Mathematica Slovaca

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Mathematica Slovaca, Vol. 51 (2001), No. 5, 565--582

Persistent URL: http://dml.cz/dmlcz/130274

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ON KELLEY'S MULTIPLICITY FUNCTION OF AN ABELIAN VON NEUMANN ALGEBRA

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. Let $\mathcal{A}$ be an abelian von Neumann algebra of operators on a Hilbert space H and let $G(\cdot)$ be its canonical spectral measure (see Definition 5) on the Borel subsets of its maximal ideal space $\mathcal{M}$. By describing Kelley's multiplicity function ϕ of $\mathcal{A}$ in terms of the uniform multiplicity function of Halmos, the basic structure theorem of Kelley [KELLEY, J. L.: *Commutative operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 598–605] is deduced from the theory of orthogonal spectral representations applied to $G(\cdot)$. When the commutant $\mathcal{A}'$ is countably decomposable, $G(\cdot)$ has CGS-property in H and in this case, ϕ is also described in terms of the multiplicity functions m_p and m_c of $G(\cdot)$ (see Definition 4).

Let $\mathcal{A}$ be an abelian von Neumann algebra of operators on a Hilbert space H with $\mathcal{A}'$ its commutant, $\mathcal{M}$ its maximal ideal space and $G(\cdot)$ its canonical spectral measure (see Definition 5). Kelley [5] defined a multiplicity function ϕ on $\mathcal{M}$ and result 5.1 of [5] (namely, the basic structure theorem) determines H and $\mathcal{A}$, up to unitary equivalence, in terms of ϕ.

In a series of papers [6], [7], [8], [9], [10], we gave a unified approach to deduce or generalize all the important results known on the problem of unitary invariance. (See [10; Introduction].) The present paper forms the last part of the series and deduces Kelley's basic structure theorem from the theory of orthogonal spectral representations developed in [8]. For this, we describe ϕ in terms of the uniform multiplicity function of Halmos [3].

When $\mathcal{A}'$ is countably decomposable, $G(\cdot)$ has CGS-property in H and hence has two multiplicity functions m_p and m_c on $\mathcal{M}$ corresponding to the discrete

2000 Mathematics Subject Classification: Primary 47B15, 46L10.
Keywords: abelian von Neumann algebra, canonical spectral measure, H-multiplicity, UH-multiplicity, primitive projection, Kelley's multiplicity function, OSD, OTSR, multiplicity functions m_p and m_c.

Supported by the C.D.C.H.T. project C-845-97-05-B of the Universidad de los Andes, Mérida, Venezuela.

part p_G and the continuous part c_G (see Definition 4). In Theorem 5 we describe ϕ in terms of m_p and m_c. Consequently, a spatial isomorphism theorem in terms of m_p and m_c holds for $\mathcal{A}$. Assuming Theorem 5, this isomorphism result has been obtained in [10; Corollary 2].

1. Preliminaries

In this section we fix notation and terminology and give some definitions and results from the literature to make the paper self-contained.

Let H, H_1 and H_2 denote (complex) Hilbert spaces of arbitrary dimension (> 0). The closed subspace spanned by a subset $\mathcal{X}$ of a Hilbert space is denoted by $[\mathcal{X}]$. $\bigoplus M_i$ is the orthogonal direct sum of a family of mutually orthogonal closed subspaces M_i of a given Hilbert space or of Hilbert spaces $\{M_i\}_i$.

If P is a projection in a von Neumann algebra $\mathcal{R}$ on H, then C_P denotes the central support of P. For $x \in H$, $[\mathcal{R}x] = [Rx : R \in \mathcal{R}]$ and, sometimes, it also denotes the orthogonal projection with range $[\mathcal{R}x]$. By an isomorphism between two von Neumann algebras we mean a $*$-isomorphism. $\sum \bigoplus \mathcal{A}_i$ denotes the direct sum of the von Neumann algebras $\mathcal{A}_i$. The rest of the terminology and notation in von Nuemann algebras is standard and we follow Dixmier [1].

Let $\mathcal{S}$ be a σ-algebra of subsets of a non empty set Ω. Let $E(\cdot)$ be a spectral measure on $\mathcal{S}$ with values in projections of H. For $x \in H$, $\rho_E(x)$ denotes the measure $\|E(\cdot)x\|^2$ on $\mathcal{S}$. Let $\Sigma(\mathcal{S})$ be the set of all finite (positive) measures on $\mathcal{S}$. For $\mu_1, \mu_2 \in \Sigma(\mathcal{S})$, we write $\mu_1 \equiv \mu_2$ if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$. Clearly, $\equiv$ is an equivalence relation on $\Sigma(\mathcal{S})$.

For $\mu \in \Sigma(\mathcal{S})$, the *projection* $C_E(\mu)$ is defined as the orthogonal projection on the closed subspace $\{x \in H : \rho_E(x) \ll \mu\}$ and it follows from [3] that $C_E(\mu) \in W$, where W is the von Neumann algebra generated by the range of $E(\cdot)$. The *multiplicity* $u_E(\mu)$ of $\mu \in \Sigma(\mathcal{S})$ relative to $E(\cdot)$ is defined by

$$u_E(\mu) = \min\{H\text{-multiplicity of } C_E(\nu) : 0 \neq \nu \ll \mu, \ \nu \in \Sigma(\mathcal{S})\}$$

if $\mu \neq 0$ and $u_E(0) = 0$, where the H-multiplicity of $C_E(\nu)$ is the multiplicity of $C_E(\nu)$ relative to $E(\cdot)$ in the sense of Halmos [3]. $\mu \in \Sigma(\mathcal{S})$ is said to have *uniform multiplicity* $u_E(\mu)$ relative to $E(\cdot)$ if $u_E(\nu) = u_E(\mu)$ for $0 \neq \nu \ll \mu$ $\nu \in \Sigma(\mathcal{S})$.

For $x \in H$, let $Z_E(x) = [E(\sigma)x : \sigma \in \mathcal{S}]$.

Now we quote some definitions and results from [7], [8].

Definition 1. A spectral measure $E(\cdot)$ on $\mathcal{S}$ is said to have *CGS-property* in H if there exists a countable set $\mathcal{X}$ in H such that $[E(\sigma)x : x \in \mathcal{X}, \sigma \in \mathcal{S}] = H$.

Definition 2. Let $E(\cdot)$ be a spectral measure on $\mathcal{S}$ with values in projections of the Hilbert space H. Then H is said to have an *ordered spectral decomposition* (briefly, *OSD*) relative to $E(\cdot)$ if

$$H = \bigoplus_{1}^{N} Z_E(x_i), \qquad N \in \mathbb{N} \cup \{\infty\},$$

where the x_i are non zero vectors in H and

$$\rho_E(x_1) \gg \rho_E(x_2) \gg \dots .$$

N is called the *OSD-multiplicity* of $E(\cdot)$. (N is uniquely determined by $E(\cdot)$ by [7; Theorem 3.11].) When $N = \infty$, we say that the OSD-multiplicity of $E(\cdot)$ is $\aleph_0$.

By [7; Theorem 3.7], H has an OSD relative to $E(\cdot)$ if and only if $E(\cdot)$ has CGS-property in H.

Notation 1. Let $\mu_j \in \Sigma(\mathcal{S})$, $\mu_j \neq 0$, $j \in J$, and let $\tilde{H} = \bigoplus_{j \in J} L_2(\mu_j)$, where $L_2(\mu_j) = L_2(\Omega, \mathcal{S}, \mu_j)$. In the sequel, by $\tilde{E}(\cdot)$ we shall denote the set function on $\mathcal{S}$ given by

$$\tilde{E}(\cdot)(f_j)_{j\in J} = \big(\chi_{(\cdot)} f_j\big)_j, \qquad (f_j)_j \in \tilde{H}.$$

Definition 3. Let $\{\mu_n\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, be non zero measures in $\Sigma(\mathcal{S})$ with $\mu_1 \gg \mu_2 \gg \dots$. An isomorphism U from H onto $K = \bigoplus_1^N L_2(\mu_n)$ is called an *ordered spectral representation* (briefly, *OSR*) of H relative to $E(\cdot)$, if $UE(\cdot)U^{-1} = \tilde{E}(\cdot)$. N is called the *OSR-multiplicity of* $E(\cdot)$ (since N is uniquely determined by $E(\cdot)$ by [7; Theorem 4.2] and it coincides with its OSD-multiplicity).

The sequence $\{\mu_n\}_1^N$ is called the *measure sequence* of the OSR U. Two OSRs U_1 and U_2 of H_1 and H_2 relative to the spectral measures $E_1(\cdot)$ and $E_2(\cdot)$ defined on the σ-algebra $\mathcal{S}$ with the corresponding measure sequences $\{\mu_j^{(1)}\}_{j=1}^{N_1}$ and $\{\mu_j^{(2)}\}_{j=1}^{N_2}$ are said to be *equivalent* if $N_1 = N_2$ and $\mu_j^{(1)} \equiv \mu_j^{(2)}$ for all j.

