Joseph Neggers; Hee Sik Kim

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ON d-ALGEBRAS

J. NEGGERS* — HEE SIK KIM**

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ABSTRACT. In this paper we introduce the notion of d-algebras which is another generalization of BCK-algebras, and investigate several relations between d-algebras and BCK-algebras. Furthermore, we show that the class of oriented digraphs corresponds in a simple way to the class of edge d-algebras and that arbitrary d-algebras also determine unique edge d-algebras in a natural manner.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([1], [2]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3], [4] Q. P. Hu and X. Li introduced a broad class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCK-algebras also have some connections with other areas: D. Mundici [6] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [5] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. We introduce the notion of d-algebras, which is another useful generalization of BCK-algebras, and then we investigate several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs.

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2. d-algebras

A *d-algebra* is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all x, y in X.

A *BCK-algebra* is a d-algebra $(X; *, 0)$ satisfying the following additional axioms:

(IV) $\big((x * y) * (x * z)\big) * (z * y) = 0$,
(V) $\big(x * (x * y)\big) * y = 0$ for all x, y, z in X.

EXAMPLE 2.1.

(a) Every BCK-algebra is a d-algebra.

(b) Let $X := \{0, 1, 2\}$ be a set with the following Table 1.

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Table 1.

Then $(X; *, 0)$ is a d-algebra, but not a BCK-algebra, since $\big(2 * (2 * 2)\big) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$.

(c) Let $\mathbb{R}$ be the set of all real numbers and define $x * y := x \cdot (x - y)$, $x, y \in \mathbb{R}$, where $\cdot$ and $-$ are the ordinary product and substraction of real numbers. Then $x * x = 0$, $0 * x = 0$, $x * 0 = x^2$. If $x * y = y * x = 0$, then $x(x - y) = 0$ and $x^2 = xy$, $y(y - x) = 0$, $y^2 = xy$. Thus if $x = 0$, $y^2 = 0$, $y = 0$; if $y = 0$, $x^2 = 0$, $x = 0$ and if $xy \neq 0$, then $x = y$. Hence $(\mathbb{R}; *, 0)$ is a d-algebra, but not a BCK-algebra, since $(2 * 0) * 2 \neq 0$.

Remark.

1. If a d-algebra $(X; *, 0)$ is associative, then $0 * x = 0 = (x * x) * x = x * (x * x) = x * 0$, and thus by (III) $x = 0$, i.e., d-algebras are the "*most non-associative*" algebras.

2. Let $(X; *, 0)$ be a d-algebra. If $S \subseteq X$ is closed under $*$, then $x \in S$ implies $x * x = 0 \in S$, so that $(S; *, 0)$ is a d-algebra.

DEFINITION 2.2. Let $(X; *, 0)$ be a d-algebra and $x \in X$. Define $x * X := \{x * a \mid a \in X\}$. X is said to be *edge* if for any x in X, $x * X = \{x, 0\}$.

Remark. If $(X, \leq)$ is an ordered set (poset), then the operation $*$ on X given by $x*y = 0$ if and only if $x \leq y$ and $x*y = x$ otherwise defines a BCK-algebra. On the other hand, from our viewpoint it has the "edge" property. Although edge d-algebras are not in general BCK-algebras, they come close to being so, as we note below.

LEMMA 2.3. *Let $(X;*,0)$ be an edge d-algebra. Then $x*0 = x$ for any $x \in X$.*

Proof. Since $(X;*,0)$ is an edge d-algebra, either $x*0 = x$ or $x*0 = 0$ for any $x \in X$. If $x \neq 0$ and $x*0 = 0$, then by (III) $x = 0$, a contradiction. □

PROPOSITION 2.4. *If $(X;*,0)$ is an edge d-algebra, then the condition* (V) *holds.*

Proof. If $x = 0$, then $\big(x*(x*y)\big)*y = 0$ by (II). Let $x \neq 0$. Assume $\big(x*(x*y)\big)*y \neq 0$ for some $y \in X$. Let $\alpha := x*(x*y)$. Then $\alpha * y \neq 0$ and $\alpha \neq 0$. This means that $x \neq x*y \in x*X = \{x, 0\}$ and hence $x*y = 0$. It follows that, by Lemma 2.3, $\big(x*(x*y)\big)*y = (x*0)*y = x*y = 0$, a contradiction. □

DEFINITION 2.5. A d-algebra $(X;*,0)$ is said to be *d-transitive* if $x*z = 0$ and $z*y = 0$ imply $x*y = 0$.

