tilde{Y}(u) = h(Y)$. Locally, let

$$\Gamma: (x^i, y^\alpha) \mapsto (x^i, y^\alpha, y_i^\alpha = a_i^\alpha(x^j, y^\beta)).$$

Then

(2)
$$\varphi = (dy^\alpha - a_k^\alpha(x, y)dx^k) \otimes \partial/\partial y^\alpha,$$
$$\psi = (\partial_\beta(a_k^\alpha)a_j^\beta + \partial_j(a_k^\alpha))dx^k \wedge dx^j \otimes \partial/\partial y^\alpha.$$

Let $X = a^i(x, y)\partial/\partial x^i + b^\alpha(x, y)\partial/\partial y^\alpha$ be a tangent vector field on E. Let $\mathcal{L}_X\varphi$ denote the Lie derivative of φ by X. Locally

(3)
$$\mathcal{L}_X\varphi = (a_k^\alpha dx^k - dy^\alpha)\partial_\alpha(a^i) \otimes \partial/\partial x^i +$$
$$+ \{\partial_\beta(b^\alpha)a_k^\beta + \partial_k(b^\alpha) - \partial_j(a_k^\alpha)a^j - \partial_\beta(a_k^\alpha)b^\beta - a_i^\alpha\partial_k(a^i))dx^k - a_k^\alpha\partial_\beta(a^k)dy^\beta\} \otimes \partial/\partial y^\alpha.$$

If X has the 1-singularity at $u \in E$, then $(\mathcal{L}_X\varphi)_u = 0$. The field X will be said to be (Γ, r)-singular at $u \in E$ if the field $\varphi(X)$ has the r-singularity at u. Recall that every horizontal tangent vector field on E is (Γ, r)-singular at any $u \in E$ and for any integer $r \geqq 0$. By [2] X is conjugate with Γ at $u \in E$ if $(\mathcal{L}_X\varphi)_u = 0$. Let Y be a tangent vector field on E. Denote by $i_Y\psi$ the morphism determined by $i_Y\psi(Z) = \psi(Y, Z)$.

Lemma 1. *Let X be a projectable tangent vector field on E. Let X be $(\Gamma, 1)$-singular at $u \in E$. Then X is conjugate with Γ at u if and only if $i_X\psi$ vanishes at u.*

Proof. Every field $X = a^i(x)\partial/\partial x^i + b^\alpha(x, y)\partial/\partial y^\alpha$ is $(\Gamma, 1)$-singular at $u = (x_0, y_0)$ if and only if

$$b^\alpha(x_0, y_0) = a_i^\alpha(x_0, y_0)a^i(x_0),$$
$$\partial_\beta(b^\alpha(x_0, y_0)) = \partial_\beta(a_i^\alpha(x_0, y_0))a^i(x_0),$$
$$\partial_k(b^\alpha(x_0, y_0)) = \partial_k(a_i^\alpha(x_0, y_0))a^i(x_0) + a_i^\alpha(x_0, y_0)\partial_k(a^i(x_0)).$$

Then the relation (2) and (3) complete our proof.

Lemma 1 gives

Proposition 2. *Let ψ be the curvature morphism of Γ. Then ψ vanishes at $u \in E$ if and only if every at u $(\Gamma, 1)$-singular projectable tangent vector field is conjugate with Γ at u.*

Proposition 3. *Let X be a projectable tangent vector field on E which is $(\Gamma, 1)$-singular at $u \in E$. Let $X(u) = 0$. Then $X^{(1)}(\Gamma(u)) = 0$.*

The proof follows from (1) and (4).

Let $p\colon F\to E$ be a vector bundle over the fibre bundle E.• Let $\tau\colon E\to F$ be a cross-section. τ will be said to be vertically r-singular at $u\in E$ if $\tau|_{E_x}$, $x=\pi u$, has the r-singularity at u. Denote by $b\colon J^1E\to E$ the fibre projection. Let us recall that $b\colon J^1E\to E$ is an affine fibre bundle associate with the vector bundle $VTE\otimes T^*M$, where VTE denotes the fibre bundle of vertical tangent vectors on E. Therefore every b-vertical tangent vectors on E. Therefore every b-vertical tangent vector $Z\in T_hJ^1E$ determines $\bar{Z}\in(VTE\otimes T^*M)_{b(h)}$.

