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STRONG LAWS OF LARGE NUMBERS FOR DOUBLE SEQUENCES OF RANDOM ELEMENTS

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ABSTRACT. Strong laws of large numbers are proved for double sequences of independent random elements in a real separable Rademacher type p Banach space under various conditions on the random elements and on the behaviour of norming constants. The results obtained are new even for real valued random elements.

1. Introduction

Let (Ω, S, P) be a probability space, B a real separable Banach space. A random element X in B is an S-measurable transformation from Ω to $(B, \mathbb{B})$, where $\mathbb{B}$ denotes the Borel σ-algebra of subsets of B. The expected value of X is defined to be the *Pettis integral.*

Let $\{a_{mn}\}$ and $\{b_{mn}\}$ be double sequences of constants with $a_{mn} \neq 0$ for all m, n and $0 < b_{mn} \to \infty$ as $m, n \to \infty$.

Let $\{X_{mn}\}$ be a double sequence of random elements in B with the expected values EX_{mn}. Then the sequence $a_{mn}(X_{mn} - EX_{mn})$ is said to obey the *strong law of large numbers with norming constants* b_{mn} if and only if the weighted sums $\frac{1}{b_{mn}} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} a_{ij}(X_{ij} - EX_{ij})$ converge to the zero in norm of B with probability 1. This will be written

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(X_{ij} - EX_{ij}) \to 0 \quad \text{a.c.}$$

The aim of the paper is to obtain strong laws of large numbers under various conditions on the constants a_{mn}, b_{mn}, the random elements X_{mn} as well as on Banach space B.

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We recall some basic definitions used throughout this paper.

DEFINITION 1. Let $\{Y_n\}$ be i.i.d. random variables with $P\{Y_1 = \pm 1\} = \frac{1}{2}$. Let B be a real separable Banach space. Put $B^\infty = B \times B \times \dots$ and define

$$C(B) = \left\{ \{x_n\}_{n=1}^{\infty} \in B^\infty : \sum_{n=1}^{\infty} Y_n x_n \text{ converges in probability} \right\}.$$

Let $1 \le p \le 2$. Then B is said to be of *Rademacher type* p if there exists a positive constant C such that

$$E\left\|\sum_{1}^{\infty} Y_n x_n\right\|^p \le C \sum_{1}^{\infty} \|x_n\|^p \qquad \text{for all} \quad \{x_n\} \in C(B).$$

According to a well-known result of Hoffmann-Jorgensen and Pisier [2], a real separable Banach space is of Rademacher type p if and only if there exists a positive constant C such that $E\left\|\sum\limits_{j=1}^{n} X_j\right\|^p \le C \sum\limits_{j=1}^{n} E\|X_j\|^p$ for every finite collection $\{X_1, \dots, X_n\}$ of independent random elements in B with $EX_j = 0$, $E\|X_j\|^p < \infty$, $1 \le j \le n$.

The definitions of independence and of identically distributed random elements are similar to those for real-valued random variables.

DEFINITION 2. The array $\{X_{mn}\}$ of random elements in B is said to be *stochastically dominated by a real-valued random variable* A if there exists a positive constant D such that

$$P\big(\|X_{mn}\| > t\big) \le DP\big(|A| > t\big) \qquad \text{for each} \quad t \ge 0,$$

and all natural numbers $m, n \ge 1$.

DEFINITION 3. The array $\{X_{mn}\}$ of random elements in B is said to be *row-wise stochastically dominated by a sequence of real-valued random variables* $\{A_m\}$ if there exists a positive constant D such that

$$P\big(\|X_{mn}\| > t\big) \le DP\big(|A_m| > t\big) \qquad \text{for each} \quad t \ge 0,$$

and all natural numbers m and n.

2. Strong laws of large numbers

We begin with three lemmas which play a key role in establishing the main theorems of the paper.

