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CENTERS IN LINE GRAPHS

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ABSTRACT. For every graph H without isolated nodes there exists a graph G such that H is the center of G and the line graph of H is the center of the line graph of G. Graphs which are the periphery of some line graph are characterized and other distance related concepts in line graphs are studied.

1. Introduction and basic results

Suppose an official has to find a suitable place for an emergency facility (such as a fire station) in a given traffic network. It is naturally to locate it in such a way that the distance to the furthest node will be as short as possible, hence to build the fire station in the center of the corresponding graph. This is one reason for which centers in graphs have been studied in many papers. It is known [2, p. 41] that for each graph H there is a graph G having the center H and containing at most four noncentral nodes. The minimum number of noncentral nodes $A(H)$ among graphs having the center H was found by Buckley, Miller and Slater [3] in the case H is a tree. Some graphs H with $A(H) = 3$ were presented by Bielak [1] and Chen [4]. Buckley, Miller and Slater [3] have also shown that for each graph H with $n \geq 9$ nodes and an integer $k \geq n + 1$ there exists a k-regular graph G having the center H. So far little is known about centers of special graphs. Clearly the center of a tree consists of either a single node or a pair of adjacent nodes. All seven central subgraphs admissible in maximal outerplanar graphs were listed by Proskurowski [10]. The greatest contains six nodes. Laskar and Shier [8] studied centers in chordal graphs. The center in the cartesian product [2, p. 23] of two graphs equals the product of their centers (Nieminen [9]). Spanning subgraphs with a prescribed central node were studied by Cheston, Farley, Hedetniemi and Proskurowski [5]. They suggested an $O(m, n)$ algorithm which for every node v in a biconnected graph G with n

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nodes and m edges produces a spanning tree T such that v is a central node in T. A good survey on centers can be found in the book [2].

In this paper we focus our attention on centers in line graphs. In general, our terminology is consistent with general usage, such as in Buckley and Harary [2]. If a graph posseses no edges, then it is said to be a *trivial graph.* Let G be a connected graph, v be its node and f be its edge. Then the *eccentricity* $e_G(v)$ or $e(v)$ *of* v is the distance to a node furthest from v. The *eccentricity* $e_G(f)$ or $e(f)$ *of the edge* f equals the eccentricity $e_{L(G)}(f)$ of the node f in the line graph of G. The *radius* $r(G)$ is the minimal eccentricity of the nodes, whereas the *diameter* $d(G)$ is their maximal eccentricity. Further, v is a *central node* if $e(v) = r(G)$ and the *center* $C(G)$ *of* G is the subgraph induced by all central nodes, while the *periphery* $\mathrm{Per}(G)$ of G is the subgraph induced by the nodes with the greatest eccentricity.

The connections in distance properties of a graph and its line graph are investigated in this paper. Relations between the eccentricity of an edge and the eccentricity of its endnodes are provided. We prove that for every graph H without isolated nodes there is a graph G such that H is the center of G and the line graph of H is the center of the line graph of G. If the line graph of a graph H has the radius at least three, then the similar result holds for the periphery instead of the center. Two conjectures on centers are presented.

We start our investigation with several observations on distances. By $d_G(x,y)$ or $d(x,y)$ we mean the distance between the nodes x and y in a graph G. Let $e = ab$ and $f = uv$ be two edges in a connected graph G. Then for their distance in the line graph of G we have $d_{L(G)}(e,f) = 0$ if $e = f$ and

$$d_{L(G)}(ab, uv) = 1 + \min\{d(a,u), d(a,v), d(b,u), d(b,v)\} \tag{1}$$

otherwise. Further, for an integer k and two nodes x and y in G, we mean by $S_k(x,y)$ the subgraph in G induced by the nodes which have the distance from both x and y at least k. Now we can express the eccentricity in a line graph in the following way:

OBSERVATION 1. *Let u and v be adjacent nodes in a connected graph G with at least three nodes. Then the eccentricity of the node uv in $L(G)$ equals the maximal $k \geq 0$ such that the subgraph $S_{k-1}(u,v)$ contains an edge.*

A node is *eccentric to* a node v if their distance equals $e(v)$. The next Lemma provides relations between the eccentricity of an edge and that of its endnodes.