Proposition 4. *Let a vertical tangent vector field X on E vanishe at $u\in E$. Then X is vertically 1-singular at u if and only if $\overline{X^{(1)}(h_1)}=\overline{X^{(1)}(h_2)}$ for any such h_1, $h_2\in J^1E$ that $b(h_1)=b(h_2)=u$.*

Proof. Let $X=b^\alpha(x,y)\ \partial/\partial y^\alpha$, $b^\alpha(u)=0$. Then (1) gives $X^{(1)}(h)=[\partial_\beta(b^\alpha(u))y_i^\beta+\partial_i(b^\alpha(u))]\partial/\partial y_i^\alpha$, $b(h)=u$. In our case X is vertically 1-singular at u if and only if $\partial_\beta(b^\alpha(u))=0$. This gives our assertion.

Let Γ, $\tilde{\Gamma}$ be generalized connections on E. It will be said that Γ and $\tilde{\Gamma}$ have at $u\in E$ the r-contact, or the vertical r-contact if $j_u^r\Gamma=j_u^r\tilde{\Gamma}$, or $j_u^r(\Gamma|_{E_{\pi u}})=j_u^r(\tilde{\Gamma}|_{E_{\pi u}})$, respectively. It is known that Γ and $\tilde{\Gamma}$ determine the cross-section $(\tilde{\Gamma}-\Gamma)$: $E\to VTE\otimes T^*M$. Obviously, the connections Γ and $\tilde{\Gamma}$ have the r-contact, or the vertical r-contact, if and only if $(\tilde{\Gamma}-\Gamma)$ is r-singular, or vertically r-singular.

Proposition 5. *Let $\Gamma^{(1)}$, $\tilde{\Gamma}^{(1)}\colon E\to\bar{J}^2E$ be the first prolongations of Γ, $\tilde{\Gamma}$. Let $(\tilde{\Gamma}-\Gamma)\colon E\to VTE\otimes T^*M$ be vertically 1-singular at $u\in E$. Then Γ and $\tilde{\Gamma}$ have the 1-contact at u if and only if $\Gamma^{(1)}(u)=\tilde{\Gamma}^{(1)}(u)$.*

Proof. Locally, Γ: $(x^i,y^\alpha)\mapsto(x^i,y^\alpha,y_i^\alpha=a_i^\alpha(x,y))$, $\tilde{\Gamma}$: $(x^i,y^\alpha)\mapsto(x^i,y^\alpha,y_i^\alpha=\tilde{a}_i^\alpha(x,y))$, $(\tilde{\Gamma}-\Gamma)\colon(x^i,y^\alpha)\mapsto(\tilde{a}_i^\alpha(x,y)-a_i^\alpha(x,y))\,\mathrm{d}x^i\otimes\partial/\partial y^\alpha$. Then $(\tilde{\Gamma}-\Gamma)$ is vertically 1-singular at u if and only if $\tilde{a}_i^\alpha(u)=a_i^\alpha(u)$, $\partial_\beta(\tilde{a}_i^\alpha(u))=\partial_\beta(a_i^\alpha(u))$. Then using the local relation

$$\Gamma^{(1)}\colon(x^i,y^\alpha)\mapsto(x^i,y^\alpha,y_i^\alpha=a_i^\alpha,\ y_{ik}^\alpha=\partial_k(a_i^\alpha)+\partial_\beta(a_i^\alpha)a_k^\beta)$$

we get our assertion.

2. Let the Lie group G act on M. Denote by $x\cdot g$, $x\in M$, $g\in G$, the action of G on M. Then G acts on T_k^rM by $u\cdot g=j_0^r(\gamma\cdot g)$, $u=j_0^r\gamma\in T_k^rM$. Let us recall (see [3]) that the isotropy group of order r at $x\in M$ is the Lie group $G_x^r=\{g\in G,\ j_x^rg=j_x^rg=j_x^r\,\mathrm{id}_M\}$.

Lemma 2. *Let $g\in G$, $x\in M$. Then $g\in G_x^r$ if and only if $u\cdot g=u$ for any $u\in T_k^rM$, $\beta u=x$, $k\geqq 1$.*

