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DIGRAPHS MAXIMAL WITH RESPECT TO ARC CONNECTIVITY

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1. Introduction

One of the most important goals of the theory of connectivity of digraphs is to compile a list of all k-connected digraphs. But this problem seems to be very difficult. Thus there were investigated (by Kameda [6] for strong connectivity, by Mader [7] for strong arc connectivity) minimally k-connected digraphs, which are "lower bound" for the class of k-connected digraphs. The dual question, maximally k-connected digraphs which are "upper bound" for the class of digraphs with connectivity k, have been studied by the author of [5] for all three invariants of point connectivity.

In this paper we shall describe constructively and determine the number of digraphs maximal with respect to the strong or weak arc connectivity, respectively. A sufficient and a necessary condition for a digraph to be maximal with respect to the unilateral arc connectivity is given.

2. Notation and terminology

The notions not defined here will be used in the sense of [4].

The strong (unilateral, weak) connectivity $\varkappa^3 = \varkappa^3(G)$ ($\varkappa^2 = \varkappa^2(G)$, $\varkappa^1 = \varkappa^1(G)$) of a digraph G is the minimum number of points whose removal results in a not strong (unilateral, weak) or trivial digraph.

Analogously, the strong (unilateral, weak) arc connectivity $\lambda^3 = \lambda^3(G)$ ($\lambda^2 = \lambda^2(G)$, $\lambda^1 = \lambda^1(G)$) of a digraph G is the minimum number of arcs whose removal results in a not strong (unilateral, weak) digraph.

Let digraphs G_1 and G_2 have disjoint sets V_1 and V_2 of points and disjoint arc sets E_1 and E_2, respectively.

Their union is the digraph $G = G_1 \cup G_2$, which has the point set $V = V_1 \cup V_2$ and the arc set $E = E_1 \cup E_2$.

Their join $G_1 + G_2$ consists of $G_1 \cup G_2$ and all arcs joining V_1 with V_2.

Their directional join $G_1 \oplus G_2$ consists of $G_1 \cup G_2$ and all arcs going from V_1 to V_2. It is clear that the directional join is not a commutative operation.

We shall denote by $\bar{D}$ the complement of a digraph D, by K_p the complete graph on p vertices, by $K_{s,r}$ the graph $\bar{K}_s + \bar{K}_r$, by D_p the complete digraph on p points, by $D_{s,r}$ the digraph $\bar{D}_s + \bar{D}_r$.

Let v be a point of a digraph G. Then $\mathrm{id}(v)$ is the indegree of v and $\mathrm{od}(v)$ is the outdegree of v. Let $\deg(v)$ be the sum of $\mathrm{id}(v)$ and $\mathrm{od}(v)$ and let $\delta(G) = \min_{v \in G} \deg(v)$.

The symbol $\Gamma(G)$ denotes the vertex(point) group of a graph (digraph) G and $\Gamma_1(G)$ denotes the edge (arc) group of G.

Further, let the symbol $A(G; n)$ denote the class of digraphs that arose from a digraph G by adding n new arcs.

We shall say that $G_1, G_2 \in A(G; n)$ are similar if there is $\varphi \in \Gamma(G)$ such that φ: $G_1 \to G_2$ is an isomorphism. If G_1, G_2 are not similar, they are called dissimilar.

Let, as usual, the symbol $Z(H)$ be the cycle index of a permutation group H. The polynomial $Z(H, 1+x)$ is determined by substituting $1 + x^k$ for each variable s_k in $Z(H)$.

3. Maximal digraphs

Let n be a nonnegative integer and let G be a non-complete digraph. Then G is called λ^i_n ($\varkappa^i_n$)*-maximal* if $\lambda^i(G) = n$ and $\lambda^i(G+x) > n$ ($\varkappa^i(G) = n$ and $\varkappa^i(G+x) > n$) for every arc $x \in E(\bar{G})$, $i = 1, 2, 3$.

In this part of our paper we shall describe λ^1_n- and λ^3_n-maximal digraphs. For λ^2_n-maximal digraphs a sufficient and a necessary condition are given.

