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Mathematica Slovaca, Vol. 54 (2004), No. 2, 151--159

Persistent URL: http://dml.cz/dmlcz/130429

Math. Slovaca, **54** (2004), No. 2, 151–159

Mathematica
Slovaca

Dedicated to Professor Sylvia Pulmannová
on the occasion of her 65th birthday

ITÔ VERSUS WORONOWICZ CALCULUS IN ITÔ HOPF ALGEBRAS

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. It is shown that the differential map in the universal enveloping algebra of a commutator Lie algebra and the right and left differential maps in the Itô Hopf algebra over an associative algebra each generate first order differential calculi in the sense of Woronowicz in which the Leibniz formula is found by absorbing the Itô term of the Leibniz-Itô formula into either the left or the right action. In general these first order calculi are not bicovariant.

1. Introduction

In [4] a differential calculus was formulated in the universal enveloping algebra $\mathcal{U}$ of a Lie algebra $\mathcal{L}$ in which the bracket is got by taking commutators in an associative multiplication, usually nonunital, over the same vector space. This calculus was abstracted from a concrete version [6] formulated in terms of operators in a Fock space using quantum stochastic calculus [5], [8]. Recently, motivated by [1], an extension of this calculus [7] has been used to describe a general method of quantisation of Lie bialgebras giving a partial simplification of the method of Etingof and Kazhdan [2].

This extension lives in the space $\mathcal{T}(\mathcal{L})$ of tensors over $\mathcal{L}$, which becomes a Hopf algebra when equipped with the noncommutative Itô extension of the shuffle product over $\mathcal{L}$ given by $\alpha\beta = \gamma$ where the nth homogeneous component of the tensor γ is given in terms of those of α and β by

$$\gamma = \sum_{A \cup B = \{1,2,\dots,n\}} \alpha^A_{|A|} \beta^B_{|B|} \,. \tag{1}$$

2000 Mathematics Subject Classification: Primary 16W30, 81S25.
Keywords: Itô, Woronowicz calculus, Hopf algebras.

Here the notation is as in [4] and [6] so that the sum is over all 3^n ordered pairs of subsets (A, B) whose union is $\{1, 2, \ldots, n\}$, the place notation $\alpha_{|A|}^{A}$ means that the homogeneous $|A|$th rank tensor $\alpha_{|A|}$ is located in the $|A|$ copies of $\mathcal{L}$ within $\bigotimes^{n} \mathcal{L}$ labelled by A with $\beta_{|B|}^{B}$ being defined analogously, and double occupancies are reduced using the multiplication in $\mathcal{L}$. If the sum in (1) is restricted to the 2^n pairs of disjoint subsets (A, B) the resulting commutative multiplication is well known as the *shuffle product.* The standard coproduct making the shuffle product algebra into a Hopf algebra, whose action on homogeneous product vectors is

$$\Delta(L_1 \otimes L_2 \otimes \cdots \otimes L_n) = \sum_{j=1}^{n} (L_1 \otimes \cdots \otimes L_j) \otimes (L_{j+1} \otimes \cdots \otimes L_n), \qquad (2)$$

remains a coproduct when the Itô terms are included in (1) so that a noncocommutative and noncommutative Hopf algebra is thereby obtained ([7]) which we call the *Itô Hopf algebra* over $\mathcal{L}$. The subspace $\mathcal{S}(\mathcal{L})$ of $\mathcal{T}(\mathcal{L})$ comprising symmetric tensors is a sub-Hopf algebra isomorphic to the Hopf algebra $\mathcal{U}$ under the universal extension v of the Lie algebra homomorphism

$$\mathcal{L} \ni L \mapsto (0, L, 0, 0, \ldots) \in \mathcal{T}(\mathcal{L}).$$

The purpose of this paper is to study the various differential calculi of [4], [6] and [7] and in particular to relate them to differential calculus in the sense of [9].