Definition 4. Let X be a Hausdorff topological space, $\mathcal{S} = \mathcal{B}(X)$, the σ-algebra of all Borel subsets of X (i.e., the σ-algebra generated by the open sets in X), and $E(\cdot)$ a spectral measure on $\mathcal{S}$ with CGS-property in H. Then the *discrete part* p_E of $E(\cdot)$ is defined as the set $\{t \in X : E(\{t\}) \neq 0\}$. The *continuous part* c_E of $E(\cdot)$ is defined as the set $X \setminus p_E$. We shall write $\mathcal{M}(E) = E(p_E)H$ and $\mathcal{R}(E) = E(c_E)H = H \ominus \mathcal{M}(E)$.

The *multiplicity function* m_p on X relative to $E(\cdot)$ is defined as $m_p(t) = 0$ if $t \in X \setminus p_E$ and $m_p(t) = \dim E(\{t\})H$ if $t \in p_E$, where $m_p(t) = \aleph_0$ if $E(\{t\})H$ is infinite dimensional.

Let $\mathcal{R}(E) = \bigoplus_1^N Z_F(y_i)$ be an OSD of $\mathcal{R}(E)$ relative to $F(\cdot) = E(\cdot)E(c_E)$. Then the *multiplicity function* m_c on X relative to $E(\cdot)$ is defined as follows:

(i) $m_c(t) = 0$ if $\mathcal{R}(E) = \{0\}$ or if $\mathcal{R}(E) \neq \{0\}$ and there exists an open set $\mathcal{U}$ containing t for which $E(\mathcal{U})y_1 = 0$.

(ii) $m_c(t) = n \in \mathbb{N}$ if y_k do exist for all $k = 1, 2, \ldots, n$ and for every open set $\mathcal{U}$ containing t we have $E(\mathcal{U})y_k \neq 0$ for $k = 1, 2, \ldots, n$, while $N = n$ or y_{n+1} does exist and $E(\mathcal{U})y_{n+1} = 0$ for some open set $\mathcal{U}$ containing t.

(iii) $m_c(t) = \aleph_0$ if $N = \infty$ and for every open set $\mathcal{U}$ containing t we have $E(\mathcal{U})y_k \neq 0$ for each $k \in \mathbb{N}$.

NOTATION 2. Let W be the von Neumann algebra generated by the range of a spectral measure $E(\cdot)$ on $\mathcal{S}$ with values in projections of H and let $W' = \sum_{n \in J} \bigoplus W'Q_n$ be the type I_n-direct sum decomposition of W'. Then by $\{Q_n\}_{n \in J}$ we denote the collection of these projections in the above decomposition and $\{n : n \in J\}$ is denoted by M_W as well as by M_E. M_W (resp. M_E) is called the *multiplicity set* of W (resp. of $E(\cdot)$). The multiplicity and uniform multiplicity of projections in W in the sense of Halmos [3; pp. 100–101] will be referred to as *H-multiplicity* and *UH-multiplicity*, respectively.

DEFINITION 5. Let $\mathcal{A}$ be an abelian von Neumann algebra on H and let $\mathcal{M}$ be its maximal ideal space. If $\mathcal{B}(\mathcal{M})$ is the σ-algebra of the Borel sets in $\mathcal{M}$, then the unique spectral measure $G(\cdot)$ on $\mathcal{B}(\mathcal{M})$ which associates (under the inverse of the Gelfand mapping) with each $\sigma \in \mathcal{B}(\mathcal{M})$, the projection operator corresponding to the characteristic function of the clopen set $\Upsilon(\sigma)$ for which $\Upsilon(\sigma)\Delta\sigma$ is meagre in $\mathcal{M}$, is called the *canonical spectral measure* of $\mathcal{A}$ (see [4; pp. 157–163]). If $\mathcal{A}_i$ is an abelian von Neumann algebra on H_i, then its maximal ideal space and canonical spectral measure are denoted by $\mathcal{M}_i$ and $G_i(\cdot)$, respectively, for $i = 1, 2$.

NOTATION 3. If $\mathcal{A}$ is an abelian von Neumann algebra on H and if $G(\cdot)$ is its canonical spectral measure, then the H-multiplicity and UH-multiplicity of a projection P in $\mathcal{A}$ are with respect to $G(\cdot)$ (since $\mathcal{A}$ is the von Neumann algebra generated by the range of $G(\cdot)$). The multiplicity set $M_{\mathcal{A}}$ is as in Notation 2 but with W replaced by $\mathcal{A}$ and $E(\cdot)$ by $G(\cdot)$.

Let $E(\cdot)$ and W be as in Notation 2. As observed in [6], a projection P' in W' is abelian if and only if it is a row projection with respect to $E(\cdot)$ (in the sense of Halmos [3]) and the column generated by a projection in W' is the same as its central support. Thus [3; Theorem 66.4] of Halmos can be reformulated as follows:

PROPOSITION 1. *A non zero projection F in W has UH-multiplicity n relative to $E(\cdot)$ if and only if there exists an orthogonal family $\{E'_\alpha\}_{\alpha \in J}$ of abelian projections in W' such that* $\operatorname{card} J = n$, $C_{E'_\alpha} = F$ *for each* $\alpha \in J$ *and* $\sum_{\alpha \in J} E'_\alpha = F$; *in other words, if and only if $W'F$ is of type I_n or, equivalently, if and only if $0 \neq F \leq Q_n$ where the Q_n are the central projections in the type I_n-direct sum decomposition of W'.*

COROLLARY 1. *If $\mathcal{A}$ is an abelian von Neumann algebra with the canonical spectral measure $G(\cdot)$, then a non zero projection $P \in \mathcal{A}$ has UH-multiplicity n (in the sense of Halmos [3]) with respect to $G(\cdot)$ if and only if $n \in M_{\mathcal{A}}$ and $P \leq Q_n$, where the Q_n are the central projections of the type I_n-direct sum decomposition of $\mathcal{A}'$.*

DEFINITION 6. Suppose $E(\cdot)$ is a spectral measure on $\mathcal{S}$ with values in projections of H. Let $\{\mu_j\}_{j \in J}$ be an orthogonal family in $\Sigma(\mathcal{S})$ with each μ_j having uniform multiplicity $u_E(\mu_j) = u_j > 0$ for $j \in J$. Let $\tilde{H} = \bigoplus_{j \in J} \bigoplus_{u_j} L_2(\mu_j)$ and suppose $U\colon H \to \tilde{H}$ is an isomorphism of H onto $\tilde{H}$. Then U is called an *orthogonal spectral representation* (briefly, *OTSR*) of H relative to $E(\cdot)$ if $UE(\cdot)U^{-1} = \tilde{E}(\cdot)$ (see Notation 1). The set $\{\mu_j\}_{j \in J}$ is called the *measure family* of the OTSR U. Suppose U_i is an OTSR of the Hilbert space H_i relative to the spectral measure $E_i(\cdot)$ defined on $\mathcal{S}$ with the measure family F_i for $i = 1, 2$. Then we say that U_1 and U_2 are *equivalent* and write $U_1 \sim U_2$ if $u_{E_1}(\mu) = u_{E_2}(\mu)$ for $\mu \in F_1 \cup F_2$ and the multiplicity functions u_{E_1} and u_{E_2} are uniform on $F_1 \cup F_2$. An OTSR U of H relative to $E(\cdot)$ with the measure family F is called a *bounded OTSR* (briefly, *BOTSR*) if $F_n = \{\mu \in F : u_E(\mu) = n\}$ is bounded in $\Sigma(\mathcal{S})$ for each $n \in M_E$. A BOTSR U of H relative to $E(\cdot)$ with the measure family F is called a *BOTSR with countable multiplicities* (briefly, *COBOTSR*) if $u_E(\mu) \leq \aleph_0$ for all $\mu \in F$.

DEFINITION 7. Let $E(\cdot)$ and W be as in Notation 2. Then $E(\cdot)$ is said to have *generalized CGS-property* in H if the central projections $\{Q_n\}$ in the type I_n-direct sum decomposition of the commutant W' are countably decomposable in W.

By [7; Theorem 5.6] we have the following result:

PROPOSITION 2. *The spectral measure $E(\cdot)$ has CGS-property (resp. generalized CGS-property) in H if and only if H admits a COBOTSR (resp. a BOTSR) relative to $E(\cdot)$; consequently, if and only if every OTSR of H relative to $E(\cdot)$ is a COBOTSR (resp. a BOTSR).*

In the sequel, $\mathcal{A}$ will denote an abelian von Neumann algebra on H, with its maximal ideal space $\mathcal{M}$ and its canonical spectral measure $G(\cdot)$.

The following proposition is proved in [10]. It plays a key role in Section 4 below.