In [5] the following result has been proved:

Theorem 1. *Let G be a digraph. Then G is*

a) $\varkappa^1_0$*-maximal iff* $G \simeq D_a \cup D_b$,

b) $\varkappa^2_0$*-maximal iff either* $G \simeq D_a \cup D_b$ *or* $G \simeq D_c \oplus (D_a \cup D_b)$ *or* $G \simeq (D_a \cup D_b) \oplus D_c$ *or* $G \simeq D_d \oplus ((D_a \cup D_b) \oplus D_c)$,

c) $\varkappa^3_0$ *maximal iff* $G \simeq D_a \oplus D_b$.

Theorem 2. *A digraph G is λ^i_0-maximal iff G is $\varkappa^i_0$-maximal, for $i = 1, 2, 3$.*

Proof. Let G be a digraph. Then $\lambda^i(G) = 0$ iff $\varkappa^i(G) = 0$, i.e. G is λ^i_0-maximal iff G is $\varkappa^i_0$-maximal, $i = 1, 2, 3$. Q.E.D.

Theorem 3. *Let G be a λ^i_n maximal digraph and n be a natural number. Then $G \in A(D; n)$, where D is λ^i_0-maximal digraph (for $i = 1, 2, 3$).*

Proof. Let G be a λ'_n-maximal digraph. Then there exists a set A of arcs of G such that $|A|=n$ and $\lambda'(G-A)=0$. Let us denote by D the digraph $G-A$. To finish our proof we must show that the digraph D is λ'_0-maximal. We shall prove it indirectly.

Let $x\in E(\bar{G})$ and $\lambda'(D+x)=0$ (i.e. $x\notin A$). Then $\lambda'(G+x)=n$ and this is a contradiction because G is λ'_n-maximal. Thus D is λ'_0-maximal. Q.E.D.

Now we shall give a sufficient condition for a digraph G to be λ^2_n-maximal.

Theorem 4. *Let G be a digraph and $G\in A(D;n)$, where D is a λ^2_0-maximal digraph and every strong component of D has at least $n+2$ points. Then G is λ^2-maximal.*

Proof. Let $G\in A(D;n)$, where the digraph D is λ^2_0 maximal and every strong component of D has at least $n+2$ points. Let $A=E(G)-E(D)$. Then $\lambda^2(G-A)$ $=\lambda^2(D)=0$ and we get $\lambda^2(G)<n$. As $\lambda^2(D+x)=1$ for $x\in E(\bar{D})$ and every strong component of D has at least $n+2$ points, we have $\lambda^2(G)=n$. By a similar reasoning we get $\lambda^2(G+x)=n+1$ for $x\in E(\bar{G})$. Q.E.D.

Before describing λ^1_n- and λ^3_n-maximal digraphs we shall state two lemmas.

Lemma 1. *Let $\lambda'(G)=c$, $B\subseteq A\subseteq E(G)$, $|A|=a$, $|B|=b$. Let $\lambda'(G-A)-c-a$. Then $\lambda'(G-B)=c-b$, for $i=1,2,3$.*

Proof. Let $\lambda'(G)=c$, $B\subseteq A\subseteq E(G)$, $|A|=a$, $|B|=b$. Let $\lambda'(G\quad A)=c-a$. It is easy to see that

$$\lambda'(G)\geqslant\lambda'(G-x)\geqslant\lambda'(G)-1\quad\text{for any}\quad x\in E(G).$$

Thus $\lambda'(G-B)\geqslant c-b$. On the other hand

$$\lambda'(G-A)+1\geqslant\lambda'((G-A)+x\geqslant\lambda'(G-A)\quad\text{for any}\quad x\in E(\overline{G-A})$$

and we have $\lambda'(G-B)\leqslant(c-a)+(a-b)<c-b$. Q.E.D.

Lemma 2. *Let $r\geqslant s$ be natural numbers and let $G\in A(D_s\cup D_r;k)$. Then $\delta(G)>k$ iff either $s\geqslant\frac{k+3}{2}$ or $s=\frac{k}{2}+1$ and each point of D_s (in the case of $r=s$ each point of G) is incident with at least one arc belonging to $\overline{D_s\cup D_r}$.*

Proof. Let $r\geqslant s$ and let $G\in A(D_s\cup D_r;k)$. It is clear that the condition is sufficient for a digraph G to have the property that $\delta(G)>k$.