Proof. Let $(x^i,x_\lambda^i,\dots,x_{\lambda_1\dots\lambda_r}^i)$ be a local chart on T_k^rM. Let $j_x^rg=(x^i,\bar{x}^i,g_j^i,\dots,g_{j_1\dots j_r}^i)$, where $g\colon M\to M$, $g(m)=m\cdot g$. Let $u\cdot g=u$. Using the composition of the jets u and g, we have

$$
\begin{gathered}
x^{i}=x^{i}\\
g_{j}^{i}x_{\lambda}^{j}=x_{\lambda}^{i},\\
g_{j_1j_2}^{i}x_{\lambda_1}^{j_1}x_{\lambda_2}^{j_2}+g_{j}^{i}x_{\lambda_1\lambda_2}^{j}=x_{\lambda_1\lambda_2}^{i}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
g_{j_1\dots j_r}^{i}x_{\lambda_1}^{j_1}\dots x_{\lambda_r}^{j_r}+\dots+g_{j}^{i}x_{\lambda_1\dots\lambda_r}^{j}=x_{\lambda_1\dots\lambda_r}^{i}.
\end{gathered}
$$

These relations are true for any u, $\beta u=x$, if and only if $g_j^i=\delta_j^i$, $g_{j_1j_2}^i=0$, $\dots$, $g_{j_1\dots j_r}^i=0$, that is if and only if $j_x^r g=j_x^r\ \mathrm{id}_M$.

Lemma 3. *Let $g\in G$. Then $g\in G_x^r$ if and only if there is such $u\in H^rM$ that $\beta u=x$ and $u\cdot g=u$.*

Proof. By Lemma 2, if $g\in G_x^r$, then $u\cdot g=u$ for $u\in(H^rM)_x$. Conversely, let there be such a $u\ H_x^rM$ that $u\cdot g=u$. Then (u is invertible) $(u\cdot g)u^{-1}=uu^{-1}$, that is $g=j_x^r\ \mathrm{id}_M$.

Let e be the unit of G. Let $c\in(T_s^rG)_e$, $c=j_0^r\xi$. Then the geometric object field

$$x\mapsto j_0^r(x\cdot\xi)\in T_s^rM$$

will be said to be a (G,r,s) — object on M.

Lemma 4. *Let σ be a (G,q,s) object on T_k^rM determined by $c\in(T_s^qG)_e$. Then $c\in T_s^q(G_x^r)$ if and only if there is such a $u\in(T_k^rM)_x$ that $\sigma(u)=O_s^q(u)$.*

Proof. If $c=j_0^q\xi\in T_s^q(G_x^r)_e$ and $u\in(T_k^rM)_x$, then $\sigma(u)=j_0^q(u\cdot\xi)=O_s^q(u)$. Conversely, let $\sigma(u)=O_s^q(u)$. Let $\beta^1c=(T_s^1G)_e$ be the 1-subjet of c. Let $c\notin T_s^q(G_x^r)_e$. Then $\beta^1c\notin T_s^1(G_x^r)_e$. Denote by $T(\beta^1c)\subset T_eG$ the tangent subspace determined by β^1c. Then there is such an $X\in T(\beta^1c)$ that $X\notin T_eG_x^r$. Let X be the fundamental tangent vector field on T_k^rM generated by X. Obviously $\bar X(u)\in T\beta^1\sigma(u)$. Therefore $\bar X(u)=0$. Let us recall (see for example 3.1 of [1]) that if $\bar X(u)=0$, then $X\in T_eG_u$, where G_u denotes the isotropy group of u. By Lemma 2 we get: G_x^r is the isotropy group of any $h\in(T_k^rM)_x$. Hence $X\in T_eG_x^r$. Therefore $c\in T_s^q(G_x^r)$.

Let (z^α) be a local chart on M, $e=(0,\dots,0)$. Let $\bar x^i=f^i(x^i,z^\alpha)$ be the equations of the action of G on M. Let $c=(c_\lambda^\alpha, c_{\lambda_1\lambda_2}^\alpha,\dots,c_{\lambda_1\dots\lambda_q}^\alpha)\in(T_s^qG)_e$. Then

$$(x^i)\mapsto(x^i,\partial_\alpha(f^i(x,e))c_\lambda^\alpha,\dots,\partial_{\alpha_1\dots\alpha_q}(f^i(x,e))c_{\lambda_1}^{\alpha_1}\dots c_{\lambda_q}^{\alpha_q}+\dots+\partial_\alpha(f^i(x,e))c_{\lambda_1\dots\lambda_q}^\alpha)$$

is the (G,q,s)-object η on M determined by c. Locally, η has the r-singularity at (x_0) if and only if