Lemma 1. *Let $\{x_{mn}\}$ be a double sequence of elements of a real Banach space, $\{b_{mn}\}$ a double sequence of positive real numbers such that $b_{mn} = c_m d_n$ for all m and n, with $0 < c_m \uparrow \infty$, $0 < b_n \uparrow \infty$. If $\sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} \frac{x_{mn}}{b_{mn}}$ converges in norm, then $\frac{1}{b_{mn}} \sum\limits_{1}^{m} \sum\limits_{1}^{n} x_{ij} \to 0$ in norm for $m, n \to \infty$.*

Proof. Put $S_{mn} = \sum\limits_{1}^{m} \sum\limits_{1}^{n} \frac{x_{ij}}{b_{ij}}$, $V_{mn} = \frac{1}{b_{mn}} \sum\limits_{1}^{m} \sum\limits_{1}^{n} x_{ij}$, and $S_{ij} = 0$ whenever i or j is zero. We have $x_{ij} = b_{ij}\left(S_{ij} - S_{i-1,j} - S_{i,j-1} + S_{i-1,j-1}\right)$. It follows that

$$V_{mn} = S_{mn} - \frac{1}{b_{mn}} \sum_{i=0}^{m-1} S_{in}(b_{i+1,n} - b_{in}) - \frac{1}{b_{mn}} \sum_{j=0}^{n-1} S_{mj}(b_{m,j+1} - b_{mj})$$
$$+ \frac{1}{b_{mn}} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} S_{ij}\left(b_{i+1,j+1} - b_{i+1,j} - b_{i,j+1} + b_{ij}\right).$$

We show that all 4 terms converge in norm to the same limit, say S. Using a one-dimensional version of Toeplitz' lemma we get immediately the result for the second and third terms. As to the last term, it follows from the assumptions on b_{mn} that it can be rewritten as $\sum\limits_{i=0}^{m-1} \sum\limits_{j=0}^{n-1} S_{ij} \frac{c_{i+1}-c_i}{c_m} \frac{d_{j+1}-d_j}{d_n}$.

We have

$$\left\| \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} S_{ij} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} - S \right\|$$
$$\leq \left\| \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} S_{ij} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} S \right\|$$
$$+ \left(1 - \left(1 - \frac{c_0}{c_m}\right) \left(1 - \frac{d_0}{d_n}\right)\right) \|S\|$$
$$\leq \sum_{i=0}^{m_0} \sum_{j=0}^{n_0} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} \|S_{ij} - S\| + \sum_{i=m_0+1}^{m-1} \sum_{j=0}^{n_0} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} \|S_{ij} - S\|$$
$$+ \sum_{i=0}^{m_0} \sum_{j=n_0+1}^{n-1} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} \|S_{ij} - S\|$$
$$+ \sum_{i=m_0+1}^{m-1} \sum_{j=n_0+1}^{n-1} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} \|S_{ij} - S\|$$
$$+ \left(1 - \left(-\frac{c_0}{c_m}\right) \left(1 - \frac{d_0}{d_n}\right)\right) \|S\|.$$

It follows from this inequality that for each $\varepsilon > 0$ there exist m_1 and n_1 such that for $m > m_1$ and $n > n_1$ we have

$$\left\| \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} S_{ij} \frac{c_{i+1} - c_i}{c_m} \frac{d_{j+1} - d_j}{d_n} - S \right\| < \varepsilon .$$

□

The proofs of the following lemmas are simple and will therefore be omitted.

LEMMA 2. *For each real random variable X we have*

$$E|X|I(|X| > x) = \int_x^\infty P(|X| > t)\, \mathrm{d}t + xP(|X| > x), \qquad x \geq 0$$

where I denotes the characteristic function of the set in brackets.