Let now $\delta(G)>k$. Let us denote by $u_1, u_2, \ldots, u_s$ the points of D_s. Let $s\leqslant\frac{k+1}{2}$. Then $\min\deg(u_i)=2(s-1)+\left[\frac{k}{s}\right]\leqslant k$ and this is a contradiction. Thus $s\geqslant\frac{k}{2}+1$. In the case of $s=\frac{k}{2}+1$ we get $\deg(u_i)=k$ and each point of D_s (in the case $s=r$ each point of G) must be incident with at least one arc of $\overline{D_s\cup D_r}$. Q.E.D.

Theorem 5. *Let n be a natural number. Then the digraph G is λ_n^1-maximal iff $G \in A(D_s \cup D_r\,;\, n)$, where either $r=1$ and $s>\frac{n}{2}$ or $\delta(G)>n$.*

Proof. Let G be a λ_n^1-maximal digraph. Then $\lambda^1(G)=n$ and according to [1] we have $\delta(G) \geqslant n$. Let v be a point of G with $\deg(v)=n$. Let us denote by A the set of arcs incident with point v. Because the digraph G is λ_n^1-maximal we have $G-A=D_1 \cup D_p$ and any arc of $\bar{G}$ must be incident with point v. Thus there can be at most two points with degree n in G. We have to consider only three cases.

I. digraph G contains exactly two points with degree n. As any arc of $\bar{G}$ must be incident with those two points we get $G \simeq D_{2+n\ 2}\ \ \{x, y\}$, where x, y are symmetric arcs, i.e.

$$G \in A(D_{1+n\ 2} \cup D_1\,;\, n) \quad \text{for } n \text{ even, and} \quad G \simeq D_{\ n+3)\ 2}-x, \quad \text{i.e.}$$

$$G \in A(D_{(n+1)\ 2} \cup D_1\,;\, n) \quad \text{for } n \text{ odd.}$$

II. The digraph G contains only one point with degree n. Then any arc of G is incident with this point. Thus $G \in A(D_1 \cup D_s\,;\, n)$, where $s \geqslant \frac{n}{2}+1$.

III. The digraph G does not contain a point with degree n. Then $\delta(G)>n$ and from Theorem 3 it follows that $G \simeq A(D_s \cup D_r\,;\, n)$.

Now we shall prove the sufficient condition.

Let $G \in A(D_1 \cup D_s\,;\, n)$, where $s>\frac{n}{2}$. Let us denote by A the set of arcs incident with point v, $v \in V(D_{s+1})$. It is easy to see that $\lambda^1(D_{s+1})=2s$. Further, $|A|=2s$ and $\lambda^1(D_{s+1}-A) = \lambda^1(D \cup D_s)=0$. Thus the assumptions of Lemma 1 hold. Hence we get $\lambda^1(G)=n$ and $\lambda^1(G+x)=n+1$ for any $x \in E(\bar{G})$, i.e. G is λ_n^1-maximal.

Let $G \in A(D_s \cup D_r\,;\, n)$ where $r \geqslant s$ and $\delta(G)>n$. From Lemma 2 it follows that either $s \geqslant \frac{n+3}{2}$ or $s=\frac{n}{2}+1$ and any point of D_s (in the case of $s=r$ any point of G) is incident with at least one arc of $\overline{D_s \cup D_r}$. Let $s \geqslant \frac{n+3}{2}$. Obviously $\lambda^1(G) \leqslant n$. Let $A \subset E(G)$, $|A|<n$. Then $G-A$ is a weakly connected digraph as $\lambda^1(D_r) \geqslant \lambda^1(D_s) \geqslant n+1$. Thus $\lambda^1(G)=n$. Similarly, $\lambda^1(G+x)=n+1$ for any $x \in E(\bar{G})$, i.e. G is λ_n^1-maximal.

Let $r \geqslant s=\frac{n}{2}+1$. As $\lambda^1(D_s)=n$ and $G \in A(D_s \cup D_r\,;\, n)$, we have $\lambda^1(G)=n$. Further, let $A \subset E(D_s)$, $|A|=n$ and let D_s-A be a disconnected digraph. Then there is a point $v \in V(D_s)$ such that $A=\{x\,;\, x \in E(D_s),\ x \text{ is incident with } v\}$.

It follows that $\lambda^1(G+x)=n+1$ as each point of D_s is incident with at least one arc of $\overline{D_s \cup D_r}$. Thus G is a λ_n^1-maximal digraph. Q.E.D.