PROPOSITION 3. *If* $H = \bigoplus_1^N Z_E(x_i)$, $N \in \mathbb{N} \cup \{\infty\}$, *is an OSD of* H *relative to a spectral measure* $E(\cdot)$ *on* $\mathcal{S}$ *with values in projections of* H, *then the following assertions hold:*

(i) $C_E(\rho_E(x_1)) = I$ *and* $C_E(\rho_E(x_j)) = I - \sum_{n \in M_E \cap \mathbb{N},\, n<j} Q_n$, $1 \le j \le N$, $j \in \mathbb{N}$.

(ii) *If* $M_E = \{n_1 < n_2 < \dots\} \cup \{\aleph_0\}$, *then*

  (a) $\rho_E(x_{n_i}) \underset{\not\equiv}{\gg} \rho_E(x_j) \equiv \rho_E(x_{n_{i+1}})$ *for* $n_i < j \le n_{i+1}$, $i = 0, 1, 2, \dots$, *where* $n_0 = 0$ *and the term corresponding to* x_{n_0} *is omitted.*

  (b) $\rho_E(x_j) \underset{\not\equiv}{\gg} \mu_{Q_{\aleph_0}}$, $j \in \mathbb{N}$, *where* $C_E(\mu_{Q_{\aleph_0}}) = Q_{\aleph_0}$.

(iii) *If* $M_E = \{n_1 < n_2 < \dots < n_k\} \cup \{\aleph_0\}$, *then* (ii)(a) *holds for* $i = 0, 1, 2, \dots, k-1$ *and*

  (c) $\rho_E(x_j) \equiv \mu_{Q_{\aleph_0}}$ *for* $j > n_k$, $j \in \mathbb{N}$, *where* $\mu_{Q_{\aleph_0}}$ *is as in* (ii)(b). *Here* $k = 0$ *is also permissible* (*in the sense that* $M_E \cap \mathbb{N} = \emptyset$). *Then, in that case,* $M_E = \{\aleph_0\}$ *and* (c) *holds for all* $j \in \mathbb{N}$.

(iv) *If* $M_E = \{n_1 < n_2 < \dots\}$, *then* (ii)(a) *holds and there does not exist* $\nu \in \Sigma(\mathcal{S})$ *with* $C_E(\nu) \ne 0$ *such that* $\rho_E(x_j) \underset{\not\equiv}{\gg} \nu$ *for all* $j \in \mathbb{N}$.

(v) *If* $M_E = \{n_1 < n_2 < \dots < n_k\}$, *then* (ii)(a) *holds for* $i = 0, 1, 2, \dots, k-1$.

(vi) *If* $N = \infty$, *then one and only one of* (ii), (iii) *or* (iv) *holds.*

2. Kelley's multiplicity function of $\mathcal{A}$, $\mathcal{A}$ – arbitrary

Kelley [5] defined a multiplicity function ϕ on the maximal ideal space $\mathcal{M}$ of the abelian von Neumann algebra $\mathcal{A}$ with values in cardinal numbers. In this section, we shall interpret the terminology and results of Kelley [5] in terms of those of Halmos [3] and then describe ϕ in terms of the projections $\{Q_n\}_{n \in M_{\mathcal{A}}}$.

If P is the *carrier* of a vector $x \in H$ with respect to $\mathcal{A}$ (in the sense of Kelley [5]), evidently $P = C_{[\mathcal{A}x]} = C_{G(\rho_G(x))}$, the last equality being due to [3; Theorem 66.2] of Halmos.

In [5], Kelley defined a non empty set J of vectors in H as an $\mathcal{A}$-*base* for a projection $P \in \mathcal{A}$ if

(i) for each $x \in J$, $0 \ne \|x\| \le 1$ and $C_{[\mathcal{A}x]} = P$,

(ii) for $x, y \in J$, $x \neq y$, $x \perp [\mathcal{A}y]$,
(iii) J is maximal with respect to (i) and (ii).

Then by [1; Corollary 2 of Proposition I.1.7] it follows that a projection $P \in \mathcal{A}$ has an $\mathcal{A}$-base J if and only if P is cyclic in $\mathcal{A}$ and in that case, $\operatorname{card} J =$ H-multiplicity of P. Consequently, if a projection $P \in \mathcal{A}$ has an $\mathcal{A}$-base, then all the $\mathcal{A}$-bases of P have the same cardinal number, which coincides with the H-multiplicity of P.

A projection $P \in \mathcal{A}$ is said be *primitive* in the sense of Kelley [5] if there exists an $\mathcal{A}$-base J for P such that $[\mathcal{A}x : x \in J] = PH$. An $\mathcal{A}$-base J for a projection $P \in \mathcal{A}$ is said to be *proper* if $\rho_G(x) = \rho_G(y)$ for $x, y \in J$, with respect to the canonical spectral measure $G(\cdot)$. Then using the results of [3] one can show that every non zero cyclic projection $P \in \mathcal{A}$ has a proper $\mathcal{A}$-base and all the proper $\mathcal{A}$-bases of P have the same cardinal number, which coincides with its H-multiplicity. Finally, it follows that a non zero projection $P \in \mathcal{A}$ is primitive if and only if P is cyclic and has UH-multiplicity. Consequently, every non zero subprojection Q $(\in \mathcal{A})$ of a primitive projection in $\mathcal{A}$ is also primitive.

NOTATION 4. Given $f \in C(\mathcal{M})$, T_f denotes the operator $A \in \mathcal{A}$ whose image under the Gelfand map is f. For a projection $F \in \mathcal{A}$ we denote by e_F the clopen set in $\mathcal{M}$ for which $T_{\chi_{e_F}} = F$. Then $G(e_F) = F$.

DEFINITION 8. For each $p \in e_F$, F being a primitive projection in $\mathcal{A}$, let $\phi_1(p)$ be the cardinal of a proper $\mathcal{A}$-base of F. Then ϕ_1 is well defined on a dense open subset of $\mathcal{M}$ (see the proof of Theorem 1 below) and as shown in [5], it has a unique continuous extension ϕ to the whole of $\mathcal{M}$. We refer to ϕ as *Kelley's multiplicity function of* $\mathcal{A}$.

THEOREM 1. *Let $\mathcal{A}$ be an abelian von Neumann algebra on H with the maximal ideal space $\mathcal{M}$, and let ϕ be Kelley's multiplicity function of $\mathcal{A}$. Let $\mathcal{D}_0 = \bigcup_{n \in M_{\mathcal{A}}} e_{Q_n}$ and let $\psi(t) = n$ if $t \in e_{Q_n}$, $n \in M_{\mathcal{A}}$. Then ψ is continuous on $\mathcal{D}_0$ and ϕ is the unique continuous extension of ψ.*

Proof. Since the e_{Q_n} are mutually disjoint for $n \in M_{\mathcal{A}}$, the function ψ is well defined on the open set $\mathcal{D}_0$. Let $t \in \mathcal{D}_0$ and let $\psi(t) = n_0$. Then $t \in e_{Q_{n_0}}$ and $\psi^{-1}(n_0) = e_{Q_{n_0}}$ is open in $\mathcal{D}_0$. Hence it follows that ψ is continuous on $\mathcal{D}_0$ when the set of cardinals $c \leq \dim H$ is given the order topology.

Let $\mathcal{D} = \bigcup \{e_F :\ F$ a primitive projection in $\mathcal{A}\}$. For $n \in M_{\mathcal{A}}$, let $x \in Q_n H$, $x \neq 0$. If $Q = [\mathcal{A}'x]$, then $Q \in \mathcal{A}$ and $0 \neq Q \leq Q_n$. Then by Corollary 1, Q has UH-multiplicity n and as Q is cyclic, Q is a primitive projection. Then by Zorn's lemma it follows that there exists a maximal orthogonal family $\mathcal{F}_n$ of primitive subprojections of Q_n. We claim that $P_n = \sum_{F \in \mathcal{F}_n} F = Q_n$. Otherwise, there would exist a non zero vector $x \in (Q_n - P_n)H$ so that the projection $Q = [\mathcal{A}'x] \in \mathcal{A}$ and $0 \neq Q \leq Q_n$. Then by Corollary 1, Q has UH-multiplicity n. Then, Q is a primitive subprojection of Q_n orthogonal to $\mathcal{F}_n$, contradicting the maximality of $\mathcal{F}_n$.

Since $\sum_{n \in M_{\mathcal{A}}} Q_n = I$, it follows that $\mathcal{D}_0$ is dense in $\mathcal{M}$. Again, since $Q_n = \sum_{F \in \mathcal{F}_n} F$ for $n \in M_{\mathcal{A}}$, we have $e_{Q_n} = \overline{\left(\bigcup_{F \in \mathcal{F}_n} e_F \right)}$. Thus

$$\mathcal{M} = \overline{\mathcal{D}_0} = \overline{\left(\bigcup_{n \in M_{\mathcal{A}}} \overline{\left(\bigcup_{F \in \mathcal{F}_n} e_F \right)} \right)} \subset \overline{\mathcal{D}} \subset \mathcal{M}$$

and hence $\mathcal{D}$ is dense in $\mathcal{M}$. Moreover, $\mathcal{D} \subset \mathcal{D}_0$ by Corollary 1. If $t \in \mathcal{D}$, then $\phi(t) = \phi_1(t) =$ the cardinal number of a proper $\mathcal{A}$-base of a primitive projection F such that $t \in e_F$. Hence $\phi_1(t)$ is the same as the UH-multiplicity of F. If $\phi_1(t) = n$, then $F \leq Q_n$ by Corollary 1. Consequently, $\psi(t) = n$. Thus, $\psi|_{\mathcal{D}} = \phi_1$. Since the function ϕ is a continuous extension of ϕ_1 and since ψ is continuous on $\mathcal{D}_0$, it follows that ϕ is also a continuous extension of ψ to the whole of $\mathcal{M}$. Since $\mathcal{D}_0$ is dense in $\mathcal{M}$, the continuous extension is unique.