LEMMA 2. *Let u and v be adjacent nodes in a connected graph G. Then $|e_{L(G)}(uv) - e_G(v)| \leq 1$ holds. Moreover, if u and v have distinct eccentricities,* *then* $|e(u) - e(v)| = 1$ *holds and the eccentricity of the edge* uv *equals the eccentricity of one its endnodes.*

Proof. Note that $|e(u)-e(v)| \leq 1$ as u and v are adjacent. Now it suffices to prove that $|e_{L(G)}(uv) - e_G(v)| \leq 1$ holds. If G has two nodes, then G is K_2 and Lemma 2 holds. From now on assume that G has at least two edges. Then there exists an edge distinct from uv and due to (1) we have $e(uv) \leq 1 + e(v)$. Further, we verify that $e(uv) \geq e(v) - 1$ holds. Let a be a node eccentric to v and distinct from u, b be a neighbour of a. Then the distance between any node from $\{u, v\}$ and any node from $\{a, b\}$ is at least $e(v) - 2$, since otherwise there will be a $v - a$ path with the length shorter than $e(v)$. As $uv \neq ab$, according to (1) we have $e(uv) \geq d(uv, ab) \geq 1 + e(v) - 2 = e(v) - 1$. □

A connected graph is *selfcentered*, if $C(G) = G$ holds. Now some consequences for the radius and the center in a line graph follow.

THEOREM 3. *For a connected graph* G *with at least three nodes we have:*

(1) : $|r(L(G)) - r(G)| \leq 1$, *moreover,* $r(L(G)) = r(G) + 1$, *if and only if for each two adjacent central nodes* x *and* y *there is an edge* f *such that both endnodes of* f *are eccentric to both* x *and* y. *Further,* $r(L(G)) = r(G) - 1$ *if and only if for each edge* f *joining central nodes and each other edge* g *at least one endnode of* f *has the distance at most* $r(G) - 2$ *to some endnode of the edge* g.

(2) : *If* G *has a nontrivial center and a greater radius than its line graph, then* $C(L(G))$ *is an induced subgraph in* $L(C(G))$. *Moreover, if* $L(G)$ *is selfcentered, then also* G *is selfcentered.*

(3) : *If* G *has a nontrivial center and a smaller radius than its line graph, then* $L(C(G))$ *is an induced subgraph in* $C(L(G))$. *Moreover, if* G *is selfcentered, then* $L(G)$ *is selfcentered, and* $L(C(G)) = C(L(G))$ *if and only if* G *is selfcentered.*

Proof. Part (1) follows directly from Lemma 2 and Observation 1. Assume G has the radius R.

(2) : Let $r(L(G)) = R - 1$. Then for a node uv in $C(L(G))$ we have $e(u) \geq R$ and $e(v) \geq R$, Lemma 2 gives $e(v) = e(u) = R$, hence uv is in $L(C(G))$. Moreover, if $L(G)$ is selfcentered, then $C(L(G)) = L(G)$ is an induced subgraph in $L(C(G))$, so G is a subgraph in $C(G)$ and G is selfcentered.

(3): Let $r(L(G)) = R + 1$. Then, for a node uv in $L(C(G))$, we have $e(u) = e(v) = R$, which gives $e(uv) \leq R + 1 = r(L(G))$ and that is why uv is in $C(L(G))$. Moreover, if G is selfcentered, then $L(G)$ is an induced subgraph in $C(L(G))$, hence $L(G)$ is selfcentered. Further, if G is selfcentered, then we have $L(C(G)) = L(G) = C(L(G))$. On the other hand, suppose that G is not selfcentered. Then it contains an edge cy joining a central node c to a noncentral node y, hence $e(c) = R$ and $e(y) = R + 1$. Then $e(cy) \leq R + 1$ due to Lemma 2 and cy is in $L(C(G))$. Hence $C(L(G)) = L(C(G))$ does not hold. □

Theorem 4 provides results on the periphery similar to those on centers.

THEOREM 4. *For a connected graph G with at least three nodes we have:*

(1) : $|d(L(G)) - d(G)| \leq 1$.

(2) : *If G has a nontrivial periphery and a greater diameter than its line graph, then $L(\mathrm{Per}(G))$ is an induced subgraph in $\mathrm{Per}(L(G))$. Moreover, if G is selfcentered, then $L(G)$ is also selfcentered and $L(\mathrm{Per}(G)) = \mathrm{Per}(L(G))$ holds if and only if G is selfcentered.*

(3) : *If G has a nontrivial periphery and a smaller diameter than its line graph, then $\mathrm{Per}(L(G))$ is an induced subgraph in $L(\mathrm{Per}(G))$. Moreover, if $L(G)$ is selfcentered, then G is also selfcentered.*

P r o o f. The part (1) follows directly from Lemma 2.

Denote by D the diameter of G.

