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ON CHEBYSHEV'S INEQUALITIES FOR BEURLING'S GENERALIZED PRIMES

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ABSTRACT. In this article we give continuous versions of generalized primes considered by R. S. Hall (see [HALL, R. S.: *Beurling generalized prime number systems in which the Chebyshev inequalities fail*, Proc. Amer. Math. Soc. **40** (1973), 79–82]).

We provide an explicit calculation of the associated zeta function. This allows us to obtain an expression for the counting function for the generated generalized integers with several terms rather than just an O-term for the error. Our generalized number systems show that, in the context of Beurling's theory, the Prime Number Theorem is not equivalent to the statement $M(x) = o(x)$, where $M(x)$ is the sum of the generalized Möbius function.

§1. Introduction

Let $P = \{p_1, p_2, \dots\}$ be a set of real numbers satisfying the three conditions

$$1 < p_1 , \qquad p_k \le p_{k+1} , \qquad \lim_{k\to\infty} p_k = \infty .$$

Let $N = \{n_1 = 1 < n_2 \le n_3 \le \dots\}$ be a set of generalized integers with P as its set of generalized primes. Thus, each n_j is a finite product of elements of P. Let

$$N(x) = \sum_{n_j \le x} 1 ,$$

be the counting function of the set N. Assume that $N(x)$ satisfies, for some γ and positive c,

$$N(x) = cx + O\left(\frac{x}{\log^\gamma x}\right) . \tag{1}$$

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Under this hypothesis A. Beurling (see [2]) proved that if $\gamma > 3/2$, then the counting function $P(x)$ of the set P, satisfies

$$P(x) \sim \frac{x}{\log x}\,. \tag{2}$$

This, of course, is the Prime Number Theorem for P. Beurling also gave an example in which (1) holds with $\gamma = 3/2$ but (2) is not true.

In [4], H. G. Diamond proved that if (1) is satisfied with $\gamma \in (1, 3/2)$, then there are positive numbers a and b such that for all $x \geq x_0$

$$a\,\frac{x}{\log x} \leq P(x) \leq b\,\frac{x}{\log x}\,. \tag{3}$$

These are Chebyshev's inequalities for P. Diamond did not provide any examples for which (1) is true with $\gamma > 1$ but (2) fails. One would expect that such examples exist. In fact, it was proved by R. S. Hall (see [6]) that for each triplet (α, β, γ) with $\alpha \in [0,1]$, $\beta \in [1, +\infty]$ and $\gamma \in [0,1)$ there is a generalized number system N such that

(i) $N(x) = cx + O\big(\frac{x}{\log^\gamma x}\big)$,

(ii) $\liminf\limits_{x\to+\infty} \frac{P(x)}{\log x/x} = \alpha$,

(iii) $\limsup\limits_{x\to+\infty} \frac{P(x)}{\log x/x} = \beta$.

In this article we will give continuous analogs of Hall's examples as well as examples for which (1) and (3) are true (but (2) fails) with $\gamma \in [1/2, 1) \cup (1, 3/2]$. All these examples allow us to conclude that the statements (2) and

$$\sum_{n_j \leq x} \mu(n_j) = o(x)\,, \tag{4}$$

are not equivalent in general, where $\mu(n_j)$ is the generalized Möbius function defined in §7. More precisely, the examples of this article provide us with generalized number systems for which (2) is false while (4) is true. In [7], W.-B. Zhang proved that (2) and (4) are not in general equivalent. He used more difficult methods.

To achieve his goal, Hall alters the distribution of ordinary primes by eliminating *all* the primes in large intervals in such a way that condition (ii) above is satisfied. The addition of primes with large multiplicity ensures (iii). We will apply the same idea to the continuous prime distribution $\tau(x) = \int\limits_1^x (1 - t^{-1}) \log^{-1} t \; dt$.

The examples of Beurling generalized numbers in this article are related to the example constructed in [1].

§2. Analogs of Hall's examples

We start by defining our "set" of continuous generalized primes. It is enough to define the cumulative distribution function of this "set". We let $P(x)$ be given by

$$P(x) = \int_1^x \frac{1 - t^{-\rho}}{\log t} K(\log t)\, \mathrm{d}t\,, \tag{5}$$

where $K(t) = K_n(t)$ is the Fejer kernel, i.e.,

$$K(t) = 1 + 2\sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \cos jt\,, \tag{6}$$

and ρ is a positive constant which we might choose suitably, if need arises. This constant ρ will determine the position of some singular points of our zeta function, and it will always be in the range $0 < \rho < 1/2$. We also remark that the parameter $n \geq 2$ in equation (6) will remain fixed.

LEMMA 1. *The Fejer kernel satisfies*

$$K(t) = \frac{1}{n}\left|\frac{\sin nt/2}{\sin t/2}\right|^2 \qquad \textit{and} \qquad \int_{-\pi}^{+\pi} K(t)\, \mathrm{d}t = 2\pi\,.$$

From this lemma it follows that $P(x)$ is increasing. With the aid of the next lemma we will prove that $P(x)$ satisfies the Chebyshev inequalities (3).

