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A NOTE ON RESTRICTED MEASURABILITY

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McMinn in [2] introduced into the general theory of measure a concept of restricted measurability and established the condition under which it is equivalent to the measurability in the usual Carathéodory sense. It is our aim to give another condition under which the two concepts of measurability are equivalent and to show that our condition is somewhat weaker than that of McMinns.

Throughout this paper we consider X to be a fixed set with respect to which we make definitions.

Definition 1. *Let μ be an outer measure on a hereditary σ-ring $\mathscr{H}$. Let $\mathscr{B} \subset \mathscr{H}$ be a hereditary family. A set E in $\mathscr{H}$ is called μ measurable $\mathscr{B}$ iff, for every set B in $\mathscr{B}$.*

$$\mu(B) = \mu(B \cap E) + \mu(B \cap E'),$$

where $E' = X - E$ is the complement of E.

It is clear that if $\mathscr{B} = \mathscr{H}$ then we obtain the usual measurability introduced by Caratheodory.

We shall denote the σ-ring of all μ measurable $\mathscr{B}$ sets by $\mathscr{S}_{\mathscr{B}}$, and the σ-ring of all μ measurable sets by $\mathscr{S}$. Recall that $\mathscr{S} = \mathscr{S}_{\mathscr{H}}$.

Clearly $\mathscr{S} \subset \mathscr{S}_{\mathscr{B}}$ holds for every hereditary $\mathscr{B} \subset \mathscr{H}$.

Definition 2. *$\mathscr{B}$ is called μ convenient iff μ is an outer measure defined on a hereditary σ-ring $\mathscr{H}$, $\mathscr{B} \subset \mathscr{H}$ is hereditary, and corresponding to each $A \in \mathscr{H}$ of a finite outer measure there exists such a sequence $\{C_i\}_{i=1}^{\infty}$ such that*

$$\mu\left(A - \bigcup_{i=0}^{\infty} C_i\right) = 0$$

and, for each integer i,

$$C_i \subset C_{i+1} \in \mathscr{B},$$

C_i is μ measurable $\mathscr{B}$.

Theorem 3. (Theorem 3.4. of [2]). *If $\mathscr{B}$ is μ convenient, then E is μ measurable whenever E is μ measurable $\mathscr{B}$, i.e. $\mathscr{S}_{\mathscr{B}} = \mathscr{S}$.*

For our next consideration we need the following:

Definition 4.

(i) *If $\mathcal{H}$ is a hereditary σ-ring and μ is an outer measure on $\mathcal{H}$, we shall denote by H the set of all elements of a finite outer measure in $\mathcal{H}$.*

(ii) *For any two elements E and F in H we shall write*

$$\varrho(E, F) = \mu(E \dot{-} F),$$

where $E \dot{-} F = (E - F) \cup (F - E)$ is the symmetric difference of the sets E and F.

(iii) *By $H_>$ we denote the following $H_> = \{E /\ E \in H,\ \mu(E) > 0\}$.*

It is easy to verify that the function ϱ is a pseudometric on H.

The following two assertions are the principal theorems of this paper.

Theorem 5. *If $\mathcal{B} \cap H$ is dense in $H_>$, i.e. $\overline{\mathcal{B} \cap H} \supset H_>$* (where $\overline{\mathcal{B} \cap H}$ denotes the closure of the set $\mathcal{B} \cap H$ in the pseudometric space (H, ϱ)), *then E is μ measurable $\mathcal{B}$ iff E is μ measurable, i.e. $\mathcal{S}_{\mathcal{B}} = \mathcal{S}$.*

Proof. If $E \in \mathcal{S}_{\mathcal{B}}$, we must show that

(1) $$\mu(A) = \mu(A \cap E) + \mu(A \cap E')$$

for every $A \in \mathcal{H}$.

If $\mu(A)$ is zero or infinite, then (1) is trivial. If now $0 < \mu(A) < \infty$, then $A \in H_>$. From the density of $\mathcal{B} \cap H$ in $H_>$ it follows that there exists a sequence of sets $B_n \in \mathcal{B} \cap H$, $n = 1, 2, \ldots$ so that $B_n \overset{\varrho}{\to} A$. The set E is μ measurable $\mathcal{B}$ from which we have

(2) $$\mu(B_n) = \mu(B_n \cap E) + \mu(B_n \cap E')$$

for every n.

We shall show that

$$\mu(A) = \lim_n \mu(B_n).$$

Clearly

$$B_n \subset (B_n - A) \cup (A - B_n) \cup A$$

and

$$A \subset (A - B_n) \cup (B_n - A) \cup B_n,$$

and so by the monotonity and subadditivity of μ we have

$$\mu(B_n) \leqq \varrho(B_n, A) + \mu(A)$$

and

$$\mu(A) \leqq \varrho(A, B_n) + \mu(B_n).$$

Since $\varrho(A, B_n) \to 0$, we obtain

$$\mu(A) = \lim_n \mu(B_n).$$

Similarly

$$\mu(B_n \cap E) \to \mu(A \cap E) \quad \text{and} \quad \mu(B_n \cap E') \to \mu(A \cap E').$$

Reference to (2) completes the proof.