THEOREM 2.6. *Let $(X;*,0)$ be a d-transitive edge d-algebra. Then $(X;*,0)$ is a BCK-algebra.*

Proof. By Proposition 2.4, it is enough to show that condition (IV) holds. Assume that $\big((x*y)*(x*z)\big)*(z*y) \neq 0$ for some $x, y, z \in X$. Since $(x*y)*(x*z) \in (x*y)*X = \{x*y, 0\}$,

$$(x*y)*(x*z) = x*y. \tag{a}$$

If $x*y = 0$, then $0 \neq \big((x*y)*(x*z)\big)*(z*y) = \big(0*(x*z)\big)*(z*y) = 0*(z*y) = 0$, a contradiction. It follows that

$$x*y = x. \tag{b}$$

Hence

$$\begin{aligned} x &= x*y && \text{[by (b)]} \\ &= (x*y)*(x*z) && \text{[by (a)]} \\ &= x*(x*z) && \text{[by (b)]} \end{aligned}$$

that is,

$$x = x*(x*z). \tag{c}$$

If $x * z \neq 0$, then $x * z = x$, since X is an edge d-algebra. By applying (III), $x = x * (x * z) = x * x = 0$. This means that

$$\begin{aligned} 0 &\neq \big((x * y) * (x * z)\big) * (z * y) \\ &= (x * x) * (z * y) && \text{[by (b) and } x * z = x\text{]} \\ &= 0 * (z * y) \\ &= 0\,, \end{aligned}$$

a contradiction. Thus we conclude

$$x * z = 0\,. \tag{d}$$

We claim that $z * y = 0$. If $z * y = z$, then

$$\begin{aligned} 0 &\neq \big((x * y) * (x * z)\big) * (z * y) \\ &= \big((x * y) * 0\big) * z && \text{[by (d) and } z * y = z\text{]} \\ &= (x * y) * z && \text{[by Lemma 2.3]} \\ &= x * z && \text{[by (b)]} \\ &= 0\,, && \text{[by (d)]} \end{aligned}$$

a contradiction. Thus we obtain that $x * z = 0$ and $z * y = 0$. Since X is d-transitive, $x * y = 0$, and hence $0 \neq \big((x * y) * (x * z)\big) * (z * y) = 0$, a contradiction. This proves the theorem. □

Remark. Both conditions, i.e., to have the d-transitive and edge properties, are necessary for a d-algebra of this type to be a BCK-algebra. Thus, arbitrary BCK-algebras do not always have the edge property even if the standard examples derived from posets do indeed possess it.

EXAMPLE 2.7. Consider the following d-algebra X with the Table 2.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Table 2.

We can easily see that $1 * 2 = 0$, $2 * 3 = 0$, but $1 * 3 = 1$, and hence $(X; *, 0)$ is non-d-transitive edge d-algebra. Since $\big((1 * 3) * (1 * 2)\big) * (2 * 3) = 1 \neq 0$, $(X; *, 0)$ is not a BCK-algebra.

EXAMPLE 2.8. Let $X := \{0, 1, 2, \dots\}$ and let the binary operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if } x \le y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $x * z = 0$, $z * y = 0$ implies $x \le z$, $z \le y$ and in particular $x \le y$, i.e., $x * y = 0$ also. Furthermore, $x * x = 0$, $0 * x = 0$ and $x * y = y * x = 0$ if $x \le y$, $y \le x$, whence $x = y$. Thus, the algebra $(X; *, 0)$ is a d-transitive non-edge d-algebra. Also, $\big(2 * (2 * 0)\big) * 0 = (2 * 1) * 0 = 1 * 0 = 1$, so that $(X; *, 0)$ is not a BCK-algebra.