(2): Let $d(L(G)) = D - 1$. For a node uv from $L(\mathrm{Per}(G))$ we have $e(u) = e(v) = D$, so $e(uv) \geq D - 1 = d(L(G))$, which gives uv is in $\mathrm{Per}(L(G))$. Further, if G is selfcentered, then $\mathrm{Per}(G) = G$ and $L(\mathrm{Per}(G)) = L(G)$ is an induced subgraph in $\mathrm{Per}(L(G))$, hence $L(G) = \mathrm{Per}(L(G))$, which means $L(G)$ is selfcentered.

If G is selfcentered, then $L(G)$ is also selfcentered, and clearly, $L(\mathrm{Per}(G)) = \mathrm{Per}(L(G))$ holds. On the other hand, if G is not selfcentered, then there is an edge py such that $e(p) = D$ and $e(y) = D - 1$, clearly, py is not in $L(\mathrm{Per}(G))$. Then, due to Lemma 2, we have $e(py) \geq D - 1 = d(L(G))$, hence py is in $\mathrm{Per}(L(G))$, so $L(\mathrm{Per}(G)) = \mathrm{Per}(L(G))$ does not hold.

(3): Let $d(L(G)) = D + 1$. For a node uv from $\mathrm{Per}(L(G))$ we have $e(uv) = D + 1$, so $e(u) \geq D$ and $e(v) \geq D$ and u and v are in $\mathrm{Per}(G)$. Hence uv is in $L(\mathrm{Per}(G))$. Moreover, if $L(G)$ is selfcentered, then $L(G) = \mathrm{Per}(G)$ and so G is a subgraph in $\mathrm{Per}(G)$, so G is selfcentered. □

2. Line graphs with a prescribed center

At first we show that each line graph can be a center of some line graph.

THEOREM 5. *Let H be a graph with n nodes and $m \geq 1$ edges. Then there is a connected graph G with at most $4n$ nodes and at most $m + n(n+1)$ edges such that $L(H)$ is the center of $L(G)$.*

Proof. Let $V(H) = \{v_1, \dots, v_n\}$ be the node-set of H. Now we will construct its supergraph G as follows. Its node-set will be the set $\{v_i,\ x_i,\ y_i,\ z_i \mid i = 1, \dots, n\}$. Further the edge set consists of the edges of H together with the edges joining x_i to each node from $V(H) - \{v_i\}$, the edges $x_i y_i$ and $y_i z_i$ for all $i = 1, \dots, n$ (see Fig. 1 for $H = P_3$).

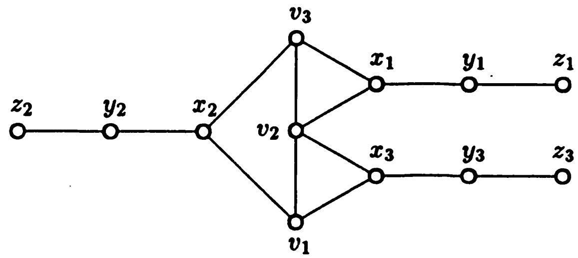

Figure 1. The line graph of the drawn graph has $L(P_3)$ as its center.

Clearly, each edge joining two central nodes has the eccentricity three, while for any other edge f let say v_1 be a central node which is nearest to f. Then its distance to $y_1 z_1$ is at least four. Hence $C\big(L(G)\big) = L(H)$ holds. □

CONJECTURE 6. *For a line graph F with n nodes and each integer $k \geq n+1$, there is a k-regular line graph with the center F.*

Simic [11] has characterized graphs G, for which the line graph transformation L and the mapping $\mathcal{K}$ which maps a graph on its clique graph commute, i.e. $L\big(\mathcal{K}(G)\big) = \mathcal{K}\big(L(G)\big)$ holds. Now we shall study a similar class of graphs, particularly connected graphs with a nontrivial center for which the mappings L and C commute, hence $L\big(C(G)\big) = C\big(L(G)\big)$. This is in a sense, according to Theorem 3, an extremal property. Denote $\Delta r(G) = r\big(L(G)\big) - r(G)$. If $\Delta r(G) = 1$, then, due to Theorem 3, the mappings L and C commute if and only if G is selfcentered. Complete graphs are examples of such graphs. But for any $i \in \{0, -1\}$ and any graph H without isolated nodes, there is a graph G with $L\big(C(G)\big) = C\big(L(G)\big)$ and $\Delta r(G) = i$, as the next theorem states.