Theorem 6. *If $\mathcal{B}$ is μ convenient, then $\mathcal{B} \cap H$ is dense in H.*

Remark. Clearly then $\mathcal{B} \cap H$ is dense in $H_>$ too.

Proof. Let A be from H. We take such a sequence $\{B_n^0\}_{n=1}^{\infty}$ that for each integer n

$$B_n^0 \subset B_{n+1}^0, \quad B_n^0 \in \mathcal{B}$$

B_n^0 is μ measurable $\mathcal{B}$ and

$$\mu\left(A - \bigcup_{n=1}^{\infty} B_n^0\right) = 0. \tag{3}$$

Let us put $B_n = B_n^0 \cap A$ for $n = 1, 2, \ldots$. We prove that $B_n \xrightarrow{\varrho} A$. We have

$$\varrho(A, B_n) = \mu(A - B_n).$$

The sets B_n^0 are from $\mathcal{S}_{\mathcal{B}}$, and so (Theorem 3) also from $\mathcal{S}$. Since μ is an outer measure on $\mathcal{H}$, μ_A defined on $\mathcal{S}$ by $\mu_A(C) = \mu(A \cap C)$ is a measure for every $A \in \mathcal{H}$. Hence

$$\lim_n \mu_A(B_n^0) = \mu_A\left(\bigcup_{n=1}^{\infty} B_n^0\right).$$

Since $B_n = B_n^0 \cap A$, it follows that

$$\lim_n \mu(B_n) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right). \tag{4}$$

B_n^0 is μ measurable, hence

$$\mu(A) = \mu(A \cap B_n^0) + \mu(A - B_n^0)$$

and so

$$\mu(A) - \mu(B_n) = \mu(A - B_n).$$

From this and from (4) we have

$$\mu(A) - \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_n \mu(A - B_n).$$

By subadditivity of μ we have

$$0 \leqq \mu(A) - \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \leqq \mu\left(A - \bigcup_{n=1}^{\infty} B_n\right) \leqq \mu\left(A - \bigcup_{n=1}^{\infty} B_n^0\right);$$

therefore by (3) we have

$$\lim_n \mu(A - B_n) = 0,$$

and so $B_n \overset{\varrho}{\rightarrow} A$.

The following example shows that Theorem 5 is stronger than Theorem 3.

Example 7. Let us put $X = \{1, 2, \ldots\}$, $\mathscr{H} = 2^X$, for $A \subset X$ $\mu(A) = 0$ if $A = \emptyset$, $\mu(A) = 1$ if A is finite and non void and $\mu(A) = \infty$ if A is an infinite set. Then μ is an outer measure on $\mathscr{H}$. Let $\mathscr{B} = \{A \subset X / A$ be finite$\}$. Then $\mathscr{B} \cap H$ is dense in H, but $\mathscr{B}$ is not μ convenient and $\mathscr{S}_{\mathscr{B}} = \{\emptyset, X\} = \mathscr{S}$.

We give one more example to show that the condition given in Theorem 5 is not necessary for $\mathscr{S} = \mathscr{S}_{\mathscr{B}}$.

Example 10. Let $X = \{a, b, c\}$, $\mathscr{H} = 2^X$, $\mu(A) = 1$ if $A \neq \emptyset$ and $\mu(\emptyset) = 0$. Let $\mathscr{B} = 2^{\{a, b\}} \cup 2^{\{a, c\}}$. Then $\mathscr{S}_{\mathscr{B}} = \mathscr{S} = \{\emptyset, X\}$ and $\mathscr{B} \cap H$ is not dense in $H_{>}$, since $\mu(X - B) = 1$ for every $B \in \mathscr{B}$.

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ЗАМЕЧАНИЕ О СУЖЕННОЙ ИЗМЕРИТЕЛЬНОСТИ

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Ретюме

Пусть μ-внешная мера на наследственном σ-кольце $\mathscr{H}$. Пусть $\mathscr{B} \subset \mathscr{H}$ наследственный класс. Множество $E \in \mathscr{H}$ Называется μ-измеримым $\mathscr{B}$, если для всякого $B \in \mathscr{B}$ имеет место соотношение

$$\mu(B) = \mu(B \cap E) + \mu(B \cap E').$$

Пусть псевдометрика ϱ_μ на множестве всех элементов конечной внешней меры определена по формуле

$$\varrho_\mu(E, A) = \mu(E \dot{-} A).$$

В статье доказывается следующая теорема:

Если внешная мера μ конечна и $\mathscr{B}$ плотно в $\mathscr{H}$ в псевдометрике ϱ_μ, то $E \in \mathscr{H}$ μ-измеримо $\mathscr{B}$ тогда и только тогда, когда E является μ-измеримым в смысле Каратеодори.