2. Differential maps

In [4] and [6] the calculus in $\mathcal{U}$ was constructed in terms of a differential map d from $\mathcal{U}$ to $\mathcal{U} \otimes \mathcal{L}$ defined as follows. First one forms the universal extension Ψ, called the *enabling map*, to the ideal $\mathcal{V}$ in $\mathcal{U}$ generated by $\mathcal{L}$, of the identity map in $\mathcal{L}$ regarded as a Lie algebra homomorphism from the Lie algebra $\mathcal{L}$ to the commutator Lie algebra of the associative algebra $\mathcal{L}$. One may then imitate the definition of the differential of a polynomial

$$\mathrm{d}f(x) = f(x + \mathrm{d}x) - f(x)$$

to define the differential map $\mathrm{d}\colon \mathcal{U} \to \mathcal{U} \otimes \mathcal{L}$ by

$$\mathrm{d}(U) = (\mathrm{id}_{\mathcal{U}} \otimes \Psi)\big(\Delta(U) - U \otimes 1_{\mathcal{U}}\big) \qquad (3)$$

where Δ denotes the coproduct of $\mathcal{U}$, that is, the universal extension of the Lie algebra homomorphism

$$\mathcal{L} \ni L \mapsto L \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes L \in \mathcal{U} \otimes \mathcal{U}.$$

Because of the cocommutativity of the coproduct Δ, this "right differential map" differs from the corresponding left differential map

$$(\tau \circ \mathrm{d})(U) = (\Psi \otimes \mathrm{id}_{\mathcal{U}})\big(\Delta(U) - 1_{\mathcal{U}} \otimes U\big) \tag{4}$$

from $\mathcal{U}$ to $\mathcal{L} \otimes \mathcal{U}$ only by intervention of the flip map τ, the linear map from $\mathcal{U} \otimes \mathcal{L}$ to $\mathcal{L} \otimes \mathcal{U}$ which exchanges components of product tensors. In [7] (see also [4]) differential maps are defined on $\mathcal{T}(\mathcal{L})$. Because the coproduct is no longer cocommutative it is necessary to distinguish right and left maps from $\mathcal{T}(\mathcal{L})$ to $\mathcal{T}(\mathcal{L}) \otimes \mathcal{L}$ and $\mathcal{L} \otimes \mathcal{T}(\mathcal{L})$ respectively, which may be defined by linear extension of their actions on homogeneous product vectors,

$$\begin{aligned}
\overrightarrow{\mathrm{d}}(L_1 \otimes L_2 \otimes \cdots \otimes L_n) &= (L_1 \otimes L_2 \otimes \cdots \otimes L_{n-1}) \otimes L_n\,,\\
\overleftarrow{\mathrm{d}}(L_1 \otimes L_2 \otimes \cdots \otimes L_n) &= L_1 \otimes (L_2 \otimes L_3 \otimes \cdots \otimes L_n)\,.
\end{aligned}$$

That the conjugates under v of the restrictions of these maps to $\mathcal{S}(\mathcal{L})$ are consistent with (3) and (4) can be seen by observing that, in view of (1), the map

$$(0, \alpha_1, \alpha_2, \ldots) \mapsto \alpha_1$$

is an associative algebra homomorphism to $\mathcal{L}$ from the ideal in $\mathcal{T}(\mathcal{L})$ consisting of elements with vanishing zero rank homogeneous components $\mathcal{L}$ whose restriction to the corresponding ideal in $\mathcal{S}(\mathcal{L})$ is conjugate under v to Ψ; see also [4].

3. The Leibniz-Itô formula

The differential map d satisfies a modification of the Leibniz rule called the Leibniz-Itô formula, namely

$$\mathrm{d}(UV) = \mathrm{d}(U)V + U\,\mathrm{d}(V) + \mathrm{d}(U)\,\mathrm{d}(V)\,. \tag{5}$$

Here, in the first two terms on the right hand side, $\mathcal{U} \otimes \mathcal{L}$ is regarded as a $\mathcal{U}$-bimodule using the natural tensorial biaction of $\mathcal{U}$ got by linearly extending multiplication on the first component of product tensors, and in the third term as an associative algebra using the natural tensor product multiplication. (5) can be verified directly using the definition (4) (see [4]). The maps $\overrightarrow{\mathrm{d}}$ and $\overleftarrow{\mathrm{d}}$ also satisfy the Leibniz-Itô formula

$$\begin{aligned}
\overrightarrow{\mathrm{d}}(\alpha\beta) &= \overrightarrow{\mathrm{d}}(\alpha)\beta + \alpha\,\overrightarrow{\mathrm{d}}(\beta) + \overrightarrow{\mathrm{d}}(\alpha)\,\overrightarrow{\mathrm{d}}(\beta)\,,\\
\overleftarrow{\mathrm{d}}(\alpha\beta) &= \overleftarrow{\mathrm{d}}(\alpha)\beta + \alpha\,\overleftarrow{\mathrm{d}}(\beta) + \overleftarrow{\mathrm{d}}(\alpha)\,\overleftarrow{\mathrm{d}}(\beta)\,,
\end{aligned}$$