LEMMA 2. *If $K(t) = K_n(t)$ is the Fejer kernel (n being a fixed natural number), and $|t| < 1/n$, then $K(t) \geq 4n/9$.*

P r o o f. If $|\theta| < \pi/2$ and $\theta \neq 0$, then

$$\frac{1}{1 + |\theta|} < \frac{\sin\theta}{\theta} \leq 1\,.$$

Hence

$$\begin{aligned} K(t) &= n\left|\frac{\sin \frac{n}{2}t/\frac{n}{2}t}{\sin \frac{1}{2}t/\frac{1}{2}t}\right|^2 \geq n\left|\frac{\sin nt/2}{nt/2}\right|^2 \\ &\geq n\left|\frac{1}{1 + |nt/2|}\right|^2 \geq n\left|\frac{1}{1 + 1/2}\right|^2 = \frac{4}{9}n\,. \end{aligned}$$

□

THEOREM 3. *If $P(x)$ is as in (5), then there are positive numbers a and b such that for $x \ge x_0$*

$$a\,\frac{x}{\log x} \le P(x) \le b\,\frac{x}{\log x}\,.$$

Remark. $P(x)$ as defined in (5) depends on n. It can be shown that the constants a, b and x_0 are independent of n.

Proof of Proposition 3. Since

$$\int_1^{\mathrm{e}} \frac{1-t^{-\rho}}{\log t} K(\log t)\,\mathrm{d}t = O(1) \qquad \text{and} \qquad \int_{\mathrm{e}}^{x} \frac{t^{-\rho}}{\log t} K(\log t)\,\mathrm{d}t \le 2nx^{1-\rho}\,,$$

we have

$$P(x) = \int_{\mathrm{e}}^{x} \frac{K(\log t)}{\log t}\,\mathrm{d}t + O\big(x^{1-\rho}\big)\,. \tag{7}$$

We will estimate the last integral, first from below.

$$\begin{aligned}
\int_{\mathrm{e}}^{x} \frac{K(\log t)}{\log t}\,\mathrm{d}t &= \int_1^{\log x} \frac{K(t)}{t}\,\mathrm{e}^t\,\mathrm{d}t \ge \frac{1}{\log x}\int_1^{\log x} K(t)\,\mathrm{e}^t\,\mathrm{d}t \\
&\ge \frac{1}{\log x}\sum_{j=1}^{J}\int_{2\pi j-1/n}^{2\pi j+1/n} K(t)\,\mathrm{e}^t\,\mathrm{d}t\,,
\end{aligned}$$

where $J \ge 1$ is such that the interval $[1, \log x]$ contains J intervals of length $2/n$ centered around 2π, 4π, … , e.g., $J = [\log x/2\pi] - 1$. By Lemma 2, we get

$$\begin{aligned}
\int_{\mathrm{e}}^{x} \frac{K(\log t)}{\log x}\,\mathrm{d}t &\ge \frac{1}{\log x}\sum_{j=1}^{J}\frac{4}{9}n\int_{2\pi j-1/n}^{2\pi j+1/n} \mathrm{e}^t\,\mathrm{d}t \\
&\ge \frac{4}{9}\,n\,\frac{1}{\log x}\sum_{j=1}^{J}\frac{2}{n}\,\mathrm{e}^{2\pi j-1/n} \\
&= \frac{8}{9}\frac{\mathrm{e}^{-1/n}}{\log x}\,\frac{\mathrm{e}^{2\pi(J+1)}-\mathrm{e}^{2\pi}}{\mathrm{e}^{2\pi}-1} \ge \frac{4}{9}\frac{\mathrm{e}^{-1/n}}{\log x}\,\frac{\mathrm{e}^{2\pi[\log x/2\pi]}}{\mathrm{e}^{2\pi}-1} \\
&\ge \frac{4}{9}\frac{\mathrm{e}^{-2\pi-1/n}}{\mathrm{e}^{2\pi}-1}\,\frac{x}{\log x}\,.
\end{aligned}$$

From this and (7) we get the lower bound for $P(x)$. Let us now prove the upper bound. Recall that $0 < \rho < 1/2$. From equation (7) we get

$$P(x) = \int_{\sqrt{x}}^{x} \frac{K(\log t)}{\log t}\,\mathrm{d}t + O\big(x^{1-\rho}\big)\,. \tag{8}$$