Theorem 6. *Let n be a natural number and let G be a digraph. Then G is λ^3_n-maximal iff $G \in A(D_s \oplus D_r; n)$, where either $s=1$, $r \geqslant n+1$ or $r=1$, $s \geqslant n+1$ or s, $r \geqslant n+2$.*

Proof. Let n be a natural number. Let G be a λ^3_n-maximal digraph. Then $\lambda^3(G)=n$ and from [1] it follows that $n \leqslant \min(\mathrm{od}(v), \mathrm{id}(v))$ for every $v \in V(G)$.

Let v be a point of G such that $\mathrm{id}(v)=n$. Let $A=\{uv; uv \in E(G)\}$. As G is λ^3_n-maximal, we have $G-A=D_1 \oplus D_p$. It follows that for any $x \in E(\bar{G})$ we have $x=uv$. Analogously, for a point with $\mathrm{od}(v)=n$ we get: if $x \in E(\bar{G})$, then $x=vu$. Further, there is no point with $\mathrm{od}(v)=\mathrm{id}(v)=n$. Thus there exist at most two points with the property that the indegree or the outdegree of them is exactly n. Let us consider three cases.

I. Let u, v be points of G such that $\mathrm{od}(v)=\mathrm{id}(u)=n$. Then $G \simeq D_{n+2}-x$, thus $G \in A(D_1 \oplus D_{n+1}; n)$ and $G \in A(D_{n+1} \oplus D_1; n)$, too.

II. Let v be a point of G with $\mathrm{od}(v)=n$ and semidegrees of all other points are greather than n. Then for any $x \in E(\bar{G})$ we have $x=vu$ and $G \in A(D_s \oplus D_1; n)$, where $s>n+1$. If $\mathrm{id}(v)=n$, we get $G \in A(D_1 \oplus D_r; n)$, where $r>n+1$.

III. Let $\mathrm{od}(v)$, $\mathrm{id}(v)>n$ for any point v of G. Then from Theorem 2 and 3 it follows that $G \in A(D_s \oplus D_r; n)$ and s, $r \geqslant n+2$.

Now we shall prove the sufficient condition.

Let $G \in A(D_s \oplus D_r; n)$. Let $r=1$, $s \geqslant n+1$ and let $v \in V(D_{s+1})$ and $A=\{uv; uv \in E(D_{s+1})\}$. It is easy to see that $\lambda^3(D_{s+1})=s$. Further, $|A|=s$ and $\lambda^2(D_{s+1}-A) = \lambda^3(D_s \oplus D_1)=0$. Thus the assumptions of Lemma 1 hold. Hence we get $\lambda^3(G)=n$ and $\lambda^3(G+x)=n+1$ for any $x \in E(\bar{G})$, i.e. G is λ^3_n-maximal. The proof for $s=1$, $r \geqslant n+1$ is analogical. Let $r \geqslant s \geqslant n+2$. As $\lambda^3(D_r) \geqslant \lambda^3(D_s) \geqslant n+1$ we get $\lambda^3(G)=n$ and $\lambda^3(G+x)=n+1$ for any $x \in E(\bar{g})$. Thus G is λ^3_n-maximal. Q.E.D.

By using Theorem 3 we prove the following inequalities.

Theorem 7. *Let G be a digraph with p points and q arcs. Let $\lambda^i(G)=n$. Then*

$$\begin{array}{lll} \text{a)} & q \leqslant (p-1)(p-2)+n & \text{for } i=1,2; \\ \text{b)} & q \leqslant (p-1)^2+n & \text{for } i=3. \end{array} \tag{1}$$

Proof. Let G be a digraph with p points and q arcs and let $\lambda^i(G)=n$. Then there exists a λ^i_n-maximal digraph H such that G is a factor of H. As H is λ^i_n-maximal, from Theorem 3 it follows that $H \in A(D; n)$, where D is λ^i_0-maximal. Thus we have

$$q(G) \leqslant q(H) = q(D)+n. \tag{2}$$

By [1] we have

$$\begin{array}{ll} q(D) \leqslant (p-1)(p-2) & \text{for } i=1,2; \\ q(D) \leqslant (p-1)^2 & \text{for } i=3. \end{array} \tag{3}$$

The statement (1) follows immediately from (2) and (3). In addition, for $G \in A(D \cup D_{p-1}; n)$, $i = 1, 2$ and for $G \in A(D_1 \oplus D_{p-1}; n)$, $i = 3$, we get an equality in (1). Q.E.D.