where now $\mathcal{T}(\mathcal{L}) \otimes \mathcal{L}$ and $\mathcal{L} \otimes \mathcal{T}(\mathcal{L})$ are regarded as $\mathcal{T}(\mathcal{L})$-bimodules using the tensorial biactions and as associative algebras using the tensor product multiplication. To see this one may use the multiplication formula for homogeneous product vectors, equivalent to (1),

$$(L_1 \otimes L_2 \otimes \cdots \otimes L_n)(L_{n+1} \otimes L_{n+2} \otimes \cdots \otimes L_{n+m}) = \sum_{P \in \mathcal{P}} (L_{P_1} \otimes L_{P_2} \otimes \cdots \otimes L_{P_k}) \,. \quad (6)$$

Here the sum is over all ordered partitions $P = (P_1, P_2, \ldots, P_k)$ of $(1, 2, \ldots \ldots, n+m)$ into subsets $P_1, P_2, \ldots, P_k$ such that each P_j is either a singleton $\{t\}$, in which case L_P is defined to be L_t, or a pair (r, s) with $r \in (1, 2, \ldots, n)$ and $s \in (n+1, n+2, \ldots, n+m)$, in which case L_P is defined to be $L_r L_s$, and such that the subsets $(1, 2, \ldots, n)$ and $(n+1, n+2, \ldots, n+m)$ retain their relative orders in the permutation $(P_1, P_2, \ldots, P_k)$ of $(1, 2, \ldots, n+m)$. The three terms on the right of the Leibniz-Itô formula are obtained from (6) by distinguishing the partitions $P = (P_1, P_2, \ldots, P_k)$ corresponding to the three possibilities that P_k is a singleton subset of $\{1, 2, \ldots, n\}$, a singleton subset of $\{n+1, n+2, \ldots, n+m\}$ or a pair.

4. Differential actions

Let D be a map from a unital algebra $\mathcal{A}$ to either $\mathcal{A} \otimes \mathcal{L}$ or $\mathcal{L} \otimes \mathcal{A}$ satisfying the Leibniz-Itô formula

$$\mathrm{D}(AB) = \mathrm{D}(A)B + A\,\mathrm{D}(B) + \mathrm{D}(A)\,\mathrm{D}(B)$$

where as before the first two terms on the right hand side refer to the tensorial biaction of $\mathcal{A}$ on the bimodule $\mathcal{M} = \mathcal{A} \otimes \mathcal{L}$ or $\mathcal{L} \otimes \mathcal{A}$, and the third to the tensor product multiplication. Note that the tensorial biaction is *bicompatible* with the multiplication, in that for arbitrary $A \in \mathcal{A}$ and $M_1, M_2 \in \mathcal{M}$

$$A(M_1 M_2) = (AM_1)M_2 \,, \quad (M_1 M_2)A = M_1(M_2 A) \,, \quad (M_1 A)M_2 = M_1(AM_2)$$

as is easily verified by bilinear extension of the case when M_1 and M_2 are product tensors.

Using the map D we define a linear map $\overset{\curvearrowleft}{\bullet}$ from $\mathcal{A} \times \mathcal{M}$ to $\mathcal{M}$ by

$$A \overset{\curvearrowleft}{\bullet} M = AM + \mathrm{D}(A)M \,, \qquad A \in \mathcal{A}, \;\; M \in \mathcal{M} \,, \qquad (7)$$

where the two terms on the right hand side have their previous meaning.

THEOREM 1. *$\overset{\leftarrow}{\bullet}$ is a left action of $\mathcal{A}$ on $\mathcal{M}$ which is compatible with the right tensorial action and with the multiplication in the associative algebra $\mathcal{M}$. That is, we have*

$$A\overset{\leftarrow}{\bullet}(B\overset{\leftarrow}{\bullet}M) = (AB)\overset{\leftarrow}{\bullet}M\,, \tag{8}$$

$$(A\overset{\leftarrow}{\bullet}M)B = A\overset{\leftarrow}{\bullet}(MB)\,, \tag{9}$$

$$A\overset{\leftarrow}{\bullet}(M_1M_2) = (A\overset{\leftarrow}{\bullet}M_1)M_2 \tag{10}$$

for arbitrary $A, B \in \mathcal{M}$ and $M, M_1, M_2 \in \mathcal{M}$.