This completes the proof of the theorem. □

3. The $\mathcal{A}$-base structure theorem of Kelley

Using the results of [7], [8] we first study some properties of the spectral representations of H relative to the canonical spectral measure $G(\cdot)$ of the abelian von Neumann algebra $\mathcal{A}$ and then, using Theorem 1, we deduce the $\mathcal{A}$-base structure theorem of Kelley [5; Result 5.1].

THEOREM 2. *For an abelian von Neumann algebra $\mathcal{A}$ on H with the canonical spectral measure $G(\cdot)$ the following assertions hold:*

(i) *If $\mathcal{A}$ is countably decomposable, then $G(\cdot)$ has generalized CGS-property in H.*

(ii) *$\mathcal{A}'$ is countably decomposable if and only if $G(\cdot)$ has CGS-property in H.*

(iii) *If $\mathcal{A}'$ is countably decomposable, then H admits OSRs relative to $G(\cdot)$. If U is an OSR of H relative to $G(\cdot)$ with the measure sequence $\{\mu_j\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, then, for $A \in \mathcal{A}$,*

$$UAU^{-1}(f_n)_1^N = (gf_n)_1^N, \qquad (f_n)_1^N \in \bigoplus_1^N L_2(\mathcal{M}, \mathcal{B}(\mathcal{M}), \mu_j) \tag{1}$$

where $g \in C(\mathcal{M})$ with $T_g = A$.

(iv) *If $\mathcal{A}'$ is countably (resp. if $\mathcal{A}$ is countably) decomposable, then every OTSR of H relative to $G(\cdot)$ is a COBOTSR (resp. BOTSR). If U is* *a COBOTSR (resp. BOTSR) of H relative to $G(\cdot)$ with the measure family $(\mu_\alpha)_{\alpha\in J}$, then, for $A \in \mathcal{A}$,*

$$\begin{aligned} &UAU^{-1}(f_{\alpha,i})_{\alpha\in J,\, i\in I_\alpha} = (gf_{\alpha,i})_{\alpha\in J,\, i\in I_\alpha}, \\ &(f_{\alpha,i})_{\alpha\in J,\, i\in I_\alpha} \in \bigoplus_{\alpha\in J}\bigoplus_{u_\alpha} L_2(\mathcal{M},\mathcal{B}(\mathcal{M}),\mu_\alpha) \end{aligned} \tag{2}$$

where $g \in C(\mathcal{M})$ with $T_g = A$, $\operatorname{card} I_\alpha = u_\alpha$ and $u_G(\mu_\alpha) = u_\alpha > 0$ for $\alpha \in J$.

(v) *If $\mathcal{A}$ is arbitrary, then* (iv) *holds with the change that U is just an OTSR of H relative to $G(\cdot)$.*

Proof. Since $\mathcal{A}$ is the von Neumann algebra generated by the range of $G(\cdot)$, (i) and (ii) are evident.

(iii) By (ii), $G(\cdot)$ has CGS-property in H and by [7; Theorem 4.2(i)], H admits OSRs relative to $G(\cdot)$. Let U be an OSR of H relative to $G(\cdot)$ with the measure sequence $\{\mu_j\}_1^N$, $N \in \mathbb{N}\cup\{\infty\}$. Then $UG(\cdot)U^{-1} = \tilde{G}(\cdot)$, where $\tilde{G}(\cdot)(f_j)_1^N = \left(\chi_{(\cdot)}f_j\right)_1^N$, $(f_j)_1^N \in K = \bigoplus_1^N L_2(\mathcal{M},\mathcal{B}(\mathcal{M}),\mu_j)$. Let $g \in C(\mathcal{M})$. If $e_n = \{t \in \mathcal{M} : |g(t)| \le n\}$ for $n \in \mathbb{N}$, then by [7; Lemma 4.7] the operator $T(g)$ defined by

$$T(g)\mathbf{f} = \lim_n \int_{e_n} g \, \mathrm{d}\tilde{G}\mathbf{f}, \qquad \mathbf{f} \in K,$$

is a bounded normal operator on K, with resolution of the identity $\tilde{G}_g$ given by $\tilde{G}_g(e) = \tilde{G}(g^{-1}(e))$ for $e \in \mathcal{B}(\mathbb{C})$. Thus

$$T(g)\mathbf{f} = \lim_n \int_{e_n} g d\tilde{G}\mathbf{f} = \int_{\mathcal{M}} g \, \mathrm{d}\tilde{G}\mathbf{f} = U\Big(\int_{\mathcal{M}} g \, \mathrm{d}G\Big)U^{-1}\mathbf{f} = UT_gU^{-1}\mathbf{f} \tag{3}$$

for $g \in C(\mathcal{M})$ and $\mathbf{f} = (f_j)_1^N \in K$. Now the last part of (iii) follows from (3) and from [7; Lemma 4.7(iii)].

(iv) This is immediate from [8; Theorem 5.6], since [7; Lemma 4.7] is quite general so that relations analogous to (1) and (3) hold in these cases too and thus (2) is true.

(v) The proof is similar to that of (iv) except that we have to appeal to [8; Theorem 3.6] instead of [8; Theorem 5.6].

This completes the proof of the theorem. □

In view of Theorem 1, the following theorem is the same as the result [5; 5.1] of Kelley (i.e., the $\mathcal{A}$-base structure theorem of Kelley). In [5], the result 5.1 is proved for a particular case only and the proof of the general case is left to the reader.

THEOREM 3 (**$\mathcal{A}$-BASE STRUCTURE THEOREM OF KELLEY**). ([5]) *There exists a maximal orthogonal family $\mathcal{F}$ of primitive projections in the abelian von Neumann algebra $\mathcal{A}$ on H. For $F \in \mathcal{F}$, let J_F be a proper $\mathcal{A}$-base for F. Let $L_2(\mathcal{M}, \mathcal{B}(\mathcal{M}), \rho_G(x)) = L_2(\rho_G(x))$. Choose $x_F \in J_F$, $F \in \mathcal{F}$. Let $\operatorname{card} J_F = n_F$. Then there exists an isomorphism U from H onto $K = \bigoplus_{F\in\mathcal{F}} \bigoplus_{n_F} L_2(\rho_G(x_F))$ such that*

$$UAU^{-1}(f_{F,j})_{F\in\mathcal{F},\, j\in J_F} = (gf_{F,j})_{F\in\mathcal{F},\, j\in J_F}, \qquad (f_{F,j})_{F\in\mathcal{F},\, j\in J_F} \in K$$

for $A \in \mathcal{A}$, where $g \in C(\mathcal{M})$ with $T_g = A$.

Proof. For $n \in M_{\mathcal{A}}$, let $\mathcal{F}_n$ be a maximal orthogonal family of primitive non zero subprojections of Q_n (see the proof of Theorem 1). Let $\mathcal{F} = \bigcup_{n\in\mathcal{M}_{\mathcal{A}}} \mathcal{F}_n$. Then $\mathcal{F}$ is a maximal orthogonal family of primitive projections in $\mathcal{A}$ since $\sum_{\mathcal{F}} F = \sum_{n\in M_{\mathcal{A}}} \sum_{F\in\mathcal{F}_n} = \sum_{n\in M_{\mathcal{A}}} Q_n = I$.

Let $F \in \mathcal{F}$. Then, as observed above (before Definition 8), F is cyclic in $\mathcal{A}$ with UH-multiplicity and the cardinality n_F of the proper $\mathcal{A}$-base J_F for F is its UH-multiplicity. By the definition of $\mathcal{A}$-base, $F = C_{[\mathcal{A}x_F]} = C_G(\rho_G(x_F))$, the last equality being due to [3; Theorem 66.2] of Halmos. If $\nu \in \Sigma(\mathcal{B}(\mathcal{M}))$ with $0 \neq \nu \ll \rho_G(x_F)$, then by [3; Theorem 65.3] of Halmos there exists a vector $y \in [\mathcal{A}x_F]$ such that $\rho_G(y) = \nu$ and hence $C_G(\nu) \neq 0$. Since $0 \neq C_G(\nu) \leq F$, by Corollary 1 the projection $C_G(\nu)$ has UH-multiplicity n_F. Thus $\rho_G(x_F)$ has uniform multiplicity $u_G(\rho_G(x_F)) = n_F > 0$ relative to $G(\cdot)$.