Let $J' = [\log x/2\pi] + 1$ and assume that $x \geq e^{2\pi}$. Recall that $\int_{-\pi}^{+\pi} K(t)\, dt = 2\pi$. Then

$$\begin{aligned}
\int_{\sqrt{x}}^{x} \frac{K(\log t)}{\log t}\, dt &\leq \frac{2}{\log x} \int_{\sqrt{x}}^{x} K(\log t)\, dt \leq \frac{2}{\log x} \int_{\pi}^{\log x} K(t)\, e^{t}\, dt \\
&\leq \frac{2}{\log x} \sum_{j=1}^{J'} \int_{2\pi j-\pi}^{2\pi j+\pi} K(t)\, e^{t}\, dt \leq \frac{2}{\log x} \sum_{j=1}^{J'} e^{2\pi j+\pi} \int_{2\pi j-\pi}^{2\pi j+\pi} K(t)\, dt \\
&= 2\pi \frac{2\, e^{\pi}}{\log x} \frac{e^{2\pi(J'+1)} - e^{2\pi}}{e^{2\pi} - 1} \leq 2\pi \frac{2}{\log x} \frac{e^{3\pi}}{e^{2\pi} - 1} e^{2\pi J'} \\
&\leq 2\pi \frac{2\, e^{5\pi}}{e^{2\pi} - 1} \frac{x}{\log x}.
\end{aligned}$$

From this and (8) we obtain the upper bound for $P(x)$. $\square$

§3. Towards an expression for $N(x)$

In this section we perform the calculations necessary to obtain an expression for the counting function $N(x)$, of the "set" of generated integers N. To this end, let us assume that we have a function

$$G(x) = 1 + 2 \sum_{j=1}^{n-1} \alpha_j \cos(jx), \tag{9}$$

such that $G(x) \geq 0$ for all x. We will also assume that the α_j's are nonzero real numbers with absolute value less than one (by integrating $(1 \pm \cos(jx))G(x)$ over $(-\pi, \pi)$ one gets that $|\alpha_j| \leq 1$ if $G(x)$ is to be non-negative). For such a function $G(x)$ we let (cf. (5))

$$P(x) = \int_{1}^{x} \frac{1 - t^{-\rho}}{\log t} G(\log t)\, dt.$$

Because $G(x) \geq 0$, we see that $P(x)$ is an increasing function which we take as the cumulative distribution function associated with our set P of generalized primes. The distribution function for N is given by

$$N(x) = \int_{1^-}^{x} dN = \int_{1^-}^{x} e^{dP},$$

where

$$e^{\mathrm{d}P} = \delta + \mathrm{d}P + \frac{1}{2!}\,\mathrm{d}P * \mathrm{d}P + \frac{1}{3!}\,\mathrm{d}P^{*3} + \cdots,$$

δ being the Dirac measure at the point 1, $*$ the multiplicative convolution of measures, and the convergence is in total variation in each finite interval. A detailed description of these notions can be found in [3]. The above procedure for obtaining N from P works when P is continuous as well as when it is a discrete distribution.

We can also determine $N(x)$ by using Perron's inversion formula:

$$N(x) = \frac{1}{2\pi\mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} \zeta(s)\frac{x^s}{s}\,\mathrm{d}s, \qquad b > 1, \tag{10}$$

where $\zeta(s)$ is defined for $s = \sigma + \mathrm{i}t$, with $\sigma > 1$, as

$$\zeta(s) = \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}N(x) = \int_{1^-}^{\infty} x^{-s}\,e^{\mathrm{d}P}(x) = \exp\left\{ \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}P(x) \right\}. \tag{11}$$

PROPOSITION 4. *If $\zeta(s)$ is as defined above, i.e., if*

$$\zeta(s) = \exp\left\{ \int_1^{\infty} t^{-s}\frac{1-t^{-\rho}}{\log t}\left(1 + 2\sum_{j=1}^{n-1} \alpha_j \cos(j\log t)\right)\mathrm{d}t \right\},$$

then, for $\sigma > 1$,

$$\zeta(s) = \frac{s+\rho-1}{s-1} \prod_{0<|j|<n} \left(1 - \frac{\rho}{s-\mathrm{i}j-1+\rho}\right)^{-\alpha_j}, \tag{12}$$

where we have set $\alpha_{-j} = \alpha_j$.

To prove this we need a lemma.

LEMMA 5. *Let j be an integer. Let $\sigma > 1$. Then the following formula holds.*

$$\int_1^{\infty} t^{-s}\frac{1-t^{-\rho}}{\log t}\cos(j\log t)\,\mathrm{d}t$$

$$= -\frac{1}{2}\log\left\{ \left(1 - \frac{\rho}{s-\mathrm{i}j-1+\rho}\right)\left(1 - \frac{\rho}{s+\mathrm{i}j-1+\rho}\right) \right\}.$$

Proof. Put $\cos(j\log t) = \frac{1}{2}\left(t^{\mathrm{i}j} + t^{-\mathrm{i}j}\right)$. Then