4. Enumeration of maximal digraphs

In this part we determine the number of λ_n^1- and λ_n^3-maximal digraphs.

The case $n = 0$ is not interesting as we obviously have:

Theorem 8. *The number of nonisomorphic λ_0^i-maximal digraphs with p points is*

a) $\left[\frac{p}{2}\right]$ *if* $i = 1$;

b) $p - 1$ *if* $i = 3$.

The number of λ_n^i-maximal digraphs for $n > 0$, $i = 1, 3$ will be determined by applying Polya's Enumeration Theorem [8].

Theorem 9. *Let G be a digraph. Then the number of dissimilar digraphs in the class $A(G; n)$ is the coefficient of x^n in $Z(\Gamma_1(\bar{G}), 1 + x)$.*

The proof of Theorem 9 is the same as that in [2] for its undirected version.

Before determining the number of λ_n^1-maximal digraphs we shall state a lemma and give the cycle index of $\Gamma_1(D_{s,r})$.

Lemma 3. *Suppose that $G_1, G_2 \in A(D_s \cup D_r; n)$, where either $s = 1$, $r > \frac{n}{2} + 1$ or $s > \frac{n}{2}$, $r > \frac{n}{2} + 1$. Then G_1 and G_2 are similar iff $G_1 \simeq G_2$.*

Proof. Let $G_1, G_2 \in A(D_s \cup D_r; n)$, where $s = 1$, $r \geqslant \frac{n}{2} + 1$ or $s > \frac{n}{2}$, $r > \frac{n}{2} + 1$. As $G_1, G_2 \in A(D_s \cup D_r; n)$, we have $V(G_1) = V(G_2) = V(D_s \cup D_r) = V(D_s) \cup V(D_r)$. To express it more clearly $V(G_i) \cap V(D_s) = A_i$, $V(G_i) \cap V(D_r) = B_i$, for $i = 1, 2$. Obviously $A_1 = A_2$, $B_1 = B_2$. It is well known that $\varphi \in \Gamma(D_s \cup D_r)$ iff the components of digraph $D_s \cup D_r$ are invariable with respect to φ for $s \neq r$, and $\varphi \in \Gamma(D_r \cup D_r)$ iff either the components of this digraph are invariant with respect to φ or φ maps any point of one component onto a point from the other component.

The necessity of the condition is traightforward. Let now $G_1 \sim G_2$. We shall show that G_1 and G_2 are similar. We shall consider two cases.

I. It is clear that in the case of $s = 1$, $r = \frac{n}{2} + 1$ the statement holds. Let $s = 1$, $r \geqslant \frac{n+3}{2}$. Then for the point $v \in A$ we have $n = \deg(v) < \deg(u)$ for every point

$u \in B_i$, $i-1, 2$. Therefore if φ: $G_1 \to G_2$ is an isomorphism, there must be $\varphi(v) = v$. Thus $\varphi \in \Gamma(D_1 \cup D_r)$ and G_1 and G_2 are similar.

II. Let $s > \frac{n}{2}$, $r > \frac{n}{2} + 1$. We shall prove indirectly that G_1 and G_2 are similar. Let us consider the case $s \neq r$. Let φ: $G_1 \to G_2$ be an isomorphism and let $\varphi \notin \Gamma(D_s \cup D_r)$. Thus there is a point $u \in A_1$ such that $\varphi(u) \in D_2$. Put

$$A = \{u\,;\, u \in A_1,\ \varphi(u) \in B_2\},$$
$$B = \{u\,;\, u \in B_1,\ \varphi(u) \in A_2\}.$$

Obviously $|A| = |B|$. Let $|A| = p$ (as $\varphi \notin \Gamma(D_s \cup D_r)$ in the case of $s = r$ there must be $p < s$). Let u, $v \in A_1$ and $\varphi(u) \in A_2$, $\varphi(v) \in B_2$. Since uv, $vu \in E(G_1)$, and φ is an isomorphism, $\varphi(u)\varphi(v)$ and $\varphi(v)\varphi(u)$ belong to $E(G_-)$. As $\varphi(u) \in A_2$ and $\varphi(v) \in B_2$, the arcs $\varphi(u)\varphi(v)$ and $\varphi(v)\varphi(u)$ do not belong to $E(D_s \cup D_r)$. Analogously for z, $w \in B_1$, $\varphi(z) \in A_2$ and $\varphi(w) \in B_2$. Therefore

