Michal Fečkan

On a certain type of functional differential equations

Mathematica Slovaca, Vol. 43 (1993), No. 1, 39--43

Persistent URL: `http://dml.cz/dmlcz/130391`

Math. Slovaca, 43 (1993), No. 1, 39–43

ON A CERTAIN TYPE OF FUNCTIONAL DIFFERENTIAL EQUATIONS

MICHAL FEČKAN

(*Communicated by Milan Medveď*)

ABSTRACT. Certain types of advanced functional-differential equations are studied by means of a simple fixed point theorem.

1. Introduction

The purpose of this paper is to give an approach to solve certain functional differential equations similar to the following

$$\begin{aligned} u'(x) &= f\big(u(u(x))\big)\,, \\ u(0) &= 0\,, \end{aligned} \tag{1.1}$$

where $f \in C^1(\mathbb{R},\mathbb{R})$. Hence we shall study equations containing the term $u\big(u(x)\big)$.

To show the existence of solutions of (1.1) we can apply the Leray-Schauder degree theory. But we would like to obtain more. We are interested in the existence of a Picard iteration method for (1.1). Then such a technique will enable us to approximate a solution of (1.1). The difficulty in the derivation of a Lipschitz like inequality for (1.1) is the term $u\big(u(x)\big)$. Since to obtain this inequality in C^0-norm we have to use C^1-norm of u. Thus it is impossible to apply the classical implicit function theorem to (1.1). To overcome this difficulty we use a simple fixed point theorem.

Similarly, we study the equation

$$u(x) = \int_0^1 G(x,t)\cdot f\big(u(u(t))\big)\,\mathrm{d}t \tag{1.2}$$

AMS Subject Classification (1991): Primary 34C23. Secondary 34K10.
Key words: Advanced functional-differential equations, Bifurcations.

and the bifurcation problem

$$-u''(x) = \lambda \cdot f\big(u(1+u(x))\big) + u \\ u(0) = u(\pi) = 0 \tag{1.3}$$

with $f(0) = 0$, $f'(0) \neq 0$, $\lambda \in \mathbb{R}$ is near to 0.

A related problem is studied in [1]. There is also explained a motivation of the investigation of such advanced differential equations. We refer the paper [2] for a philosophical background.

2. A simple fixed point theorem

Let $X \subset C^0([0,1],\mathbb{R})$ be a closed nonempty subset. Consider an operator $T\colon X \to X$ such that

i) T is continuous.
ii) $T(X) \subset C^1([0,1],\mathbb{R})$.
iii) $\|T(u_1) - T(u_2)\| \leq \alpha\big(\max\{\|u_1'\|,\|u_2'\|\}\big) \cdot \|u_1 - u_2\|$ for u_1, $u_2 \in C^1([0,1],\mathbb{R})$.
iv) $\|(T(u))'\| \leq \beta(\|u'\|)$,

for α, β continuous functions. Here $\|\cdot\|$ is the C^0-norm.

THEOREM 2.1. *If there is $r > 0$ such that $\beta(t) \leq r$ and $\alpha(t) < 1$ for each $0 \leq t \leq r$, then there is a u satisfying $u = T(u)$. Moreover, $u_n \to u$ in C^0-norm for $u_n = T^n(0)$. This convergence is geometrical.*

Proof. We take

$$Y = \{u \in X\,, \quad \|u'\| \leq r\}.$$

Then $T(Y) \subset Y$ and

$$\|T(u_1) - T(u_2)\| \leq \gamma \cdot \|u_1 - u_2\|$$

for each u_1, $u_2 \in Y$ with $\gamma = \max\limits_{[0,r]} \alpha$. We know that $\gamma < 1$. Hence T is a contraction and the proof is finished.

Remark 2.2. In our applications usually $\beta(\cdot) = \beta$ is constant and α is increasing. Then the assumptions of this theorem have the form $\alpha(\beta) < 1$.

3. Results

First of all, we need the following estimate for u_1, $u_2 \in C^1([0,1],[0,1])$

$$\begin{aligned} &\left|u_1\big(u_1(x)\big) - u_2\big(u_2(x)\big)\right| \\ &\leq \left|u_1\big(u_1(x)\big) - u_1\big(u_2(x)\big)\right| + \left|u_1\big(u_2(x)\big) - u_2\big(u_2(x)\big)\right| \\ &\leq \|u_1'\| \cdot \|u_1 - u_2\| + \|u_1 - u_2\| \\ &= \big(1 + \|u_1'\|\big) \cdot \|u_1 - u_2\| . \end{aligned} \tag{3.1}$$

Now we shall study the problem (1.1).

THEOREM 3.1. *Let $|f(0)| < 1$. Then there is $c > 0$ such that* (1.1) *has a solution $u \in C^1([-c,c],[-c,c])$ and $u_n \to u$ uniformly on $[-c,c]$ for*

$$u_0 = 0, \quad u_{n+1}(t) = \int_0^t f\big(u_n(u_n(s))\big)\,\mathrm{d}s, \qquad t \in [-c,c].$$

P r o o f. Since $|f(0)| < 1$, there is $c > 0$ such that

$$|f(t)| < 1 \tag{3.2}$$

for $|t| \leq c$. We take

$$X = C^0([-c,c],[-c,c]),$$

$$T(u) = \int_0^t f\big(u(u(s))\big)\,\mathrm{d}s.$$

By (3.2) $T(X) \subset X$. If c is sufficiently small, then Remark 2.2 holds and the proof is finished.