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}s}\int_1^\infty t^{-s}\frac{1-t^{-\rho}}{\log t}\cos(j\log t)\ \mathrm{d}t \\
&= \int_1^\infty t^{-s}\left(1-t^{-\rho}\right)\frac{t^{\mathrm{i}j}+t^{-\mathrm{i}j}}{2}\ \mathrm{d}t \\
&= \frac{1}{2}\int_1^\infty \left\{t^{-s+\mathrm{i}j} - t^{-\rho-s+\mathrm{i}j} + t^{-s-\mathrm{i}j} - t^{-\rho-s-\mathrm{i}j}\right\}\ \mathrm{d}t \\
&= \frac{1}{2}\left\{\frac{1}{s-\mathrm{i}j-1} - \frac{1}{s-\mathrm{i}j-1+\rho} + \frac{1}{s+\mathrm{i}j-1} - \frac{1}{s+\mathrm{i}j-1+\rho}\right\} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\log\left\{\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)\left(1-\frac{\rho}{s+\mathrm{i}j-1+\rho}\right)\right\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_1^\infty t^{-s}\frac{1-t^{-\rho}}{\log t}\cos(j\log t)\ \mathrm{d}t \\
&= -\frac{1}{2}\log\left\{\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)\left(1-\frac{\rho}{s+\mathrm{i}j-1+\rho}\right)\right\} + \text{Constant}.
\end{aligned}
$$

By taking the limit as $\mathrm{Re}(s)$ tends to infinity we see that the constant of integration is zero. □

Proof of Proposition 4. We use the preceding lemma with $0 \le j < n$. For $\sigma > 1$ we have

$$
\begin{aligned}
\log\zeta(s) &= \int_1^\infty t^{-s}\frac{1-t^{-\rho}}{\log t}G(\log t)\ \mathrm{d}t \\
&= \int_1^\infty t^{-s}\frac{1-t^{-\rho}}{\log t}\left\{1 + 2\sum_{j=1}^{n-1}\alpha_j\cos(j\log t)\right\}\ \mathrm{d}t \\
&= \log\frac{s+\rho-1}{s-1} - \sum_{j=1}^{n-1}\alpha_j\log\left[\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)\left(1-\frac{\rho}{s+\mathrm{i}j-1+\rho}\right)\right] \\
&= \log\left\{\frac{s+\rho-1}{s-1}\prod_{0<|j|<n}\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)^{-\alpha_j}\right\}.
\end{aligned}
$$

Since $0 < |\alpha_j| < 1$, the finite product on the right hand side of (12) has branch points at $1+\mathrm{i}j$ and $1-\rho+\mathrm{i}j$, for $0<|j|<n$. Let $\widetilde{C}_j$ be the horizontal line segments joining the pairs of points $1-\rho+\mathrm{i}j$ and $1+\mathrm{i}j$. If we remove these branch cuts $\widetilde{C}_j$ from the complex plane and denote the resulting set by $\mathcal{D} = \mathcal{D}_G$, then the product on the right hand side of (12) is a single valued analytic function with $\mathcal{D}$ as its domain of definition. Thus, equation (12) gives an analytic continuation of $\zeta(s)$ to $\mathcal{D}$. □

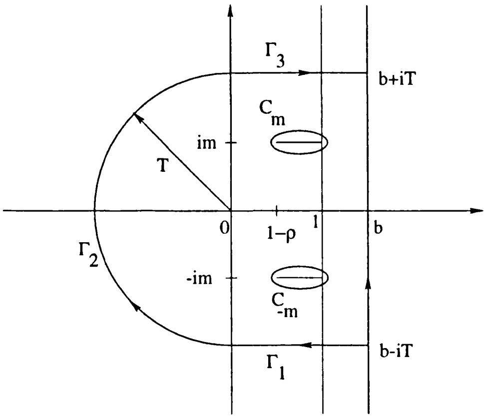

FIGURE 2.1.

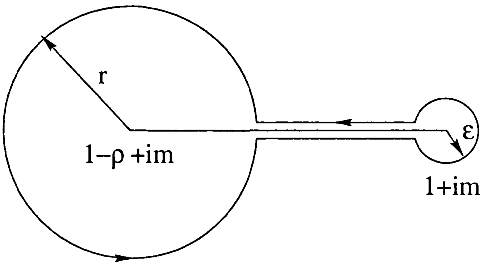

FIGURE 2.2.