$$n \geq 2(r-p)p + 2(s-p)p.$$

On the other hand for $r > s \geq \frac{n+1}{2}$, $1 \leq p \leq s$ (in the case $r = s \geq \frac{n}{2} + 1$, $1 \leq p \leq s-1$) we have

$$2(r-p)p + 2(s-p)p > n \tag{4}$$

and we have a contradiction. Thus $\varphi \in \Gamma(D_s \cup D_r)$, i.e. G_1 and G_2 are similar.

Now we shall consider the case of $s = \frac{n}{2}$, $r \geq \frac{n}{2} + 1$. The statement (4) holds for $r > \frac{n}{2} + 1$, $s = \frac{n}{2}$, $1 \leq p \leq s$ and for $r = \frac{n}{2} + 1$, $s = \frac{n}{2}$, $p < s$, too. But for $r = \frac{n}{2} + 1$, $s = \frac{n}{2}$, $p = s$ we have only the equality in (4). However, then there must be $G_1 \simeq G_2 \simeq (D_s \cup D_s) + D_1$ (i.e. G_1, $G_2 \in A(D_s \cup D_{s+1}\,;\, n)$), hence we get that G_1, G_2 are similar. Q.E.D.

Theorem 10. *Let n, m be natural numbers. Then for $n \neq m$ we have*

$$Z(\Gamma_1(D_{n,m})) = Z(n, m) = \frac{1}{n!\,m!} \sum_{(\alpha,\beta)} \prod_{r,t=1}^{n,m} s_{[r,t]}^{2(r,t) j_r(\alpha) j_t(\beta)}, \tag{5}$$

where $\alpha \in S_n$, $\beta \in S_m$ and $j_k(\varphi)$ is the number of cycles of length k in the disjoint cycle decomposition of φ,

$$Z(\Gamma_1(D_{n,n})) = \frac{1}{2}\,(Z(n, n) + Z'_n), \tag{6}$$

where

$$Z'_n = \frac{1}{n!} \sum_{(j)} \frac{n!}{\prod k^{j_k} j_k!} \prod_{k\ \mathrm{odd}} s_{2k}^{j_k} \prod s_{2k}^{2(k(\ )+[\frac{k}{2}]\ j_k)} \prod_{r<t} s_{2[r,t]}^{2(r,t) j_r j_t},$$

and the sum is over all partitions $(j) = (j_1, j_2, \ldots, j_n)$ *of* n.

Proof. Let n, m be natural numbers. From [2] it follows that

$$Z(\Gamma_1(K_{n,m})) - Z(S_n \times S_m) = \frac{1}{n!m!} \sum_{(\alpha)(\beta)} \prod_{r}^{n} \prod_{t}^{m} s_{[r,t]}^{(r,t) j_r(\alpha) j_t(\beta)} \tag{7}$$

for $n \neq m$,

$$Z(\Gamma(K_{n,n})) - \frac{1}{2}(Z(S_n \times S_n) + Z_n), \tag{8}$$

where

$$Z_n = \frac{1}{n!} \sum_{(j)} \frac{n!}{\prod k^{j_k} j_k!} \prod_{k\ \mathrm{odd}} s_k^{j_k} \prod_k s_2^{k(\ )+[\frac{k}{2}]\ j_k} \prod_{r<t} s_{2[r,t]}^{(r,t) j_r j_t}.$$

The cycle index of $\Gamma(D_{n,m})$ will be determined using (7) and (8). As the groups $\Gamma(K_{n,m})$ and $\Gamma(D_{n,m})$ are identical, the group $\Gamma_1(D_{n,m})$ is induced by the group $\Gamma(K_{n,m})$, too.

Let us denote the maximal indenpendent sets of vertices of the graph $K_{n,m}$ by A and B. Thus $A \cup B = V(K_{n,m})$, $A \cap B = \emptyset$. Now we indicate the correspondence between the terms of the cycle indices of $\Gamma_1(K_{n,m})$ and $\Gamma_1(D_{n,m})$.