3. Construction of edge d-algebras

Suppose that $(X; *, 0)$ is an arbitrary d-algebra. Assume that $(X; *, 0)$ is not an edge d-algebra. Define a binary operation $\oplus \colon X \times X \to X$ by

$$x \oplus y := \begin{cases} x & \text{if } x * y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can see easily that $(X; \oplus, 0)$ is a d-algebra. Suppose now that $x \oplus X = \{0\}$. Then $x * y = 0$ for all $y \in X$. In particular, $x * 0 = 0 = 0 * x$, so that also $x = 0$. Hence, if $x \neq 0$, then $x \oplus X = \{x, 0\}$. We summarize:

THEOREM 3.1. *Given a d-algebra $(X; *, 0)$ we can construct an edge d-algebra $(X; \oplus, 0)$, called the extended edge d-algebra.*

PROPOSITION 3.2. *A d-algebra $(X; *, 0)$ is d-transitive if and only if its extended edge d-algebra $(X; \oplus, 0)$ is d-transitive.*

Proof. If $(X; *, 0)$ is d-transitive then $x \oplus z = 0$ and $z \oplus y = 0$ imply $x * z = 0 = z * y$, so that $x * y = 0$ and $x \oplus y = 0$ as well. Conversely, if $(X; \oplus, 0)$ is d-transitive, then $x * z = 0$ and $z * y = 0$ imply $x \oplus z = 0 = z \oplus x$, so that $x \oplus y = 0$ and $x * y = 0$ as well. □

EXAMPLE 3.3. There are 27 d-algebras as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	u
b	b	v	0	0
c	c	0	w	0

where $u, v, w \in \{a, b, c\}$. All of these algebras have the same unique edge d-algebra as follows:

$\oplus$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	0	c	0

This d-algebra is not d-transitive since $a \oplus b = b \oplus c = 0$, while $a \oplus c = a \neq 0$. It also has the following *d-chain property*: $x \oplus y \neq 0$ implies $y \oplus x = 0$.

Properties of edge d-algebras are properties of d-algebras having the edge property. This may be a useful observation in the reduction of certain questions from d-algebras to the simpler situation where one has to deal with edge d-algebras.

Let Γ' be a digraph such that if (x, y) is an edge, then (y, x) is not an edge. Such a digraph Γ' is said to be *oriented*. Let 0 be adjoined to Γ' and denote by Γ the oriented digraph with edges $(0, x)$, $x \in \Gamma'$ added to those of Γ'. On Γ define the operation $*$ by $x * x = 0$; $x * 0 = x$; $0 * x = 0$ and $x * y = x$ if (x, y) is not an edge, $x * y = 0$ if (x, y) is an edge. Then $(\Gamma; *, 0)$ is an edge d-algebra. We summarize:

THEOREM 3.4. *Every edge d-algebra $(X; *, 0)$ produces an oriented digraph $\Gamma = \Gamma' \cup \{0\}$ and conversely.*

4. Direct sum(product) of d-algebras

Let $\{(X_i; *, 0_i) \mid i \in I\}$ be a non-empty family of d-algebras and let $\prod_{i \in I} X_i$ consist of all vectors $(x_i)_{i \in I}$, $x_i \in X_i$. Then $(x_i = 0_i)_{i \in I} = 0$ serves as 0 if we define $(x_i)_{i \in I} * (y_i)_{i \in I} := (x_i * y_i)_{i \in I}$ and $\left(\prod_{i \in I} X_i; *, 0\right)$ is a d-algebra, called the *direct product* of the d-algebras $\{(X_i; *, 0) \mid i \in I\}$. Similarly, $\bigoplus_{i \in I} X_i$ consisting of all vectors $(x_i)_{i \in I}$, $x_i \in X_i$, such that $x_i = 0_i$ except for a finite number of i, is a subset of $\prod_{i \in I} X_i$ which is closed under $*$, whence $\left(\bigoplus_{i \in I} X_i; *, 0\right)$ is a d-algebra, called the *direct sum* of the d-algebras $\{(X_i; *, 0) \mid i \in I\}$. Let $(X; *, 0)$ and $(Y; *, 0)$ be d-algebras. A mapping $f: X \to Y$ is called a

d-morphism if $f(x*y)=f(x)*f(y)$ for any $x,y\in X$. Note that $f(0_X)=0_Y$. Using this concept we study some edge properties.

PROPOSITION 4.1. *Let $f\colon (X;*,0)\to(Y;*,0)$ be an onto d-morphism and let $(X;*,0)$ be an edge d-algebra. Then $(Y;*,0)$ is also an edge d-algebra.*

Proof. Consider $y=f(x)$, $b=f(a)$. Then $y*b=f(x)*f(a)=f(x*a)\in\{f(x),f(a)\}=\{y,0\}$, whence the conclusion follows. □

Even though $(X;*,0)$ and $(Y;*,0)$ are edge d-algebras, their direct sum $X\oplus Y$ need not have the edge property.