Let us consider the contour of integration in Figure 2.1. In this figure we have used C_m to denote a closed contour as shown in Figure 2.2, where ε is the radius of the small circle and r is the radius of the large one. For every piece Γ of the contour in Figure 2.1, we write

$$I_\Gamma = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \zeta(s) \frac{x^s}{s}\, \mathrm{d}s\,.$$

We also let $c_1 x$ and c_0 be the residues of $\zeta(s)x^s/s$ at, respectively, $s=1$ and $s=0$. Suppose that $T>n$. From Cauchy's Theorem we have

$$\frac{1}{2\pi \mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T}\zeta(s)\frac{x^s}{s}\,\mathrm{d}s = c_1x+c_0+I_{\Gamma_1}+I_{\Gamma_2}+I_{\Gamma_3}+\sum_{0<|m|<n} I_{C_m}\,.$$

From equation (12) it is clear that $|\zeta(s)|\to 1$ as $|s|\to\infty$. Therefore

$$|I_{\Gamma_2}| \ll \int_{-3\pi/2}^{-\pi/2}\left|\frac{x^{T\exp(\mathrm{i}\theta)}}{T\exp(\mathrm{i}\theta)}T\,\mathrm{i}e^{\mathrm{i}\theta}\right|\,\mathrm{d}\theta \le \int_{-3\pi/2}^{-\pi/2} x^{T\cos\theta}\,\mathrm{d}\theta \to 0 \qquad \text{as} \quad T\to\infty\,.$$

I_{Γ_1} and I_{Γ_3} also tend to zero:

$$|I_{\Gamma_1}| \ll \int_0^b \frac{x^\sigma}{|\sigma+\mathrm{i}T|}\,\mathrm{d}\sigma \le \frac{bx^b}{T}\to 0 \qquad \text{as} \quad T\to\infty\,.$$

(The same estimate holds for I_{Γ_3}.)

The above considerations together with (10) imply that

$$N(x) = c_1x + c_0 + \sum_{0<|m|<n} I_{C_m}\,. \tag{13}$$

PROPOSITION 6. *Let $\zeta(s)$ be as in* (12). *For $m\in(0,n)$ set*

$$I_{C_m} = \frac{1}{2\pi\mathrm{i}}\int_{C_m}\zeta(s)\frac{x^s}{s}\,\mathrm{d}s\,,$$

with C_m being the contour in Figure 2.2. *Then for every such m, there is a real number A_m distinct from zero such that*

$$I_{C_m}+I_{C_{-m}} = A_m x\cos(m\log x)\left(\frac{1}{\log x}\right)^{1-\alpha_m} + O\left(\frac{x\log\log x}{\log^{2-\alpha_m}x}\right).$$

P r o o f . Consider the contour C_m in Figure 2.2. It is easy to see that the integral

$$\frac{1}{2\pi\mathrm{i}}\int\zeta(s)\frac{x^s}{s}\,\mathrm{d}s$$

over the circle of radius r is $O(x^{1-\rho+r})$. Also, the integral over the circle of radius ε tends to zero as $\varepsilon\to 0$ (because the "pole" of $\zeta(s)$ at $s=1+\mathrm{i}m$ is of order less than one). Therefore, if C'_m and C''_m are the line segments lying respectively above and below the branch cut $\widetilde{C}_m$ and joining the points $1-\rho+r+\mathrm{i}m$ and $1+\mathrm{i}m$, then

$$I_{C_m} = \frac{1}{2\pi\,\mathrm{i}} \int\limits_{C'_m} + \int\limits_{C''_m} \zeta(s)\frac{x^s}{s}\,\mathrm{d}s + O\big(x^{1-\rho+r}\big)\,. \tag{14}$$

Writing

$$s = 1+\mathrm{i}m + t\,\mathrm{e}^{\mathrm{i}\theta}\,, \qquad -\pi \le \theta < \pi\,,$$

and letting $\theta=-\pi$, and t run from 0 to $\rho-r$, we obtain a parametrization of C''_m with its direction reversed:

$$-C''_m : \begin{cases} \theta = -\pi\,, \\ s = 1+\mathrm{i}m - t\,, \\ \mathrm{d}s = -\,\mathrm{d}t\,, \\ 0 \le t \le \rho - r\,. \end{cases}$$

Before we use this parametrization of C''_m, let us rewrite the integrand in (14):

$$\begin{aligned} \zeta(s)\frac{x^s}{s} &= \frac{x^s}{s}\frac{s+\rho-1}{s-1}\prod_{0<|j|<n}\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)^{-\alpha_j} \\ &= \frac{x^s}{s}\frac{s+\rho-1}{s-1}\left(1-\frac{\rho}{s-\mathrm{i}m-1+\rho}\right)^{-\alpha_m}\prod_{\substack{0<|j|<n\\ j\neq m}}\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)^{-\alpha_j} \\ &= (s-1-\mathrm{i}m)^{-\alpha_m}\frac{x^s}{s}\frac{s+\rho-1}{s-1}\,\frac{\prod_{\substack{0<|j|<n\\ j\neq m}}\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)^{-\alpha_j}}{(s-\mathrm{i}m-1+\rho)^{-\alpha_m}}\,. \end{aligned}$$

Let

$$f_m(s) = \frac{(s+\rho-1)}{s(s-1)(s-\mathrm{i}m-1+\rho)^{-\alpha_m}}\prod_{\substack{0<|j|<n\\ j\neq m}}\left(1-\frac{\rho}{s-\mathrm{i}j-1+\rho}\right)^{-\alpha_j}\,.$$