Let $\varphi \in \Gamma(K_{n,m})$ and let for any vertex $u \in A$ we have $\varphi(u) \in A$ (then for $u \in B$ we have $\varphi(u)B$). Let $\varphi' \in \Gamma_1(K_{n,m})$ and $\varphi'' \in \Gamma_1(D_{n,m})$ be automorphisms induced by φ. Let $\Pi s_i^{j_i}$ be an addend of $Z(\Gamma_1(K_{n,m}))$ corresponding to φ. Then $\Pi s_i^{2j_i}$ is the addend from $Z(\Gamma_1(D_{n,m}))$ corresponding to φ'' as any cycle of length k in the disjoint cycle decomposition of φ' corresponds to two cycles of length k of φ''. One of them consists of arcs going from A to B, the other from symmetrics arcs, i.e. from arcs going from B to A. Thus from (7) we get the statement (5).

Now let $\varphi \in \Gamma(K_{n,n})$ such that for any $u \in A$, $\varphi(u) \in B$. Let $\varphi' \in \Gamma_1(K_{n,n})$ and $\varphi'' \in \Gamma_1(D_{n,n})$ be automorphisms induced by φ. Let $\Pi s_i^{j_i}$ be an addend from $\Gamma_1(K_{n,n})$ corresponding to φ'. Then $\prod_{\mathrm{odd}} s_{2i}^{j_i} \prod_{i\ \mathrm{even}} s_i^{2j_i}$ is the addend of $\Gamma_1(D_{n,n})$ corresponding to φ'' as φ'' maps an arc going from A to B onto an arc going from B to A. Thus any cycle of length k of φ' corresponds to two cycles of length k for k even, and a cycle of length $2k$ for k odd of φ''.

From the above and from Z_n we get that the contribution of automorphisms, which maps any vertex of A onto a vertex of B in $Z(\Gamma_1(D_{n,n}))$ is equal to Z'_n. As it is clear that $Z(\Gamma(D_{n,n})) - \frac{1}{2}(Z(n, n) + Z_n)$, the proof is complete.

Theorem 11. *Let p, n be natural numbers, $p \geq \frac{n+3}{2}$. Then the number of λ_n^1-maximal digraphs with p points is*

a) 1 *for* $p = \frac{n+3}{2}$,

b) *the coefficient of* x^n *in* $Z(\Gamma_1(D_{1,p-1}), 1+x)$, *where either* $\frac{n+4}{2} \leq p < n+2$ *or* $p = n+2$ *and n is odd*

c) *the coefficient of* x^n *in*

$$Z(\Gamma_1(D_{1,p-1}), 1+x) - \frac{(-1)^n+1}{2} Z(\Gamma_1(D_{\frac{n}{2},p-\frac{n}{2}-1}), 1+x) + \sum_{i=[\frac{n+3}{2}]}^{[p/2]} Z(\Gamma_1(D_{i,p-i}), 1+x),$$

where either $n+2 < p$ *or* $p = n+2$ *and n is even.*

Proof. Let G be a digraph with p points and let $p \geq \frac{n+3}{2}$. From Theorem 5 it is easy to see that there is only one λ_n^1-maximal digraph for $p = \frac{n+3}{2}$.

Let now either $\frac{n+4}{2} \leq p < n+2$ or $p = n+2$ and n is odd. From Theorem 5 it follows that G is λ_n^1-maximal iff $G \in A(D_1 \cup D_{p-1}; n)$. By Theorem 9 we have that the number of dissimilar digraphs in $A(D_1 \cup D_{p-1}; n)$ is the coefficient of x^n in $Z(\Gamma_1(\overline{D_1 \cup D_{p-1}}), 1+x) = Z(\Gamma_1(D_{1,p-1}), 1+x)$ and by Lemma 3 we get that the number of dissimilar digraphs in $A(D_1 \cup D_{p-1}; n)$ is equal to the number of nonisomorphic digraphs in this class. Thus we get part b).