THEOREM 7. *Let H be a graph with n nodes, $m \geq 1$ edges without isolated nodes and i be either 0 or -1. Then there exist connected graphs G_i such that $H = C(G_i)$, $L\big(C(G_i)\big) = C\big(L(G_i)\big)$ and $i = \Delta r(G)$ holds. Moreover, G_0 has $4n + 6$ nodes and $m + n^2 + 4n + 4$ edges.*

Proof. Let $V(H) = \{v_1, \dots, v_n\}$ be the node-set of H and let $E(H)$ be its edge-set. We shall construct graphs G_0 and G_{-1} with the requested properties and start with G_0.

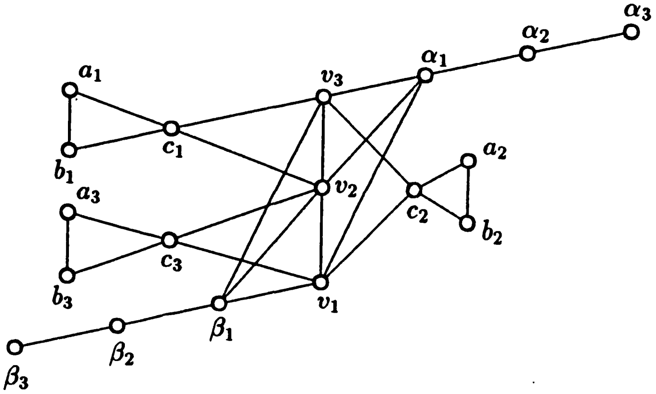

Figure 2a. A graph G_0 with $C(G_0) = P_3$ and $C(L(G_0)) = L(C(G_0))$.

The node-set of G_0 equals $V(H) \cup \{a_i,\ b_i,\ c_i \mid i = 1, \dots, n\} \cup \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3\}$. Its edge set is $E(H) \cup \{a_ib_i,\ a_ic_i,\ b_ic_i,\ \alpha_1v_i,\ \beta_1v_i \mid i = 1, \dots, n\} \cup \{c_iv_j \mid i \neq j\} \cup \{\alpha_1\alpha_2,\ \alpha_2\alpha_3,\ \beta_1\beta_2,\ \beta_2\beta_3\}$ (see Fig. 2a). One can check that $r(G_0) = 3$ and $H = C(G_0)$, since every node outside H has the distance at least four to some of the nodes α_3 and β_3. Further, any edge from $C(G_0)$ has the eccentricity three (its distance to a_1b_1 is three). As a center of $L(G_0)$ lies in a single block of $L(G_0)$, if any edge not from H lies in $C(L(G_0))$, then one its endnode, say v_1, is in H. Then its distance to a_1b_1 is four, hence $C(L(G_0)) = L(C(G_0))$ holds.

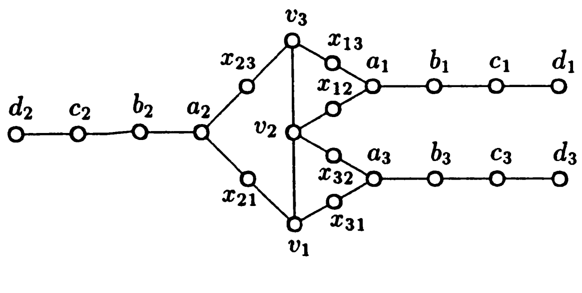

Figure 2b. A graph G_{-1} with $C(G_{-1}) = P_3$ and $C(L(G_{-1})) = L(C(G_{-1}))$.

Now we shall construct the graph G_{-1}. The node set of G_{-1} will contain $V(H) \cup \{a_i, b_i, c_i, d_i \mid i = 1, \dots, n\} \cup \{x_{ij} \mid i \neq j, \ i = 1, \dots, n, \ j = 1, \dots, n\}$. Its edge-set consists of $E(H) \cup \{x_{ij}a_i, x_{ij}v_j \mid i \neq j\} \cup \{a_ib_i, b_ic_i, c_id_i \mid i = 1, \dots, n\}$ (see Fig. 2b).

Obviously, H is the center of G_{-1} and $r(G_{-1}) = 6$, as $d(v_i, d_i) = 6$. Note that every edge joining central nodes has the eccentricity five due to (1). Further, if an edge is adjacent to exactly one central node, say v_1, then its distance to the edge c_1d_1 is at least six. Finally, if an edge f is adjacent to no central node, then its distance to some edge of the form c_id_i is also at least six, so $r(L(G_{-1})) = 5 = r(G_{-1}) + 1$ and $L(H) = C(L(G_{-1}))$ holds. □

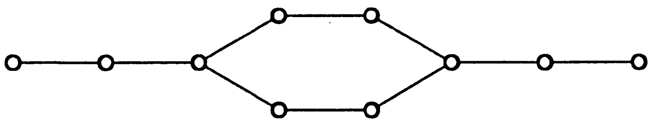

Figure 3. A graph having the center $2K_2$.