$$
\begin{gathered}
(5)\qquad \partial_\alpha(f^i(x_0,e)c_\lambda^\alpha=0,\dots,\partial_{\alpha_1\dots\alpha_q}(f^i)c_{\lambda_1}^{\alpha_1}\dots c_{\lambda_q}^{\alpha_q}+\dots+\partial_\alpha(f^i)c_{\lambda_1\dots\lambda_q}^\alpha=0,\\
\partial_\alpha\partial_k(f^i)c_j^\alpha=0,\dots,\partial_{\alpha_1\dots\alpha_q}\partial_k(f^i)c_{\lambda_1}^{\alpha_1}\dots c_{\lambda_q}^{\alpha_q}+\dots+\partial_\alpha\partial_k(f^i)c_{\lambda_1\dots\lambda_q}^\alpha=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\partial_\alpha\partial_{k_1\dots k_r}(f^i)c_\lambda^\alpha=0,\dots,\partial_{\alpha_1\dots\alpha_q}\partial_{k_1\dots k_r}(f^i)c_{\lambda_1}^{\alpha_1}\dots c_{\lambda_q}^{\alpha_q}+\dots+\partial_\alpha\partial_{k_1\dots k_r}(f^i)c_{\lambda_1\dots\lambda_q}^\alpha=0.
\end{gathered}
$$

Let $h=(x^i,x_t^i,\dots,x_{t_1\dots t_r}^i)\in T_k^r(M)$, $u=j_0^r\varphi$. Let $g=(g^\alpha)\in G$. Then

(6) $$h \cdot g = j_0^r((v^t) \mapsto f^i(\varphi^k(v^t), g^\alpha)) = (f(x^i, g^\alpha),$$
$$\partial_j(f^i(x^i, g^\alpha))x^j_t, \ldots, \partial_{j_1 \ldots j_r}(f^i(x^i, g^\alpha))x^{j_1}_{t_1} \ldots x^{j_r}_{t_r} + \ldots + \partial_j(f^i(x^j, g^\alpha))x^j_{t_1 \ldots t_r}).$$

Calculating the coordinate form of the (G, q, s)-object on $T^r_k M$ determined by $c = j^q_0 \xi \in (T^q_s G)_e$ we get

Proposition 6. *Let ξ be a (G, q, s)-object on M determined by $c \in (T^q_s G)_e$. Then ξ has the singularity of order r at $x \in M$ in and only if $\sigma(u) = O^q_s(u)$ for any $u \in (T^r_k M)_x$, where σ is the (G, q, s)-object on $T^r_k M$ determined by c.*

Now Proposition 6 and Lemma 4 give

Proposition 7. *Let a (G, q, s)-object ξ on M be generated by $c \in (T^q_s G)_e$. Then ξ has at $x \in M$ the singularity of the order r if and only if $c \in (T^q_s G^r_x)_e$.*

Using (6) we have

Proposition 8. *A subgroup $H \subset G$ is the isotropy group of order r at $x \in M$ if and only if H is the isotropy group of order $r - q$ at any $u \in (T^q_s M)_x$, $q \leqq r$.*

Let Φ be a Lie groupoid of operators on the fibre manifold E. Let us recall (see [4]) that every section ξ of the Lie algebroid depl Φ determines the tangent vector field X on E. It follows from Proposition 7 that X is vertically r-singular at $u \in E_x$, $\pi u = x$, if and only if $\xi(x) \in T_e(G_x)_h$, where $(G_x)_h \subset \Phi$ is the isotropy group of the order r at u. Let γ_1, γ_2 be connections on Φ. Let Γ_1, or Γ_2, be the connection on E determined by γ_1, or γ_2, respectively. By Proposition 7, Γ_1 and Γ_2 have the vertical r-contact at $u \in E$ if and only if $\gamma_2 - \gamma_1 \in T(G_x)_u \otimes T^*_x M$, $\pi u = x$.

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ОСОБВЕННОСТИ ВЫСШЕГО ПОРЯДКА НЕКОТОРЫХ ПОЛЕЙ ГЕОМЕТРИЧЕСКИХ ОБЪЕКТОВ

Антон Декрет

Резюме

Пусть M дифференцируемое многообразие. Пусть T_k^r пространство всех r-струей отображений $R^k \to M$ с началом в $O \in R^k$. Поле геометрических объектов $\sigma\colon M \to T_k^r M$ имеет в точке $x \in M$ p-особенность, если $j_x^p \sigma = j_x^p O_k^r$, где $O_k^r\colon M \to T_k^r M$ такой геометрический объект, что $O_k^r(y) = j_0^r \gamma$, $\gamma(z) = y$ для каждого $z \in R^k$. В статье найдены некоторые достаточные и необходимые условия для p-особенностей некоторых полей геометрических объектов.