Further, we investigate the equation (1.2) under the following assumptions: $G \geq 0$ and it is C^1-smooth, $f\colon [0,1] \to [0,\infty)$ is continuous, Lipschitz with the constant M. We put

$$m = \max_{[0,1]} f(t),$$

$$K = \max_{[0,1]} \int_0^1 G(x,t)\,\mathrm{d}t,$$

$$\tilde{K} = \max_{[0,1]} \int_0^1 \frac{\partial}{\partial x} G(x,t)\,\mathrm{d}t.$$

THEOREM 3.2. *If*

$$M \cdot K \cdot (1 + m \cdot \tilde{K}) < 1\,,$$
$$m \cdot K \leq 1\,,$$

then (2.1) *has a solution* $u\colon [0,1] \to [0,1]$ *and* $u_n \to u$ *uniformly on* $[0,1]$ *for*

$$u_0 = 0\,,$$
$$u_{n+1}(x) = \int_0^1 G(x,t) \cdot f\big(u_n(u_n(t))\big)\,\mathrm{d}t\,, \qquad x \in [0,1]\,.$$

P r o o f . We take the space $X = C^0\big([0,1],[0,1]\big)$ and

$$Tu = \int_0^1 G(\cdot,t) \cdot f\big(u(u(t))\big)\,\mathrm{d}t\,.$$

Since $m \cdot K \leq 1$, we have $T(X) \subset X$. The first assumption of this theorem ensures the validity of the condition iii) of Section 2. Indeed, by (3.1) we have

$$\|Tu_1 - Tu_2\| \leq M \cdot K \cdot \big(1 + \|u_2'\|\big) \cdot \|u_1 - u_2\|\,.$$

Moreover, it is clear that

$$\|(Tu)'\| \leq \tilde{K} \cdot m\,.$$

Hence we can choose

$$\alpha(t) = m \cdot K \cdot (1+t)\,,$$
$$\beta(t) = \tilde{K} \cdot m\,.$$

Since $\alpha(m \cdot \tilde{K}) < 1$ the proof is finished.

Finally we shall solve the bifurcation problem (1.3).

We take $w = v + c \cdot \sin t$, where c is small and

$$v \in X = \left\{ y \in C^0\big([0,\pi],[-1/2,1/2]\big)\,, \quad \int_0^\pi y(t) \cdot \sin t\,\mathrm{d}t = 0 \right\}.$$

We decompose (1.3) in the following way

$$v = \lambda \cdot \mathcal{K} \cdot \tilde{Q} f\big(v(1+v+c\cdot\sin t) + c\cdot\sin(1+v+c\cdot\sin t)\big)\,, \tag{3.3}$$

$$0 = Pf\big(v(1+v+c\cdot\sin t) + c\cdot\sin(1+v+c\cdot\sin t)\big)\,, \tag{3.4}$$

where $\mathcal{K}$ is the "inverse" of $u \to -u'' - u$, $u(0) = u(\pi) = 0$, $\tilde{Q} = Id - P$ and $Pu = \frac{2}{\pi}\int\limits_0^\pi u(t)\cdot\sin t \,\mathrm{d}t \cdot \sin t$.

For c small we can solve (3.3) by applying Theorem 2.1 to obtain a solution $v(\lambda, c) \in X$. Using the standard arguments we see that v depends on λ, c continuously and $v(0,c) = 0$. We put this solution into (3.4) and have

$$Q(\lambda,c) = Pf\big(v(\lambda,c)(1+v(\lambda,c)+c\cdot\sin t) + c\cdot\sin\big(1+v(\lambda,c)+c\cdot\sin t\big)\big) = 0\,.$$

Hence $Q(0,c) = \frac{2}{\pi}\int\limits_0^\pi f\big(c\cdot\sin(1+c\cdot\sin t)\big)\,\mathrm{d}t$. We have

$$Q(0,0) = 0\,, \qquad \frac{\partial}{\partial c}Q(0,0) = 2\cdot f'(0)\cdot\sin 1 \neq 0\,.$$

This implies the existence of a small solution c of $Q(\lambda,c) = 0$ for each λ small. Summing up we obtain

THEOREM 3.3. *Under the above conditions the problem* (1.3) *has a small nontrivial solution bifurcating from* $u_0 = 0$.

Of course, a similar approach can be applied to the following problem:

$$-u'' = f\big(u'(u(x))\big)\,,$$
$$u(0) = u(1) = 0\,.$$

REFERENCES

[1] EDER, E.: *The functional differential equation* $x' = x(x(t))$, J. Differential Equations **54** (1984), 390–400.

[2] DRIVER, R. D.: *Can the future influence the present?*, Phys. Rev. D (3) **19** (1979).

Received December 5, 1991

Revised February 20, 1992

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
Bratislava
Slovakia