Since $\sum_{F\in\mathcal{F}} C_G(\rho_G(x_F)) = \sum_{\mathcal{F}} F = I$ and $\{\rho_G(x_F)\}_{F\in\mathcal{F}}$ is an orthogonal family in $\Sigma(\mathcal{B}(\mathcal{M}))$ with $u(\rho_G(x_F)) > 0$ being uniform, by [8; Theorem 3.5] there exists an OTSR of H relative to $G(\cdot)$ with the measure family $\{\rho_G(x_F)\}_{F\in\mathcal{F}}$. Now the result follows from Theorem 2(v).

This completes the proof of the theorem. □

Now we shall give some characterizations of spatial isomorphisms of abelian von Neumann algebras involving the equivalence of certain spectral representations.

THEOREM 4. *Let $\mathcal{A}_i$ be an abelian von Neumann algebra on H_i for $i = 1, 2$, and let Φ be an isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$. If $G_1(\cdot)$ is the canonical spectral measure of $\mathcal{A}_1$ and $F_2(\cdot) = \Phi \circ G_1(\cdot)$, then the following hold:*

(i) *If $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are countably decomposable, then Φ is spatial if and only if any two OSRs (resp. any two COBOTSRs) of H_1 and H_2 relative to $G_1(\cdot)$ and $F_2(\cdot)$, respectively, are equivalent.*

(ii) *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are countably decomposable, then Φ is spatial if and only if any two BOTSRs of H_1 and H_2 relative to $G_1(\cdot)$ and $F_2(\cdot)$, respectively, are equivalent.*

(iii) *Φ is spatial if and only if any two OTSRs of H_1 and H_2 relative to $G_1(\cdot)$ and $F_2(\cdot)$, respectively, are equivalent.*

Proof. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are the norm closures of the linear subspaces generated by the ranges of $G_1(\cdot)$ and $F_2(\cdot)$, respectively, it follows that Φ is spatial if and only if the spectral measures $G_1(\cdot)$ and $F_2(\cdot)$ are unitarily equivalent.

(i) follows from Theorem 2(ii) above and from [7; Theorem 4.2(iv)] (resp. and from [8; Theorem 5.8]).

(ii) holds by Theorem 2(i) above and by [8; Theorem 5.8].

(iii) is due to [8; Theorem 4.6].

This completes the proof of the theorem. $\square$

4. Kelley's multiplicity function of $\mathcal{A}$ with $\mathcal{A}'$ countably decomposable

Suppose the abelian von Neumann algebra $\mathcal{A}$ on H has its commutant $\mathcal{A}'$ countably decomposable. Then by Theorem 2 (ii) the canonical spectral measure $G(\cdot)$ of $\mathcal{A}$ has CGS-property in H. Consequently, as in Definition 4, we can associate with $G(\cdot)$ the multiplicity functions m_p and m_c defined on $\mathcal{M}$. Now we shall study the relation between Kelley's multiplicity function ϕ of $\mathcal{A}$ and the multiplicity functions m_p and m_c.

LEMMA 1. *Let $\mathcal{A}'$ be countably decomposable with the OSD-multiplicity of $G(\cdot)$ $\aleph_0$ and let the discrete part p_G be void. If $H = \bigoplus_1^\infty Z_G(x_i)$ is an OSD of H relative to $G(\cdot)$, let $e_i = e_{C_G(\rho_G(x_i))}$ $(= e_{[\mathcal{A}'x_i]})$, the clopen set in $\mathcal{M}$ corresponding to the projection $C_G(\rho_G(x_i))$. Let $e_0 = \bigcap_{i=1}^\infty e_i$. Then the following assertions hold:*

(i) *e_0 is non void and closed.*

(ii) *If $M_{\mathcal{A}} \cap \mathbb{N} = \{n_j\}_1^\infty$, then for $t \in e_{n_j} \backslash e_{n_{j+1}}$, $m_c(t) = n_j$; if $M_{\mathcal{A}} \cap \mathbb{N} = \{n_j\}_1^k$, then for $t \in e_{n_j} \backslash e_{n_{j+1}}$, $m_c(t) = n_j$, $j = 1,2,\ldots,k-1$, and for $t \in e_{n_k} \backslash e_{n_k+1}$, $m_c(t) = n_k$; and if $M_{\mathcal{A}} \cap \mathbb{N} = \emptyset$, then $e_0 = \mathcal{M}$ and $m_c(t) = \aleph_0$ for $t \in \mathcal{M}$. Finally, $t \in e_0$ if and only if $m_c(t) = \aleph_0$.*

(iii) *Kelley's multiplicity function ϕ of $\mathcal{A}$ coincides with the multiplicity function m_c on $\mathcal{M}$ and consequently, m_c is continuous on $\mathcal{M}$.*

(iv) *$e_0 = \{t : \phi(t) = \aleph_0\} = \{t : m_c(t) = \aleph_0\}$.*

(v) *$\aleph_0 \in M_{\mathcal{A}}$ if and only if $\operatorname{int} e_0 \neq \emptyset$ (equivalently, $\aleph_0 \notin M_{\mathcal{A}}$ if and only if e_0 is nowhere dense in $\mathcal{M}$). If $\aleph_0 \in M_{\mathcal{A}}$, then $G(e_0) = Q_{\aleph_0}$. If*

$M_{\mathcal{A}} \cap \mathbb{N}$ is non void and finite (resp. if $M_{\mathcal{A}} \cap \mathbb{N} = \emptyset$), then $\aleph_0 \in M_{\mathcal{A}}$ and $e_0 = e_{Q_{\aleph_0}}$ (resp. $e_0 = e_{Q_{\aleph_0}} = \mathcal{M}$) so that e_0 is clopen.

(vi) *$M_{\mathcal{A}}$ coincides with the range of ϕ $(= m_c)$ if and only if* $\operatorname{int} e_0 \neq \emptyset$.

Proof. Let $M_{\mathcal{A}} \cap \mathbb{N} = \{n_1 < n_2 < \dots\} = \{n_j\}_1^k$, where $k \in \mathbb{N} \cup \{\infty\}$. Then by Proposition 3 we have: If $k = \infty$, then

$$\rho_G(x_{n_j}) \underset{\not\equiv}{\gg} \rho_G(x_i) \equiv \rho_G(x_{n_{j+1}}) \qquad \text{for} \quad n_j < i \le n_{j+1}\,, \quad j = 0, 1, 2, \dots,$$

with $n_0 = 0$ and $\rho_G(x_0)$ omitted; and if k is finite, then

$$\rho_G(x_{n_j}) \underset{\not\equiv}{\gg} \rho_G(x_i) \equiv \rho_G(x_{n_{j+1}}) \qquad \text{for} \quad n_j < i \le n_{j+1}\,, \quad 0 \le j \le k-1\,.$$

with $n_0 = 0$ and $\rho_G(x_0)$ omitted and

$$\rho_G(x_{n_k}) \underset{\not\equiv}{\gg} \rho_G(x_{n_k+1}) \equiv \rho_G(x_{n_k+2}) \equiv \dots\,.$$

Then by [3; Theorem 65.2] of Halmos, $\{e_i\}_1^\infty$ is a non increasing sequence of clopen sets in $\mathcal{M}$ and these clopen sets satisfy the following relations: If $k = \infty$ and $n_0 = 0$, then

$$e_\ell = e_{n_j} \qquad \text{for} \quad n_{j-1} < \ell \le n_j\,, \quad j \in \mathbb{N}.$$

If k is finite and $n_0 = 0$, then

$$e_\ell = e_{n_j} \qquad \text{for} \quad n_{j-1} < \ell \le n_j\,, \quad 1 \le j \le k$$

and

$$e_\ell = e_{n_k+1} \qquad \text{for} \quad l > n_k\,.$$

Moreover, by Proposition 3(i),

$$C_G(\rho_G(x_1)) = I \qquad \text{and} \qquad C_G(\rho_G(x_j)) = I - \sum_{n<j,\, n \in M_{\mathcal{A}} \cap \mathbb{N}} Q_n \tag{4}$$

for $j > 1$.

Hence, if $k = \infty$, then

$$e_{Q_{n_j}} = e_{n_j} \backslash e_{n_{j+1}} \qquad \text{for} \quad j \in \mathbb{N} \tag{5}$$

and if k is finite, then

$$e_{Q_{n_j}} = e_{n_j} \backslash e_{n_{j+1}} \quad \text{for } 1 \le j \le k-1 \qquad \text{and} \qquad e_{Q_{n_k}} = e_{n_k} \backslash e_{n_k+1}\,. \qquad 6)$$

Claim 1. For a non void open set U in $\mathcal{M}$, $G(U) \neq 0$.

In fact, $\overline{U}$ is clopen and $\overline{U} \backslash U$ is nowhere dense in $\mathcal{M}$. Hence $G(\overline{U} \backslash U) = 0$ and therefore, $G(U) = G(\overline{U}) = T_{\lambda_{\overline{U}}} \neq 0$ (see Notation 4).