Let $x\in X$ and $y\in Y$, and let $x*a=x$, $y*b=0$ for some $a\in X$ and $b\in Y$. Then $(x,y)*(a,b)=(x*a,y*b)=(x,0)\notin\{(x,y),(0,0)\}$ if $y\neq 0$. In order for a Cartesian product of two d-algebras to have the edge property, we introduce a new binary operation $\circledast$. Let $(X;*,0)$ and $(Y;*,0)$ be d-algebras. Define the binary operation $\circledast$ on $X\times Y$ as follows: $(x,y)\circledast(a,b):=(x,y)$ unless $x*a=0=y*b$, when $(x,y)\circledast(a,b):=(0,0)$. Then we can easily see that $(X\times Y;\circledast,0_{X\times Y})$ is an edge d-algebra, denoted by $X\circledast Y$, and called the *edge product* of d-algebras $(X;*,0)$ and $(Y;*,0)$. Given $X\oplus Y$ and $X\circledast Y$, there are inclusion mappings ι_X and ι_Y, and projections π_X and π_Y. Now, $\iota_X(x*a)=(x*a,0)=(x,0)*(a,0)=\iota_X(x)*\iota_X(a)$. Similarly, $\iota_Y(y*b)=\iota_Y(y)*\iota_Y(b)$. Moreover, $\pi_X(x*a,y*b)=x*a=\pi_X(x,y)*\pi_X(a,b)$. Similarly, $\pi_Y(x*a,y*b)=\pi_Y(x,y)*\pi_Y(a,b)$. We summarize:

PROPOSITION 4.2. *The inclusion mappings and projections relative to $X\oplus Y$ are d-morphisms.*

THEOREM 4.3. *Let $(X;*,0)$ and $(Y;*,0)$ be d-algebras. Then X (or Y, respectively) is an edge d-algebra if and only if the inclusion mapping ι_X (or ι_Y, respectively) is a d-morphism relative to $X\circledast Y$.*

Proof. Suppose ι_X is a d-morphism relative to $X\circledast Y$. Then $(x*a,0)=\iota_X(x*a)=\iota_X(x)\circledast\iota_X(a)=(x,0)\circledast(a,0)$, and hence $x*a\in\{x,0\}$ for any $a\in X$. This means $x*X=\{x,0\}$ for all $x\in X$. Thus X is an edge d-algebra. Similarly, if ι_Y is a d-morphism relative to $X\circledast Y$, then Y is an edge d-algebra. Conversely, assume X is an edge d-algebra. Consider the inclusion mapping ι_X relative to $X\circledast Y$. Then $\iota_X(x*a)=(x*a,0)=(x,0)$ or $(0,0)$, and $\iota_X(x)\circledast\iota_X(a)=(x,0)\circledast(a,0)=(x,0)$ or $(0,0)$ both according as to $x*a=x$ or $x*a=0$. Thus ι_X is a d-morphism. Similarly, if Y is an edge d-algebra, then ι_Y is a d-morphism. □

Since $X\circledast Y$ is an edge d-algebra, the following proposition is an immediate consequence of Proposition 4.1.

PROPOSITION 4.4. *If the projection π_X (or π_Y, respectively) is a d-morphism relative to $X \circledast Y$, then X (or Y, respectively) is an edge d-algebra.*

Remark. *Even though X and Y are edge d-algebras, the projections π_X and π_Y relative to $X \circledast Y$ need not be d-morphisms.*

Indeed, suppose that $y * 0 = y \neq 0$. Then $\pi_X\big((x,y) \circledast (a,0)\big) = \pi_X(x,y) = x$. On the other hand, $\pi_X(x,y) * \pi_X(a,0) = x * a$, so that if $x * a = 0$, then $x \neq 0$ implies $\pi_X\big((x,y) \circledast (a,0)\big) \neq \pi_X(x,y) * \pi_X(a,0)$, i.e., π_X is not a d-morphism, nor is π_Y a d-morphism.

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* *Department of Mathematics*
University of Alabama
Tuscaloosa, AL 35487-0350
U. S. A.
E-mail: jneggers@gp.as.ua.edu

** *Department of Mathematics*
Hanyang University
Seoul 133-791
KOREA
E-mail: heekim@email.hanyang.ac.kr