P r o o f . To prove (8) we write the left hand side as

$$\begin{aligned}
A\overset{\leftarrow}{\bullet}(B\overset{\leftarrow}{\bullet}M) &= A\big(BM + \mathrm{D}(B)M\big) + \mathrm{D}(A)\big(BM + \mathrm{D}(B)M\big) \\
&= \big(AB + \mathrm{D}(A)B + A\,\mathrm{D}(B) + \mathrm{D}(A)\,\mathrm{D}(B)\big)M \\
&= \big(AB + \mathrm{D}(AB)\big)M \\
&= (AB)\overset{\leftarrow}{\bullet}M
\end{aligned}$$

where we used the compatibility of the tensorial action with multiplication in $\mathcal{M}$ (5). Also

$$\begin{aligned}
(A\overset{\leftarrow}{\bullet}M)B &= \big(AM + \mathrm{D}(A)M\big)B \\
&= A(MB) + \mathrm{D}(A)(MB) \\
&= A\overset{\leftarrow}{\bullet}(MB)\,.
\end{aligned}$$

Finally

$$\begin{aligned}
A\overset{\leftarrow}{\bullet}(M_1M_2) &= A(M_1M_2) + \mathrm{D}(A)(M_1M_2) \\
&= (AM_1)M_2 + \big(\mathrm{D}(A)M_1\big)M_2 \\
&= \big(AM_1 + \mathrm{D}(A)M_1\big)M_2 \\
&= (A\overset{\leftarrow}{\bullet}M_1)M_2
\end{aligned}$$

using the compatibility of the left tensorial action with multiplication in $\mathcal{M}$ and the associativity of multiplication in $\mathcal{M}$. □

Defining the map $\overset{\rightarrow}{\bullet}$ from $\mathcal{M} \times \mathcal{A}$ to $\mathcal{M}$ by

$$M\overset{\rightarrow}{\bullet}A = MA + M\,\mathrm{D}(A)\,, \qquad A \in \mathcal{A}, \;\; M \in \mathcal{M}\,,$$

we prove in exactly the same way:

THEOREM 2. *$\overset{\rightarrow}{\bullet}$ is a right action of $\mathcal{A}$ on $\mathcal{M}$ compatible with the left tensorial action and with the multiplication in the associative algebra $\mathcal{M}$.*

Finally we prove:

THEOREM 3. *The left and right actions* $\overset{\leftarrow}{\bullet}$ *and* $\overset{\rightarrow}{\bullet}$ *are mutually compatible and bicompatible with the multiplication in the associative algebra* $\mathcal{M}$, *that is,*

$$(A \overset{\leftarrow}{\bullet} M) \overset{\rightarrow}{\bullet} B = A \overset{\leftarrow}{\bullet} (M \overset{\rightarrow}{\bullet} B),$$
$$(M_1 \overset{\rightarrow}{\bullet} A) M_2 = M_1 (A \overset{\leftarrow}{\bullet} M_2)$$

for arbitrary $A, B \in \mathcal{A}$ *and* $M, M_1, M_2 \in \mathcal{M}$.

Proof. We have, using the bicompatibility of the tensorial biaction and the associativity of multiplication in $\mathcal{U} \otimes \mathcal{L}$,

$$\begin{aligned}(A \overset{\leftarrow}{\bullet} M) \overset{\rightarrow}{\bullet} B &= \big(AM + \mathrm{D}(A)M\big)B + \big(AM + \mathrm{D}(A)M\big)\,\mathrm{D}(B) \\ &= (AM)B + \big(\mathrm{D}(A)M\big)B + (AM)\,\mathrm{D}(B) + \big(\mathrm{D}(A)M\big)\,\mathrm{D}(B) \\ &= A(MB) + \mathrm{D}(A)(MB) + A\big(M\,\mathrm{D}(B)\big) + \mathrm{D}(A)\big(M\,\mathrm{D}(B)\big) \\ &= A\big(MB + M\,\mathrm{D}(B)\big) + \mathrm{D}(A)\big(MB + M\,\mathrm{D}(B)\big) \\ &= A \overset{\leftarrow}{\bullet} (M \overset{\rightarrow}{\bullet} B).\end{aligned}$$