(i) Since $\{e_j\}_1^\infty$ is a non increasing sequence of non void closed sets in the compact space $\mathcal{M}$, it follows that e_0 is a non void closed set.

(ii) Let $t \in e_{n_j} \backslash e_{n_{j+1}}$ for $j \in \mathbb{N}$ if $k = \infty$ and for $j = 1, 2, \ldots, k-1$ if $k \in \mathbb{N}$. Then for $1 \leq i \leq n_j$ and for an open neighborhood U of t, by [3; Theorem 66.2] we have $C_{Z_G(G(U)x_i)} = C_{G(U)Z_G(x_i)} = G(U)C_{Z_G(x_i)} = G(U)C_G(\rho_G(x_i)) = G(U)G(e_i) = G(U \cap e_i) \neq 0$ by Claim 1 as $U \cap e_i$ is an open neighborhood of t. Consequently, $G(U)x_i \neq 0$. Let $k = \infty$. As $t \in e_{n_j} \backslash e_{n_{j+1}}$, $U_0 = e_{n_j} \backslash e_{n_{j+1}}$ is a clopen neighborhood of t, and as $U_0 \cap e_{n_{j+1}} = \emptyset$, we have $C_{Z_G(G(U_0)x_{n_j+1})} = C_{G(U_0)Z_G(x_{n_j+1})} = G(U_0)C_G(\rho_G(x_{n_j+1})) = G(U_0)G(e_{n_{j+1}}) = G(U_0 \cap e_{n_{j+1}}) = 0$ so that $G(U_0)x_{n_j+1} = 0$. Thus $m_c(t) = n_j$ if $t \in e_{n_j} \backslash e_{n_{j+1}}$ (see Definition 4). When $k \in \mathbb{N}$, one can similarly show that $m_c(t) = n_k$ for $t \in e_{n_k} \backslash e_{n_k+1}$ and $m_c(t) = n_j$ if $t \in e_{n_j} \backslash e_{n_{j+1}}$, $1 \leq j \leq k-1$.

Suppose $t \in e_0$. Then $t \in e_i$ for each $i \in \mathbb{N}$ and consequently, as in the above, $G(U)x_i \neq 0$ for each open neighborhood U of t and for each $i \in \mathbb{N}$. Hence, by Definition 4, $m_c(t) = \aleph_0$ for $t \in e_0$. Conversely, suppose $m_c(t) = \aleph_0$ for some $t \in \mathcal{M}$. Then for each open neighborhood U of t, $G(U)x_i \neq 0$ for each $i \in \mathbb{N}$. Then, clearly, $G(U)Z_G(x_i) \neq 0$ so that $0 \neq C_{G(U)Z_G(x_i)} = G(U)C_G(\rho_G(x_i)) = G(U)G(e_i) = G(U \cap e_i)$ and hence $U \cap e_i \neq \emptyset$ for each open neighborhood U of t and for each $i \in \mathbb{N}$. Thus $t \in \overline{e}_i = e_i$ for each $i \in \mathbb{N}$. Therefore, $t \in e_0$.

If $M_{\mathcal{A}} \cap \mathbb{N} = \emptyset$, then by Proposition 3(iii) we have $M_{\mathcal{A}} = \{\aleph_0\}$ and $\rho_G(x_1) \equiv \rho_G(x_2) \equiv \ldots$. Consequently, $C(\rho_G(x_j)) = Q_{\aleph_0} = I$ for all $j \in \mathbb{N}$. Thus $e_0 = \mathcal{M}$.

(iii) If $k = \infty$, then by (5) we have $e_{Q_{n_j}} = e_{n_j} \backslash e_{n_{j+1}}$ for $j \in \mathbb{N}$ and consequently, by Theorem 1, $\phi(t) = n_j$ for $t \in e_{n_j} \backslash e_{n_{j+1}}$, for $j \in \mathbb{N}$. Thus, by (ii), $\phi(t) = m_c(t)$ for $t \in \mathcal{M} \backslash e_0$. If k is finite, then $e_0 = e_{n_k+1}$ and $\mathcal{M} \backslash e_0 = \bigcup_{j=1}^{k-1} (e_{n_j} \backslash e_{n_{j+1}}) \cup (e_{n_k} \backslash e_{n_k+1})$ and thus by (6), $\mathcal{M} \backslash e_0 = \bigcup_1^k e_{Q_{n_j}}$. Then by (ii) and Theorem 1 it follows that $\phi(t) = m_c(t)$ for $t \in \mathcal{M} \backslash e_0$.

Now let $t \in e_0$. If $M_{\mathcal{A}} \cap \mathbb{N} \neq \emptyset$ and k is finite, then as seen above, $e_0 = e_{n_k+1}$ and by Proposition 3(i), $\aleph_0 \in M_{\mathcal{A}}$ and $e_{n_k+1} = e_{Q_{\aleph_0}}$. If $M_{\mathcal{A}} \cap \mathbb{N} = \emptyset$, then as seen above $e_j = \mathcal{M}$ for $j \in \mathbb{N}$ so that $e_0 = \mathcal{M} = e_{Q_{\aleph_0}}$ since $Q_{\aleph_0} = I$. Then, in both cases, by Theorem 1, $\phi(t) = \aleph_0$ and consequently, by (ii), $m_c(t) = \aleph_0 = \phi(t)$. If $k = \infty$, note that by (i), $e_0 \neq \emptyset$ and hence such t exists in e_0. Now we consider the following two cases.

Case 1. $\aleph_0 \in M_{\mathcal{A}}$.

Then $Q_{\aleph_0} \neq 0$ and by Proposition 3, $e_0 \supset e_{Q_{\aleph_0}}$, and hence, by Theorem 1, $\phi(t) = \aleph_0$ for $t \in e_{Q_{\aleph_0}}$. Let $t \in (e_0 \backslash e_{Q_{\aleph_0}})$. Since $\mathcal{M} = \overline{\left(\bigcup_{j=1}^{\infty} e_{Q_{n_j}} \cup e_{Q_{\aleph_0}} \right)}$, there exists a net $\{t_\alpha\}$ in $\left(\bigcup_{j=1}^{\infty} e_{Q_{n_j}}\right) \cup e_{Q_{\aleph_0}}$ such that $t_\alpha \to t$. Since the function ϕ is continuous on $\mathcal{M}$, we have $\phi(t) = \lim_\alpha \phi(t_\alpha)$. If $\phi(t) = n \in \mathbb{N}$, then clearly $\phi(t_\alpha) = n$ eventually and moreover, $n = n_j$ for some $j \in \mathbb{N}$. This means that $t_\alpha \in e_{Q_{n_j}}$ eventually and hence $t \in e_{Q_{n_j}}$. This is impossible since $\mathcal{M} \backslash e_0 = \bigcup_{j=1}^{\infty} e_{Q_{n_j}}$. Thus the cardinals $\phi(t_\alpha)$ converge to $\aleph_0$ and hence $\phi(t) = \aleph_0$. Then by (ii) we have $\phi(t) = m_c(t)$ for all $t \in e_0$ and consequently, $\phi = m_c$ on $\mathcal{M}$. Since ϕ is continuous on $\mathcal{M}$, m_c is also continuous on $\mathcal{M}$.

Case 2. $\aleph_0 \notin M_{\mathcal{A}}$.

Let $t \in e_0$. In this case, $\mathcal{M} = \overline{\left(\bigcup_{j=1}^{\infty} e_{Q_{n_j}}\right)}$ and hence there exists a net $\{t_\alpha\} \subset \bigcup_{j=1}^{\infty} e_{Q_{n_j}}$ such that $t_\alpha \to t$. As in the previous case, we have $\phi(t) = \lim_\alpha \phi(t_\alpha)$ and $\phi(t) \neq n$ for any $n \in \mathbb{N}$. Thus $\phi(t) = \aleph_0$. The rest of the argument is as in Case 1 and hence (iii) holds in this case also.

(iv) This is immediate from the second part of (ii) and from (iii).

(v) If $\aleph_0 \in M_{\mathcal{A}}$, then $Q_{\aleph_0} \neq 0$ and by Proposition 3, $e_{Q_{\aleph_0}} \subset e_j$ for all $j \in \mathbb{N}$. Hence $\operatorname{int} e_0 \neq \emptyset$. Conversely, let $e = \operatorname{int} e_0 \neq \emptyset$. Then $G(e) = G(\overline{e}) \neq 0$ by Claim 1, and by (4) we have $G(e) \le G(e_j) = C_G(\rho_G(x_j)) = I - \sum_{p \in M_{\mathcal{A}} \cap \mathbb{N},\, p<j} Q_p$, for $j \in \mathbb{N}$. Hence

$$G(e) \le \bigwedge_{j=1}^{\infty}\left(I - \sum_{p \in M_{\mathcal{A}} \cap \mathbb{N},\, p<j} Q_p\right) = I - \sum_{n_j \in M_{\mathcal{A}} \cap \mathbb{N}} Q_{n_j}.$$

Since $\sum_{n \in M_{\mathcal{A}}} Q_n = I$, it follows that $\aleph_0 \in M_{\mathcal{A}}$ and that $G(e) \le Q_{\aleph_0}$. Moreover, by Proposition 3 we have $e_{Q_{\aleph_0}} \subset e_j$ for $j \in \mathbb{N}$ and hence $e_{Q_{\aleph_0}} \subset e_0$. Consequently, $e_{Q_{\aleph_0}} \subset e$. Thus $G(e) = Q_{\aleph_0}$. As $e_0 \backslash e$ is nowhere dense, it follows that $G(e_0) = Q_{\aleph_0}$.