Notice that $f_m(s)$ is an analytic function at $s=1+\mathrm{i}m$. As a power series it has radius of convergence equal to ρ. Also notice that $f_m(1+\mathrm{i}m)\neq 0$. Let us write

$$f_m(s) = \sum_{j=0}^{\infty} a_{m,j}(s-1-\mathrm{i}m)^j\,.$$

Then we have

$$\begin{aligned}
\frac{1}{2\pi\,\mathrm{i}}\int\limits_{C''_m}\zeta(s)\frac{x^s}{s}\,\mathrm{d}s &= \frac{1}{2\pi\,\mathrm{i}}\int\limits_{C''_m} x^s(s-1-\mathrm{i}m)^{-\alpha_m} f_m(s)\,\mathrm{d}s\\
&= \frac{1}{2\pi\,\mathrm{i}}\int\limits_{C''_m} x^s\sum_{j=0}^{\infty} a_{m,j}(s-1-\mathrm{i}mr)^{j-\alpha_m}\,\mathrm{d}s\\
&= \sum_{j=0}^{\infty} a_{m,j}\,\frac{1}{2\pi\,\mathrm{i}}\int\limits_{C''_m} x^s(s-1-\mathrm{i}m)^{j-\alpha_m}\,\mathrm{d}s\\
&= \sum_{j=0}^{\infty} a_{m,j}\,\frac{1}{2\pi\,\mathrm{i}}\int\limits_{0}^{\rho-r} x^{1-\mathrm{i}m-t}\left(\mathrm{e}^{-\,\mathrm{i}\pi}\,t\right)^{j-\alpha_m}\,\mathrm{d}t\\
&= \frac{1}{2\pi\,\mathrm{i}}\,x^{1-\mathrm{i}m}\,\mathrm{e}^{\mathrm{i}\pi\alpha_m}\sum_{j=0}^{\infty}(-1)^j a_{m,j}\int\limits_{0}^{\rho-r} x^{-t}t^{j-\alpha_m}\,\mathrm{d}t\,.
\end{aligned}$$

Similarly, one obtains, replacing C''_m by C'_m

$$\frac{1}{2\pi\,\mathrm{i}}\int\limits_{C'_m}\zeta(s)\frac{x^s}{s}\,\mathrm{d}s = \frac{-1}{2\pi\,\mathrm{i}}\,x^{1-\mathrm{i}m}\,\mathrm{e}^{-\,\mathrm{i}\pi\alpha_m}\sum_{j=0}^{\infty}(-1)^j a_{m,j}\int\limits_{0}^{\rho-r} x^{-t}t^{j-\alpha_m}\,\mathrm{d}t\,.$$

From (14) we now get

$$I_{C_m} = \frac{x^{1-\mathrm{i}m}}{\pi}\sin(\pi\alpha_n)\sum_{j=0}^{\infty}(-1)^j a_{m,j}\int\limits_{0}^{\rho-r} x^{-t}t^{j-\alpha_m}\,\mathrm{d}t + O\left(x^{1-\rho+r}\right). \tag{15}$$

We consider now the last sum:

$$\begin{aligned}
&\sum_{j=0}^{\infty}(-1)^j a_{m,j}\int\limits_{0}^{\rho-r} x^{-t}t^{j-\alpha_m}\,\mathrm{d}t\\
&= \sum_{j=0}^{\infty}(-1)^j a_{m,j}\int\limits_{0}^{(\rho-r)\log x} \mathrm{e}^{-t}\left(\frac{t}{\log x}\right)^{j-\alpha_m}\frac{\mathrm{d}t}{\log x}\\
&= \left(\frac{1}{\log x}\right)^{1-\alpha_m}\sum_{j=0}^{\infty} a_{m,j}\left(\frac{-1}{\log x}\right)^{j}\int\limits_{0}^{(\rho-r)\log x}\mathrm{e}^{-t}\,t^{j-\alpha_m}\,\mathrm{d}t\\
&= \left(\frac{1}{\log x}\right)^{1-\alpha_m}\left\{a_{m,0}\int\limits_{0}^{(\rho-r)\log x}\mathrm{e}^{-t}\,t^{-\alpha_m}\,\mathrm{d}t + R\right\}.
\end{aligned}$$

It will be convenient to take $r = \rho - 3\log\log x/\log x$. Now since $\sum a_{m,j} z^j$ converges whenever $|z| < \rho$, it follows that $|a_{m,j}| \lesssim M^j$ for some $M > 1/\rho$. Hence

$$\begin{aligned}|R| &\le \sum_{j=1}^{\infty} |a_{m,j}| \left(\frac{1}{\log x}\right)^j \int_0^{3\log\log x} \mathrm{e}^{-t}\, t^{j-\alpha_m}\, \mathrm{d}t \\ &\le \sum_{j=1}^{\infty} |a_{m,j}| \left(\frac{1}{\log x}\right)^j (3\log\log x)^j \int_0^{3\log\log x} \mathrm{e}^{-t}\, t^{-\alpha_m}\, \mathrm{d}t \\ &\ll \sum_{j=1}^{\infty} |a_{m,j}| \left(\frac{3\log\log x}{\log x}\right)^j \ll \frac{3\log\log x}{\log x} \sum_{j=0}^{\infty} \left(\frac{3M\log\log x}{\log x}\right)^j \\ &\ll \frac{\log\log x}{\log x}.\end{aligned}$$