If H is bipartite, it suffices to add only six nodes in order to secure the property in question.

THEOREM 8. *Let H be a bipartite graph on n nodes and $m \geq 1$ edges without isolated nodes. Then there is a graph G with $n+6$ nodes and $m+n+4$ edges having the center H and satisfying $C(L(G)) = L(C(G))$.*

P r o o f. Let A and B be disjoint sets of nodes in H, such that adjacent nodes lie in distinct sets. We obtain G after the addition of the new nodes a, a_1, a_2, b, b_1 and b_2 such that a is adjacent to a_1 and to all nodes in A, b is adjacent to b_1 and to all nodes in B and a_2a_1 and b_1b_2 are also adjacent (see Fig. 3). Clearly, G has the desired property. □

Nevertheless, the problem of finding a graph G with the minimal number $f(H)$ of nodes such that $C(L(G)) = L(C(G))$ holds and G has a given center H seems to be far from its final solution.

3. The periphery in line graphs

Now we shall study the existence of line graphs with a given periphery. Note that $r(\mathrm{Per}(G)) \geq d(G)$ holds for each graph G.

THEOREM 9. *Let H be such a nontrivial graph that $L(H)$ has the radius at most two. Then $L(H)$ is the periphery of some line graph if and only if either $L(H)$ is selfcentered or H contains two nodes which are not endnodes and each edge is adjacent to just one of them.*

Proof. If $L(H)$ is selfcentered, then $L(H) = \mathrm{Per}(L(H))$ holds. Now assume H contains two nodes x and y which are not endnodes and such that each edge is adjacent to just one of the nodes x and y. Then $L(H) = \mathrm{Per}(L(H+xy))$ as if we add the edge xy to H, then its eccentricity will be one, while each other edge has the eccentricity two, since we have $d_{H+xy}(xa, yb) = 2$ for pairwise distinct nodes a, b, x, and y.

Assume now that there exists a graph G such that $L(H)$ is the periphery of $L(G)$. Then we have $2 \geq r(L(H)) \geq r(\mathrm{Per}(L(G))) \geq d(L(G))$. Hence $L(G)$ is either selfcentered or has the diameter two and the radius one. If $L(G)$ is selfcentered, then $L(H) = \mathrm{Per}(L(G)) = L(G)$, hence $L(H)$ has to be selfcentered. Assume now the latter case holds. Then G contains an edge xy with the eccentricity one and so each edge is adjacent to x or y. Note that $L(H)$ has the radius two as we have $r(L(H)) \geq d(L(G)) = 2$. Hence the edge xy is not in H. Further, if x is an endnode in H and a is the only its neighbour, then as H is connected there exists a node b, $b \neq x$ adjacent to ay. But $b = y$ as the edge ab is adjacent to either x or y and a is distinct from x and y. Hence ay has the eccentricity one which contradicts to $r(L(H)) = 2$. So x and similarly y is not an endnode, which completes the proof. □

If we prescribe a line graph with the radius at least three as a periphery, then an even stronger result holds.

THEOREM 10. *Let the line graph of a nontrivial graph H have the radius at least three. Then $H = \mathrm{Per}(H + K_1)$ and $L(H) = \mathrm{Per}(L(H + K_1))$ holds.*

Proof. Let v be a node in $H + K_1$ which is adjacent to all other nodes. As $r(L(H)) \geq 3$ holds, we have $r(H) \geq 2$ from Theorem 3. Hence v is the only node with the eccentricity one and each other node has the eccentricity two, so $H = \mathrm{Per}(H + K_1)$ holds. Further, it is easy to verify that each edge in H has the eccentricity three in $H + K_1$ and all edges adjacent to v have the eccentricity two, that is why $L(H) = \mathrm{Per}(L(H + K_1))$ holds. □

Note that line graphs are characterized by nine forbidden induced subgraphs. It would be of some interest to study centers in classes of graphs which are characterized by finite number of forbidden induced subgraphs, as Tomasta[1] suggested. We conclude with a conjecture which predicts a result analogous to Theorem 5. By the *claw* we mean the star with four nodes.

CONJECTURE 11. *Every claw-free graph is the center of some claw-free graph.*

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