Now, let $T = M_{\mathcal{A}} \cap \mathbb{N}$. Then by Proposition 3(vi), $M_{\mathcal{A}}$ is of the form $M_{\mathcal{A}} = \{n_j\}_1^k \cup \{\aleph_0\}$ with $k \in \mathbb{N}$ when $T \neq \emptyset$ and finite, and $M_{\mathcal{A}} = \{\aleph_0\}$ when $T = \emptyset$. Then by (4) we have $e_{n_k+1} = e_{Q_{\aleph_0}}$, where $n_k = 0$ if $T = \emptyset$. Thus $e_0 = e_{Q_{\aleph_0}}$ and hence e_0 is clopen.

(vi) By (ii) and (iii), $M_{\mathcal{A}} \cap \mathbb{N} = \{\phi(t) : t \in \mathcal{M} \backslash e_0\} = \{m_c(t) : t \in \mathcal{M} \backslash e_0\}$. By (v), $\aleph_0 \in M_{\mathcal{A}}$ if and only if $\operatorname{int} e_0 \neq \emptyset$. Since by (iv), $\phi(t) = m_c(t) = \aleph_0$ for $t \in e_0$, (vi) holds.

This completes the proof of the lemma. □

The proof of the following lemma is similar to that of Lemma 1 and is left to the reader.

LEMMA 2. *With $\mathcal{A}'$ countably decomposable, suppose the OSD-multiplicity of $G(\cdot)$ is finite and suppose $H = \bigoplus_1^n Z_G(x_i)$ is an OSD of H relative to $G(\cdot)$. Let the discrete part $p_G = \emptyset$ and let $e_i = e_{C_G(\rho_G(x_i))}$ for $1 \le i \le n$. Then the following assertions hold:*

(i) *Kelley's multiplicity function ϕ is the same as m_c on $\mathcal{M}$.*
(ii) *$M_{\mathcal{A}}$ is given by the range of ϕ $(= m_c)$.*

LEMMA 3. *Let Q be a non zero projection in $\mathcal{A}$. Then the maximal ideal space $\mathcal{M}_Q$ of $\mathcal{A}Q$ is homeomorphic to e_Q, the clopen set in $\mathcal{M}$ corresponding to Q. Consequently, if $\mathcal{M}_Q$ and e_Q are identified and if G_Q is the canonical spectral measure of $\mathcal{A}Q$, then $G_Q(\sigma) = G(\sigma)Q$ for $\sigma \in \mathcal{B}(e_Q)$.*

Proof. Let $\Psi\colon C(\mathcal{M}) \to \mathcal{A}$ and $\Psi_Q\colon C(\mathcal{M}_Q) \to \mathcal{A}Q$ be the inverses of the Gelfand isomorphisms so that $\Psi(f) = T_f$ and $\Psi_Q(g) = T_g$ for $f \in C(\mathcal{M})$ and $g \in C(\mathcal{M}_Q)$. For $h \in C(e_Q)$, let us define: $\hat{h}(t) = h(t)$ if $t \in e_Q$ and $\hat{h}(t) = 0$ if $t \in \mathcal{M} \setminus e_Q$. Then, as e_Q is clopen in $\mathcal{M}$, $\hat{h} \in C(\mathcal{M})$ and $\hat{h} = \hat{h}\chi_{e_Q}$. Let $\hat{\Psi}(h) = \Psi(\hat{h})$. Then $\hat{\Psi}(h) = \Psi(\hat{h})Q$ and clearly, $\hat{\Psi}$ is an involution preserving algebraic isomorphism from $C(e_Q)$ into $\mathcal{A}Q$. Given $A \in \mathcal{A}$, let $f \in C(\mathcal{M})$ such that $\Psi(f) = A$. Let $h = (f\chi_{e_Q})|_{e_Q}$. Then $h \in C(e_Q)$ and $\hat{\Psi}(h) = \Psi(f)Q = AQ$. Thus the isomorphism $\hat{\Psi}\colon C(e_Q) \to \mathcal{A}Q$ is onto. Moreover, $\|\hat{\Psi}(h)\| = \|\Psi(\hat{h})Q\| = \sup_{t\in\mathcal{M}} |\hat{h}(t)| = \sup_{t\in e_Q} |h(t)|$ for $h \in C(e_Q)$ and hence $\hat{\Psi}$ is an isometric isomorphism of $C(e_Q)$ onto $\mathcal{A}Q$. Consequently, $F = \hat{\Psi}^{-1} \circ \Psi_Q\colon C(\mathcal{M}_Q) \to C(e_Q)$ is an onto isometric isomorphism and hence, by [2; Theorem IV.6.26] there exists a bijective bicontinuous mapping $\Phi\colon e_Q \to \mathcal{M}_Q$ such that $(Ff)(t) = f(\Phi(t))$ for $f \in C(\mathcal{M}_Q)$ and $t \in e_Q$. Thus, for e clopen in $\mathcal{M}_Q$, we have $(F\chi_e)(t) = \chi_e(\Phi(t))$ for $t \in e_Q$. In other words, $F\chi_e = \chi_{\Phi^{-1}(e)}$ for e clopen in $\mathcal{M}_Q$. For $\sigma \in \mathcal{B}(\mathcal{M}_Q)$, let $\Upsilon(\sigma)$ be the clopen set associated with σ for which $\Upsilon(\sigma)\Delta\sigma$ is meagre. Then evidently, $\Upsilon(\Phi^{-1}(\sigma)) = \Phi^{-1}(\Upsilon(\sigma))$ as Φ is a homeomorphism. Consequently, for $\sigma \in \mathcal{B}(\mathcal{M}_Q)$ we have $G_Q(\sigma) = \Psi_Q(\chi_{\Upsilon(\sigma)}) = \Psi_Q \circ F^{-1}(\chi_{\Phi^{-1}(\Upsilon(\sigma))}) = \hat{\Psi}(\chi_{\Phi^{-1}(\Upsilon(\sigma))}) = G(\Phi^{-1}(\Upsilon(\sigma)))Q = G(\Upsilon(\Phi^{-1}(\sigma)))Q = G(\Phi^{-1}(\sigma))Q$.

This completes the proof of the lemma. □

THEOREM 5. *Suppose the abelian von Neumann algebra $\mathcal{A}$ on H has its commutant countably decomposable. Let $G(\cdot)$ be the canonical spectral measure of $\mathcal{A}$ defined on $\mathcal{B}(\mathcal{M})$, $\mathcal{M}$ being the maximal ideal space of $\mathcal{A}$. If ϕ is Kelley's mul-*

tiplicity function of $\mathcal{A}$ and m_p and m_c are the multiplicity functions associated with $G(\cdot)$ as in Definition 4, then the following assertions hold:

(i) *For $t \in p_G$, $\{t\}$ is clopen in $\mathcal{M}$. Consequently, p_G is open in $\mathcal{M}$.*

(ii) *For $t \in \mathcal{M} \setminus p_G$, $m_p(t) = 0$; for $t \in \overline{p}_G$, $m_c(t) = 0$.*

(iii) *$m_p(t) = \phi(t) \in M_{\mathcal{A}}$ for $t \in p_G$; $m_c(t) = \phi(t)$ for $t \in \mathcal{M} \setminus \overline{p}_G$. Thus*

$$\phi(t) = \max\big(m_p(t), m_c(t)\big)\,, \qquad t \in \mathcal{M} \setminus \big(\overline{p}_G \setminus p_G\big)\,.$$

(iv) *The following statements are equivalent:*
- (a) $\phi = \max(m_p, m_c)$.
- (b) p_G *is closed.*
- (c) $\max(m_p, m_c)$ *is continuous.*

(v) *The following assertions hold:*
- (a) $M_{\mathcal{A}} \cap \mathbb{N} = \{\phi(t) :\ t \in \mathcal{M}\} \cap \mathbb{N}$.
- (b) $\aleph_0 \in M_{\mathcal{A}}$ *if and only if* $e = \phi^{-1}(\aleph_0)$ *has non void interior.*
- (c) *When the range of ϕ is a finite set, $\aleph_0 \in M_{\mathcal{A}}$ if and only if $\phi^{-1}(\aleph_0)$ is a non void open set.*

Proof.