Similarly

$$\begin{aligned}(M_1 \overset{\rightarrow}{\bullet} A) M_2 &= \big(M_1 A + M_1\,\mathrm{D}(A)\big)M_2 \\ &= (M_1 A)M_2 + \big(M_1\,\mathrm{D}(A)\big)M_2 \\ &= M_1(AM_2) + M_1\big(\mathrm{D}(A)M_2\big) \\ &= M_1\big(AM_2 + \mathrm{D}(A)M_2\big) \\ &= M_1(A \overset{\leftarrow}{\bullet} M_2).\end{aligned}$$

□

5. Woronowicz differential calculus

Following Woronowicz [9], a *first order differential calculus* over a unital algebra $\mathcal{A}$ is an $\mathcal{A}$-bimodule $\mathcal{M}$ together with a linear map d from $\mathcal{A}$ to $\mathcal{M}$ satisfying the Leibniz formula

$$\mathrm{d}(AB) = \mathrm{d}(A)B + A\,\mathrm{d}(B), \qquad A, B \in \mathcal{A},$$

and such that every element of $\mathcal{M}$ can be expressed as a finite sum $\sum_j A_j\,\mathrm{d}(B_j)$ with the $A_j, B_j \in \mathcal{A}$. The latter condition is equivalent to every element of $\mathcal{M}$ being of the form $\sum_k \mathrm{d}(C_j)D_k$ since, by the Leibniz formula,

$$\sum_j A_j\,\mathrm{d}(B_j) = \sum_j \mathrm{d}(A_j B_j) - \sum_j \mathrm{d}(A_j)B_j = \sum_j \mathrm{d}(A_j B_j)1_{\mathcal{A}} - \sum_j \mathrm{d}(A_j)B_j.$$

If $\mathcal{A}$ is a Hopf algebra with coproduct Δ, then the first order differential calculus defined by the map d is said to be *left* (*right*) *covariant* if, whenever $\sum_j A_j\,\mathrm{d}(B_j) = 0$, it follows that $\sum_j \Delta(A_j)(\mathrm{id}_{\mathcal{A}}\otimes \mathrm{d})\big(\Delta(B_j)\big) = 0$ (resp. $\sum_j \Delta(A_j)(\mathrm{d}\otimes \mathrm{id}_{\mathcal{A}})\big(\Delta(B_j)\big) = 0$). A bicovariant first order differential calculus can be lifted uniquely to give a noncommutative generalization of the exterior algebra of differential forms ([9]).

Let us return to the Itô Hopf algebra over $\mathcal{L}$. We consider first the case when the multiplication in the algebra $\mathcal{L}$ is the trivial one in which all products are zero. Then the Itô shuffle product (1) reduces to the ordinary shuffle product. The Leibniz-Itô formulas for the right and left differential maps in $\mathcal{T}(\mathcal{L})$ reduce to the Leibniz formulas

$$\overrightarrow{\mathrm{d}}(\alpha\beta) = \overrightarrow{\mathrm{d}}(\alpha)\beta + \alpha\,\overrightarrow{\mathrm{d}}(\beta)\,, \qquad \overleftarrow{\mathrm{d}}(\alpha\beta) = \overleftarrow{\mathrm{d}}(\alpha)\beta + \alpha\,\overleftarrow{\mathrm{d}}(\beta)\,.$$

Since, from the definitions of $\overrightarrow{\mathrm{d}}$ and $\overleftarrow{\mathrm{d}}$, $\overrightarrow{\mathrm{d}}(L) = 1_{\mathcal{T}(\mathcal{L})}\otimes L$ and $\overleftarrow{\mathrm{d}}(L) = L\otimes 1_{\mathcal{T}(\mathcal{L})}$ for arbitrary $L\in\mathcal{L}\subset\mathcal{T}(\mathcal{L})$, arbitrary elements of the bimodules $\mathcal{T}(\mathcal{L})\otimes\mathcal{L}$ and $\mathcal{L}\otimes\mathcal{T}(\mathcal{L})$ can be expressed as sums $\sum_j \alpha_j\,\overrightarrow{\mathrm{d}}(\beta_j)$ and $\sum_j \alpha_j\,\overleftarrow{\mathrm{d}}(\beta_j)$ respectively by taking the β_j to be basis elements of $\mathcal{L}$. Thus $\overrightarrow{\mathrm{d}}$ and $\overleftarrow{\mathrm{d}}$ each determine a first-order differential calculus in the Woronowicz sense. Using the relations which follow from the definitions of the actions of Δ and $\overrightarrow{\mathrm{d}}$ on product vectors,