LEMMA 3. *Let X and Y be real random variables such that $P(|X| > t) \leq DP(|Y| > t)$ for each $t \geq 0$ and some D. Then*

$$E|X|I(|X| > x) \leq DE|Y|I(|Y| > x) \qquad x \geq 0 .$$

THEOREM 1. *Let B be a real separable, Rademacher type p $(1 < p \leq 2)$ Banach space. Let $\{X_{mn}\}$ be a double sequence of independent random elements in B which is row-wise stochastically dominated by a sequence of real-valued random variables $\{A_m\}$ in the sense of Definition 3. Let*

(i) $\sum_{m=1}^{\infty} \frac{E|A_m|^q}{m^q} < \infty$ *for some q, $1 < q < p$.*

Let $\{a_{mn}\}$ and $\{b_{mn}\}$ be double sequences of positive real numbers such that $b_{mn} \to \infty$ and

(ii) $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = O(b_{mn})$,

(iii) $\frac{a_{mn}}{b_{mn}} = O(m^{-1}n^{-1})$.

Then

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(X_{ij} - EX_{ij}) \to 0 \quad a.c.$$

P r o o f . First we prove that

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} (X_{ij} - EY_{ij}) \to 0 \quad \text{a.c.}$$

where

$$Y_{mn} = X_{mn} I\big(\|X_{mn}\| \le c_{mn}\big)\,, \qquad c_{mn} = \frac{b_{mn}}{a_{mn}}\,.$$

We have

$$\begin{aligned}
\sum_m \sum_n P(X_{mn} \ne Y_{mn}) &= \sum_m \sum_n P\big(\|X_{mn}\| > c_{mn}\big) \\
&\le D \sum_m \sum_n P\big(|A_m| > c_{mn}\big) \le D \sum_m \sum_n \frac{E|A_m|^q}{c_{mn}^q} \\
&\le C \sum_m \frac{E|A_m|^q}{m^q} \sum_n \frac{1}{n^q} < \infty\,.
\end{aligned}$$

Thus all we have to prove is the a.c. convergence to 0 of $\frac{1}{b_{mn}} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} a_{ij}(Y_{ij} - EY_{ij})$.

Let $s_{mn} = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \frac{Y_{ij} - EY_{ij}}{c_{ij}}$. We show that s_{mn} is fundamental in L^p. Since all X_{mn} take their values in a Rademacher type p Banach space, we have

$$\begin{aligned}
E\|s_{mn} - s_{m_0 n_0}\|^p &= E\left\| \sum_{i=m_0}^{m} \sum_{j=n_0}^{n} \frac{Y_{ij} - EY_{ij}}{c_{ij}} \right\|^p \\
&\le C \sum_{i=m_0}^{m} \sum_{j=n_0}^{n} \frac{\|Y_{ij} - EY_{ij}\|^p}{c_{ij}^p} \le 2^p C \sum_{i=m_0}^{m} \sum_{j=n_0}^{n} \frac{E\|Y_{ij}\|^p}{c_{ij}^p}\,.
\end{aligned}$$

Therefore we only need to show that $\sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} \frac{E\|Y_{mn}\|^p}{c_{mn}^p} < \infty$. To this end we have

$$\begin{aligned}
E\|Y_{ij}\|^p &= \int\limits_{\|X_{ij}\| \le c_{ij}} \|X_{ij}\|^p \, \mathrm{d}P \\
&= c_{ij}^p P\big(\|X_{ij}\| \le c_{ij}\big) - p \int\limits_0^{c_{ij}} t^{p-1} P\big(\|X_{ij}\| \le t\big) \, \mathrm{d}t \\
&= -c_{ij}^p P\big(\|X_{ij}\| > c_{ij}\big) + p \int\limits_0^{c_{ij}} t^{p-1} P\big(\|X_{ij}\| > t\big) \, \mathrm{d}t \\
&\le p \int\limits_0^{c_{ij}} t^{p-1} P\big(\|X_{ij}\| > t\big) \, \mathrm{d}t \\
&\le Cp \int\limits_0^{c_{ij}} t^{p-1} P\big(|A_i| > t\big) \, \mathrm{d}t = C \int\limits_0^{1} c_{ij}^p P\big(|A_i| > c_{ij} z^{1/p}\big) \, \mathrm{d}z
\end{aligned}$$