Suppose that $n+2 < p$ and n is even. From Theorem 5 it follows that a digraph G is λ_n^1-maximal iff either $G \in A(D_1 \cup D_{p-1}; n)$ or $G \in A(D_s \cup D_{p-s}; n)$ where $\delta(G) > n$. The number of digraphs of the class $A(D_{\frac{n}{2}+1} \cup D_{p-\frac{n}{2}-1}; n)$ with $\delta(G) > n$ is equal to

$$W = |A(D_{\frac{n}{2}+1} \cup D_{p-\frac{n}{2}-1}; n)| - |A(D_{\frac{n}{2}} \cup D_{p-\frac{n}{2}-1}; n)|.$$

Further, let $G_1 \in A(D_s \cup D_{p-s}; n)$, $G_2 \in A(D_r \cup D_{p-r}; n)$, where $r \neq s \neq p-r$. Then G_1 cannot be isomorphic to G_2 because $|E(G_1)| \neq |E(G_2)|$. Thus the number of λ_n^1-maximal digraphs is equal to

$$W + |A(D_1 \cup D_{p-1}; n)| + \sum_{i=\frac{n}{2}+2}^{[p/2]} |A(D_i \cup D_{p-i}; n)|.$$

As all of these classes of digraphs satisfy the assumptions of Lemma 3 we get from Theorem 9 that the number of nonisomorphic digraphs in these classes is the coefficient of x^n in

$$Z(\Gamma_1(D_{1\ p\ 1}), 1+x) - Z(\Gamma_1(D_{\frac{n}{2}, p\ \frac{n}{2}\ 1}), 1+x) + \sum_{i=\frac{n}{2}+1}^{[p\ 2]} Z(\Gamma_1(D_{i\ p\ i}), i+x. \quad (9)$$

Similarly for $p > n+2$, n odd we get that the number of λ_n^1-maximal digraphs with p points is the coefficient of x^n in

$$Z(\Gamma_1(D_{1\ p\ 1}), 1+x) + \sum_{i\ \frac{n+3}{2}}^{[p\ 2]} Z(\Gamma_1(D_{i\ p\ i}), 1+x). \quad (10)$$

The part c) follows immediately from (9) and (10). Q.E.D.

Lemma 4. *Let G_1, $G_2 \in A(D_r \oplus D_s; n)$, where $r, s \geqslant n+2$. Then G_1 and G_2 are similar iff $G_1 \simeq G_2$.*

The proof of Lemma 4 is analogical to that of Lemma 3.

Theorem 13. *Let n, p be natural numbers, $p \geqslant n+2$. Then the number of λ_n^3-maximal digraphs with p points is*

a) 2 *for* $n+2 < p < 2n+3$;

b) *the coefficient of x^n in*

$$2x^n + \sum_{i\ n+2}^{p\ n\ 2} Z(S_i \times S_{p\ i}, 1+x) \quad \textit{for} \quad 2n+4 \leqslant p,$$

where $Z(S_n \times S_m)$ is given by (7).

Proof. Let G be a digraph with p points. Let $n+2 \leqslant p \leqslant 2n+3$. By Theorem 6 we have that there are two λ_n^3-maximal digraphs.

Suppose that $2n+4 \leqslant p$. Let $G_1 \in A(D_s \oplus D_{p\ s}; n)$, $G_2 \in A(D_r \oplus D_{p\ r}; n)$, where $r \neq s \neq p-r$. Then G_1 cannot be isomorphic to G_2 as $|E(G_1)| \neq |E(G_2)|$. From the above and from Theorem 6 if follows that there are

$$2 + \sum_{i\ n+2}^{p\ n\ 2} |A(D_i \oplus D_{p\ i}; n)|$$

λ_n^3-maximal digraphs with p points. As all these classes of digraphs satisfy the assumptions of Lemma 4 we get from Theorem 9 that the number of λ_n^3-maximal digraphs with p points is the coefficient of x^n in

$$2x^n + \sum_{i\ n+2}^{p\ n\ 2} Z(\Gamma_1(\overline{D_i \oplus D_{p\ i}}), 1+x).$$

As $Z(\Gamma_1(\overline{D_i \oplus D_{p\ i}}) = Z(\Gamma_1(\bar{D}_{p\ i} \oplus D_i) = Z(S_{p\ i} \times S_i)$, the proof is complete.

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ОРГРАФЫ, МАКСИМАЛЬНЫЕ ОТНОСИТЕЛЬНО СВЯЗНОСТИ

Петер Горак

Резюме

Сильной (односторонней, слабой) реберной связностью орграфа называется наименьшее число ребер, удаление которых приводит к не сильному (не одностороннему, не слабому, соответственно) орграфу.

Конструктивно описано и определено число орграфов, максимальных относительно сильной или слабой связности. В случае орграфов, максимальных относительно односторонней связности, показано одно необходимое и одно достаточное условие.