Since $x^{1-\rho+r} = x/\log^3 x$ and

$$\int_0^{3\log\log x} \mathrm{e}^{-t}\, t^{-\alpha_m}\, \mathrm{d}t = \Gamma(1-\alpha_n) + O\left(\frac{\log\log x}{\log^3 x}\right),$$

we can write (15) as

$$I_{C_m} = x^{1-\mathrm{i}m} \frac{\sin(\pi\alpha_m)}{\pi} a_{m,0}\, \Gamma(1-\alpha_m) \left(\frac{1}{\log x}\right)^{1-\alpha_m} + O\left(\frac{x\log\log x}{\log^{2-\alpha_m} x}\right).$$

Finally, since $I_{C_{-m}}$ is the complex conjugate of I_{C_m}, we have

$$I_{C_m} + I_{C_{-m}} = A_m x \cos(m\log x) \left(\frac{1}{\log x}\right)^{1-\alpha_m} + O\left(\frac{x\log\log x}{\log^{2-\alpha_m} x}\right),$$

where $A_m = 2\pi^{-1}\sin(\pi\alpha_m) a_{m,0}\, \Gamma(1-\alpha_m)$, which is distinct from zero because

$$a_{m,0} = f_m(1+\mathrm{i}m) \neq 0 \text{ and } 0 < |\alpha_m| < 1 \implies \sin(\pi\alpha_m) \neq 0.$$

This finishes the proof of Proposition 6. □

§4. Hall's examples again

THEOREM 7. *Let*

$$P(x) = \int_1^x \frac{1-t^{-\rho}}{\log t} K(\log t)\, \mathrm{d}t,$$

where $K(x) = K_n(x)$ *is the Fejer kernel* (*cf. equation* (6)). *Let*

$$N(x) = \int_{1^-}^{x} e^{dP} .$$

Then there exist nonzero constants A_j *such that*

$$N(x) = c_1 x + \sum_{j=1}^{n-1} A_j x \cos(j \log x) \left(\frac{1}{\log x}\right)^{j/n} + O\left(\frac{x}{\log x}\right). \tag{16}$$

The constant c_1 *is positive:*

$$c_1 = \rho \prod_{0<|j|<n} \left(1 + \frac{\rho^2}{j^2}\right)^{1-j/n} > 0 .$$

Proof. In §3 we can take $G(x) = K(x)$ and get from equation (13)

$$N(x) = c_1 x + c_0 + \sum_{j=1}^{n-1} I_{C_j} + I_{C_{-j}} .$$

From Proposition 6, since $\alpha_j = 1 - \frac{|j|}{n}$, (cf. (5), (6) and (9)), we have

$$I_{C_j} + I_{C_{-j}} = A_j x \cos(j \log x) \left(\frac{1}{\log x}\right)^{j/n} + O\left(\frac{x}{\log x}\right), \qquad A_j \neq 0 .$$

□

Remark. In addition to (16) we showed in §2 that Chebyshev's inequalities hold:

$$\frac{x}{\log x} \ll P(x) \ll \frac{x}{\log x} . \tag{17}$$

This finishes our discussion of continuous analogs of Hall's examples.

§5. More examples

We also have the following proposition.

THEOREM 8. *Let*

$$P(x) = \int_{1}^{x} \frac{1 - t^{-\rho}}{\log t} G(\log t)\, dt \qquad \textit{with} \quad G(x) = 1 + 2\alpha \cos(x) ,$$

where α is a fixed real number such that $0 < |\alpha| \le 1/2$. (Note that $G(x) \ge 0$.) Let

$$N(x) = \int_{1^-}^{x} \mathrm{e}^{\mathrm{d}P} .$$

Then

$$N(x) = c_1 x + A_1' x \cos(\log x)\left(\frac{1}{\log x}\right)^{1-\alpha} + O\left(\frac{x \log\log x}{\log^{2-\alpha} x}\right).$$

P r o o f . This follows from equation (13) and Proposition 6. □

Thus, for every $\gamma \in [1/2, 1) \cup (1, 3/2]$ we have a continuous number system for which

$$N(x) = cx + O\left(\frac{x}{\log^{\gamma} x}\right),$$

and for which the Chebyshev inequalities (17) are true. ($\gamma \in [1/2, 1)$ when $0 < \alpha \le 1/2$, and $\gamma \in (1, 3/2]$ when $-1/2 \le \alpha < 0$.)