(i) Let $t \in p_G$ and let $\Upsilon(\{t\})$ be the clopen set in $\mathcal{M}$ corresponding to $\{t\}$ so that $\{t\}\Delta\Upsilon(\{t\})$ is meagre in $\mathcal{M}$. Then $G\big(\Upsilon(\{t\})\setminus\{t\}\big) = 0$. As $\Upsilon(\{t\})\setminus\{t\}$ is open, by Claim 1 in the proof of Lemma 1 we conclude that $\Upsilon(\{t\}) = \{t\}$ and hence (i) holds.

(ii) By the definition of p_G, $m_p(t) = 0$ for $t \in \mathcal{M} \setminus p_G$. Now let $t \in \overline{p}_G$. By (i), $\overline{p}_G$ is clopen and moreover, $\overline{p}_G \setminus p_G$ is nowhere dense in $\mathcal{M}$. Thus $G(p_G) = G(\overline{p}_G)$ so that $G(c_G) = G(\mathcal{M} \setminus p_G) = G(\mathcal{M} \setminus \overline{p}_G)$. Since $\overline{p}_G$ is clopen, $U = \overline{p}_G$ is an open neighborhood of t such that $U \cap (\mathcal{M} \setminus \overline{p}_G) = \emptyset$. Then $G(U)G(c_G) = 0$ and hence $m_c(t) = 0$ by Definition 4.

(iii) Let $t \in p_G$. Suppose $m_p(t) = n \in \mathbb{N}$ (resp. $m_p(t) = \aleph_0$). Then there exists an orthonormal basis $\{x_k\}_1^n$ (resp. $\{x_k\}_1^\infty$) in $G(\{t\})H$. Then, as $G(\sigma)x_k = 0$ if $t \notin \sigma$ and $G(\sigma)x_k = x_k$ if $t \in \sigma$, it follows that $Z_G(x_k) = [x_k]$ and hence $G(\{t\})H = \bigoplus_1^n Z_G(x_k)$ (resp. $= \bigoplus_1^\infty Z_G(x_k)$). Moreover, $\rho_G(x_k) = \delta_t$ where δ_t denotes the Dirac measure at $\{t\}$ on $\mathcal{B}(\mathcal{M})$ and hence $\rho_G(x_1) = \rho_G(x_2) \cdots = \rho_G(x_n)$ (resp. $\rho_G(x_1) = \rho_G(x_2) = \ldots$). Thus $G(\{t\})H = \bigoplus_1^n Z_G(x_i)$ (resp. $= \bigoplus_1^\infty Z_G(x_i)$) is an OSD of $G(\{t\})H$. Then, as $C_G(\rho_G(x_1)) = G(\{t\})$, we have $C_{Z_G(x_k)} = C_G(\rho_G(x_k)) = G(\{t\})$ for $k = 1, 2, \ldots, n$ (resp. for $k = 1, 2, \ldots$). Moreover, by [3; Theorem 60.2] of Halmos, $\{Z_G(x_k)\}_1^n$

(resp. $\{Z_G(x_k)\}_1^{\infty}$) is an orthogonal family of abelian projections in $\mathcal{A}'$. Hence the projection $G(\{t\})$ has UH-multiplicity n (resp. $\aleph_0$). Thus n (resp. $\aleph_0$) $\in M_{\mathcal{A}}$ and moreover, $G(\{t\}) \leq Q_n$ (resp. $\leq Q_{\aleph_0}$) by Corollary 1. Therefore, $t \in e_{Q_n}$ (resp. $\in e_{Q_{\aleph_0}}$) and hence, by Theorem 1 we have $\phi(t) = m_p(t)$. Since $e_{G(c_G)} = \mathcal{M} \setminus \overline{p}_G$ (see the proof of (ii)), by Lemma 3 the maximal ideal space $\mathcal{M}_1$ of $\mathcal{A}G(c_G)$ can be identified with $\mathcal{M} \setminus \overline{p}_G$ and by the last part of Lemma 3 it follows that the discrete part $p_{GG(c_G)} = \emptyset$. Consequently, by Lemmas 1 and 2 we have $m_c(t) = \phi(t)$ for $t \in \mathcal{M} \setminus \overline{p}_G$. Hence (iii) holds.

(iv) From the definition of ϕ it is clear that $\phi(t) \neq 0$ for any $t \in \mathcal{M}$. Moreover, (ii) implies that $\max(m_p(t), m_c(t)) = 0$ for $t \in \overline{p}_G \setminus p_G$. These observations show that (a) $\Longrightarrow$ (b). If p_G is closed, then by (iii) we have $\phi = \max(m_p, m_c)$ on $\mathcal{M}$ and hence (b) $\Longrightarrow$ (a) & (c). Finally, since $\mathcal{M} \setminus (\overline{p}_G \setminus p_G)$ is dense in $\mathcal{M}$ and ϕ is continuous on $\mathcal{M}$, (c) $\Longrightarrow$ (a) by (iii).

(v)(a) First let us observe that $M_{\mathcal{A}} = \{n \in M_{\mathcal{A}} : Q_n G(p_G) \neq 0\} \cup \{n \in M_{\mathcal{A}} : Q_n G(c_G) \neq 0\}$. Thus $M_{\mathcal{A}} \cap \mathbb{N} = (M_{\mathcal{A}G(p_G)} \cap \mathbb{N}) \cup (M_{\mathcal{A}G(c_G)} \cap \mathbb{N})$. As shown in the proof of (iii), $\{\phi(t) : \ t \in p_G\} \cap \mathbb{N} = M_{\mathcal{A}G(p_G)} \cap \mathbb{N}$. By Lemmas 1, 2 and 3 it follows that $\{\phi(t) : \ t \in \mathcal{M} \setminus \overline{p}_G\} \cap \mathbb{N} = M_{\mathcal{A}G(c_G)} \cap \mathbb{N}$. Thus we have $\{\phi(t) : \ t \in \mathcal{M} \setminus (\overline{p}_G \setminus p_G)\} \cap \mathbb{N} = M_{\mathcal{A}} \cap \mathbb{N}$. Now let $t \in \overline{p}_G \setminus p_G$. Then there exists a net $\{t_\alpha\}$ in p_G such that $t_\alpha \to t$ and consequently, by the continuity of ϕ, $\phi(t) = \lim_\alpha \phi(t_\alpha)$. If $\phi(t) = n \in \mathbb{N}$, then $\phi(t_\alpha) = n$ eventually and hence $t_\alpha \in e_{Q_n}$ eventually. Thus $n \in \{\phi(u) : \ u \in p_G\} \cap \mathbb{N} \subset \mathcal{M}_{\mathcal{A}} \cap \mathbb{N}$. The other possibility is $\phi(t) = \aleph_0$ and in that case, clearly $\phi(t) \notin \mathcal{M}_{\mathcal{A}} \cap \mathbb{N}$. Thus $M_{\mathcal{A}} \cap \mathbb{N} = \{\phi(t) : \ t \in \mathcal{M}\} \cap \mathbb{N}$.

(v)(b) Now let $\aleph_0 \in M_{\mathcal{A}}$. Then the clopen set $e_{Q_{\aleph_0}}$ is non void and by Theorem 1 we have $\phi(t) = \aleph_0$ for all $t \in e_{Q_{\aleph_0}}$. Thus $e_{Q_{\aleph_0}} \subset e$ and hence e has non void interior. Conversely, suppose e has non void interior U. Since ϕ is continuous, e is closed and hence $\overline{U} \subset e$. Let $Q = G(\overline{U}) = G(U)$. Then the non zero projection Q belongs to $\mathcal{A}$ and the clopen set $e_Q = \overline{U} \subset e$. Hence $\phi(t) = \aleph_0$ for all $t \in e_Q$. If $QQ_n \neq 0$ for some $n \in \mathbb{N}$, then for $t \in e_{QQ_n} = e_Q \cap e_{Q_n}$ we have $\phi(t) = n$ by Theorem 1. This contradiction shows that $\aleph_0 \in M_{\mathcal{A}}$ and $Q \leq Q_{\aleph_0}$. (Note that $Q = Q_{\aleph_0}$, since by Theorem 1, $\phi(t) = \aleph_0$ for $t \in e_{Q_{\aleph_0}}$.)

(v)(c) Let us now suppose that the range of ϕ is a finite set. If $\aleph_0 \in M_{\mathcal{A}}$, then $Q_{\aleph_0} = I - \sum_{n \in M_{\mathcal{A}} \cap \mathbb{N}} Q_n$. By hypothesis and by (v)(a) we can assume $M_{\mathcal{A}} \cap \mathbb{N} = \{n_j\}_1^k$ (resp. $= \emptyset$). Then $\mathcal{M} = \bigcup_{j=1}^{k} e_{Q_{n_j}} \cup e_{Q_{\aleph_0}}$ (resp. $= e_{Q_{\aleph_0}}$) so that, by Theorem 1, $\phi^{-1}(\aleph_0) = e_{Q_{\aleph_0}}$ is clopen and non void in $\mathcal{M}$. Conversely, if $\phi^{-1}(\aleph_0)$ is a non void open set, then $\aleph_0 \in M_{\mathcal{A}}$ by (v)(b).

This completes the proof of the theorem. □

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Received November 2, 2000

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