$$\big(\mathrm{id}_{\mathcal{T}(\mathcal{L})}\otimes\overrightarrow{\mathrm{d}}\,\big)\Delta = \big(\Delta\otimes\mathrm{id}_{\mathcal{L}}\big)\overrightarrow{\mathrm{d}}\,, \quad \big(\,\overrightarrow{\mathrm{d}}\otimes\mathrm{id}_{\mathcal{T}(\mathcal{L})}\big)\Delta = \tau_{(1,3,2)}\big(\Delta\otimes\mathrm{id}_{\mathcal{L}}\big)\overrightarrow{\mathrm{d}} \tag{11}$$

where $\tau_{(1,3,2)}$ is the permutation map corresponding to the permutation $(1,3,2)$ from $\mathcal{T}(\mathcal{L})\otimes\mathcal{T}(\mathcal{L})\otimes\mathcal{L}$ to $\mathcal{T}(\mathcal{L})\otimes\mathcal{L}\otimes\mathcal{T}(\mathcal{L})$ which appropriately permutes the components of product vectors, together with corresponding relations for $\overleftarrow{\mathrm{d}}$ it can be verified that these first order calculi are bicovariant (see the proof of Theorem 4 below). But since the underlying algebra $\mathcal{T}(\mathcal{L})$ is commutative they are essentially classical and the corresponding higher order calculus is the usual one of exterior differential forms.

Now consider the case when the multiplication in $\mathcal{L}$ is nontrivial so that the third term is present in the Leibniz-Itô formula. Using Theorems 1 and 2 of the previous section we can absorb this term in either the left or the right action and write the Leibniz-Itô formula for $\overrightarrow{\mathrm{d}}$ in the alternative forms

$$\overrightarrow{\mathrm{d}}(\alpha\beta) = \overrightarrow{\mathrm{d}}(\alpha)\overrightarrow{\bullet}\beta + \alpha\,\overrightarrow{\mathrm{d}}(\beta)\,, \tag{12}$$

$$\overrightarrow{\mathrm{d}}(\alpha\beta) = \overrightarrow{\mathrm{d}}(\alpha)\beta + \alpha\overleftarrow{\bullet}\,\overrightarrow{\mathrm{d}}(\beta)\,. \tag{13}$$

The argument above that every element $\mathcal{T}(\mathcal{L})\otimes\mathcal{L}$ can be expressed as a sum $\sum_j \alpha_j\,\overrightarrow{\mathrm{d}}(\beta_j)$ or equivalently as $\sum_k \overrightarrow{\mathrm{d}}(\gamma_k)\delta_k$ shows that either of these equations defines a first order differential calculus in the Woronowicz sense.

THEOREM 4. *The first order differential calculus defined by* (12) (*resp.* (13)) *is left* (*resp. right*) *covariant.*

Proof. We use (11). To prove that (12) defines a left covariant calculus, suppose that $\sum_j \alpha_j \overrightarrow{\mathrm{d}}(\beta_j) = 0$. Then, using (11),

$$\begin{aligned}\sum_j \Delta(\alpha_j)\big(\mathrm{id}_{\mathcal{T}(\mathcal{L})}\otimes \mathrm{d}\big)\big(\Delta(\beta_j)\big) &= \sum_j \Delta(\alpha_j)\big(\Delta\otimes \mathrm{id}_{\mathcal{L}}\big)\overrightarrow{\mathrm{d}}(\beta_j)\\ &= \big(\Delta\otimes \mathrm{id}_{\mathcal{L}}\big)\sum_j \alpha_j \overrightarrow{\mathrm{d}}(\beta_j)\\ &= 0\end{aligned}$$

where we use the multiplicativity of Δ and the form of the tensorial left action. That (13) defines a right covariant calculus is proved similarly. □

Analogous results hold for the left differential map $\overleftarrow{\mathrm{d}}$.

Clearly if the multiplication in $\mathcal{L}$ is nontrivial neither first order calculus defined by (12) and (13) is bicovariant and higher order differential forms cannot be constructed in the manner of [9]. However [7] shows the usefulness of the Itô type calculus defined by $\overrightarrow{\mathrm{d}}$ and $\overleftarrow{\mathrm{d}}$.

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Received September 3, 2003

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