when we have used the substitution $z = (c_{ij}^{-1}t)^p$. But

$$\int_0^1 P\big(|A_i| > c_{ij}z^{1/p}\big)\,\mathrm{d}z \le \int_0^1 \frac{E|A_i|^q}{c_{ij}^q z^{q/p}}\,\mathrm{d}z$$

and consequently

$$\sum_m\sum_n \frac{E\|Y_{mn}\|^p}{c_{mn}^p} \le C\sum_m\sum_n \int_0^1 z^{-q/p}\,\mathrm{d}z\,\frac{E|A_m|^q}{c_{mn}^q} < \infty\,.$$

Since $\{s_{mn}\}$ is fundamental in L^p, there exists a random element V such that $E\|s_{mn}-V\|^p \to 0$. Thus s_{mn} converges in probability. However the convergence in probability and a.c. convergence are equivalent for sums of independent random elements in a real separable Banach space (see [1], [3]). Hence we obtain that $\sum\limits_m\sum\limits_n \frac{Y_{mn}-EY_{mn}}{c_{mn}}$ converges a.c. and by Lemma 1

$$\frac{1}{b_{mn}}\sum_{i=1}^m\sum_{j=1}^n a_{ij}(X_{ij}-EY_{ij}) \to 0 \quad \text{a.c.}$$

Now it only remains to prove that

$$\frac{1}{b_{mn}}\sum_{i=1}^m\sum_{j=1}^n a_{ij}EZ_{ij} \to 0 \qquad \text{where} \quad Z_{ij} = X_{ij}I\big(\|X_{ij}\| > c_{ij}\big)$$

and this will be done by proving that the series $\sum\limits_m\sum\limits_n \frac{EZ_{mn}}{c_{mn}}$ converges absolutely. We have

$$\begin{aligned}
\sum_m\sum_n \frac{\|EZ_{mn}\|}{c_{mn}} &\le C\sum_m\sum_n \frac{E|A_m|I\big(|A_m| > c_{mn}\big)}{c_{mn}}\\
&\le C\sum_m\sum_n \frac{\int_{c_{mn}}^{\infty} P\big(|A_m| > t\big)\,\mathrm{d}t + c_{mn}P\big(|A_m| > c_{mn}\big)}{c_{mn}}\\
&= C\sum_m\sum_n P\big(|A_m| > c_{mn}\big) + C\sum_m\sum_n \frac{1}{c_{mn}}\int_{c_{mn}}^{\infty} P\big(|A_m| > t\big)\,\mathrm{d}t\\
&= C\sum_m\sum_n P\big(|A_m| > c_{mn}\big) + C\sum_m\sum_n \int_1^{\infty} P\big(|A_m| > c_{mn}z\big)\,\mathrm{d}z
\end{aligned}$$

whence

$$\sum_m \sum_n \frac{\|EZ_{mn}\|}{c_{mn}}$$

$$\leq C \sum_m \sum_n P(|A_m| > c_{mn}) + C \sum_m \sum_n \int_1^\infty \frac{E|A_m|^q}{c_{mn}^q z^q} \, \mathrm{d}z$$

$$\leq C \sum_m \sum_n P(|A_m| > c_{mn}) + C_1 \int_1^\infty z^{-q} \, \mathrm{d}z \sum_m \frac{E|A_m|^q}{m^q} \sum_n \frac{1}{n^q} < \infty ,$$

completing the proof. □

COROLLARY 1. *Let B be a real separable Rademacher type p $(1 < p \leq 2)$ Banach space. Let $\{X_{mn}\}$ be a double sequence of independent random elements row-wise stochastically dominated by a sequence $\{A_m\}$ of real valued random variables such that $\sum_m \frac{E|A_m|^q}{m^q} < \infty$ for some q, $1 < q < p$. Then*

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (X_{ij} - EX_{ij}) \to 0 \quad a.c.$$

COROLLARY 2. *Let $\{X_{mn}\}$ be a double sequence of independent real-valued random variables row-wise stochastically dominated by a sequence $\{A_m\}$ of real valued random variables. Let*

$$\sum_m \frac{E|A_m|^q}{m^q} < \infty$$

for some q, $1 < q < 2$.

Let $\{a_{mn}\}$, $\{b_{mn}\}$ be double sequences of positive real numbers such that $b_{mn} \to \infty$, $\sum_{i=1}^m \sum_{j=1}^n a_{ij} = O(b_{mn})$, $\frac{a_{mn}}{b_{mn}} = O(m^{-1}n^{-1})$.

Then

$$\frac{1}{b_{mn}} \sum_{i=1}^m \sum_{j=1}^n a_{ij}(X_{ij} - EX_{ij}) \to 0 \quad a.c.$$

Compared with Theorem 1, the following theorem imposes weaker conditions on a_{mn} and b_{mn}. On the other hand, the condition on X_{mn} is stronger.

THEOREM 2. *Let B be a real separable Rademacher type p $(1 < p \leq 2)$ Banach space. Let $\{X_{mn}\}$ be a double sequence of independent random elements in B stochastically dominated by a real-valued random variable X such that*

$E|X|^q < \infty$ for some q, $1 \le q < \infty$. Let $\{a_{mn}\}$, $\{b_{mn}\}$ be double sequences of positive real numbers with $0 < b_{mn} \to \infty$ and such that

i) $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} a_{ij} = O(b_{mn})$,

ii) $\frac{a_{mn}}{b_{mn}} = O(m^{-1/t} n^{-1/s})$, $0 < t$, $s < \min(p,q)$.

Then

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(X_{ij} - EX_{ij}) \to 0 \quad a.c.$$

Proof. Put $Y_{mn} = X_{mn} I(\|X_{mn}\| \le c_{mn})$. Following the first part of the proof of Theorem 1 we obtain that the series $\sum\limits_{m} \sum\limits_{n} \frac{a_{mn}(Y_{mn} - EY_{mn})}{b_{mn}}$ converges and, by using Lemma 1, that

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(X_{ij} - EY_{ij}) \to 0 \quad \text{a.c.}$$

It is now sufficient to demonstrate that

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} EZ_{ij} \to 0 \quad \text{in norm.}$$

We have

$$\begin{aligned} \frac{1}{b_{mn}} \left\| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} EZ_{ij} \right\| &\le \frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E\|X\| I(\|X\| > c_{ij}) \\ &\le \frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E\|X\| I(\|X\| > K_1 (ij)^{1/q}) \end{aligned}$$

as $c_{mn} \ge K_1 m^{1/t} n^{1/s} \ge K_1 (mn)^{1/q}$ where K_1 means a suitable constant.

Consequently the Lebesgue dominated convergence theorem and the Toeplitz lemma for double sequences imply that the last expression is smaller than given $\varepsilon > 0$ for m, n sufficiently large. □

COROLLARY 3. *Let $\{X_{mn}\}$ be a double sequence of independent real-valued random variables stochastically dominated by a real-valued random variable X with $E|X|^q < \infty$ for some q, $1 \le q < \infty$. Let $\{a_{mn}\}$, $\{b_{mn}\}$ be double sequences of real numbers satisfying the assumptions of Theorem 2.*

Then

$$\frac{1}{b_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(X_{ij} - EX_{ij}) \to 0 \quad a.c.$$

3. Conclusions

One of the pioneering works on stochastically dominated real-valued random variables is probably that of Rohatgi [4]. For random elements in a real separable Banach space the problem was studied by Woyczynski in [5]. In the current paper versions of Rohatgi's result have been obtained for double sequences of Banach space valued random elements under more general conditions on the norming constants and less restrictive conditions on the stochastic domination of the random elements. While Theorem 2 imposes a moment condition on the dominating random variable, Theorem 1 holds for a double sequence of random elements which need not be dominated by a single function. Moreover Corollary 2 seems to be a new result even for real valued random variables.

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