§6. Discrete examples

In this section we construct discrete versions of the examples considered above; that is, we construct generalized primes $p_1, p_2, \ldots$ whose counting function

$$p^*(x) = \sum_{p_j \le x} 1$$

is close to

$$P(x) = \int_{1}^{x} \frac{1 - t^{-\rho}}{\log t} G(\log t)\, \mathrm{d}t ,$$

with $G(x)$ as in (9).

Let x_0 be such that $P(x_0) > 1$. We define, for $j = 1, 2, \ldots$, our jth prime to be

$$p_j = P^{-1}(x_0 + j) .$$

We also let

$$P^*(x) = \sum_{n} \frac{1}{n} p^*(x^{1/n}) .$$

As in earlier sections, we define

$$N^*(x) = \int_{1^-}^{x} \mathrm{e}^{\mathrm{d}P^*}$$

and

$$\zeta^*(s) = \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}N^*(x) = \exp\left\{ \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}P^*(x) \right\}. \tag{18}$$

Proposition 9. *Let $\zeta^*(s)$ be as in equation (18) and $\zeta(s)$ as in (11). There exists a function $\varphi(s)$ analytic in $\mathrm{Re}(s) > 1/2$ such that*

$$\zeta^*(s) = \mathrm{e}^{\varphi(s)}\,\zeta(s) = \mathrm{e}^{\varphi(s)}\,\frac{s+\rho-1}{s-1} \prod_{0<|j|<n} \left(1 - \frac{\rho}{s - \mathrm{i}j - 1 + \rho}\right)^{-\alpha_j}.$$

The following lemma is needed.

Lemma 10. $P(x) - P^*(x) = O(\sqrt{x}\,)$.

Proof. We first notice that $P^*(x) - p^*(x) = O(\sqrt{x}\,)$. The lemma follows from

$$\begin{aligned} p^*(x) &= \sum_{p_j \le x} 1 = \sum_{P^{-1}(x_0+j)\le x} 1 = \sum_{x_0+j\le P(x)} 1 \\ &= P(x) + O(1). \end{aligned}$$

□

Proof of Proposition 9. For $\sigma > 1$ we have

$$\begin{aligned} \frac{\zeta^*(s)}{\zeta(s)} &= \frac{\exp\left\{ \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}P^*(x) \right\}}{\exp\left\{ \int_{1^-}^{\infty} x^{-s}\,\mathrm{d}P(x) \right\}} \\ &= \exp\left\{ \int_{1^-}^{\infty} x^{-s}\big(\mathrm{d}P^*(x) - \mathrm{d}P(x)\big) \right\} \\ &= \exp\left\{ s\int_{1^-}^{\infty} \frac{P^*(x) - P(x)}{x^{s+1}}\,\mathrm{d}x \right\} =: \mathrm{e}^{\varphi(s)}. \end{aligned}$$

From the preceding lemma, it follows that

$$\varphi(s) = s\int_{1^-}^{\infty} \frac{P^*(x) - P(x)}{x^{s+1}}\,\mathrm{d}x$$

is as stated in the proposition. □

LEMMA 11. *Let $\zeta^*(s)$ be as in Proposition 9. There exist constants B and K such that for all $s=\sigma+\mathrm{i}t$ satisfying*

$$1-\frac{1}{\log(|t|+8)}\le\sigma\le 2\,,$$

the following inequalities hold:

a) $|\zeta^*(\sigma+\mathrm{i}t)|\le B\log^K\big(|t|+3\big)$,
b) $|1/\zeta^*(\sigma+\mathrm{i}t)|\le B\log^K(|t|+3)$.

Proof. Let $\varphi(s)$ be as in Proposition 9. It is enough to show that there is a constant K such that $|\varphi(\sigma+\mathrm{i}t)|\le K\log\log\big(|t|+8\big)$, when σ and t are as stated in the lemma. For such σ and t we have

$$x^{-\sigma}=\frac{x^{1-\sigma}}{x}=\frac{1}{x}\exp\{(1-\sigma)\log x\}\le\frac{1}{x}\exp\left\{\frac{\log x}{\log(|t|+8)}\right\}\ll\frac{1}{x}\,,$$

whenever $1\le x\le t^2$. Now, by Lemma 10

$$\begin{aligned}\varphi(s)&=\int_{1-}^{\infty}x^{-s}\,\mathrm{d}(P^*-P)(x)\\&=\int_{1-}^{t^2}x^{-s}\,\mathrm{d}(P^*-P)(x)+\int_{t^2}^{\infty}x^{-s}\,\mathrm{d}(P^*-P)(x)\\&=\int_{1-}^{t^2}x^{-s}\,\mathrm{d}(P^*-P)(x)-\frac{P^*(t^2)-P(t^2)}{t^{2s}}+s\int_{t^2}^{\infty}\frac{P^*(x)-P(x)}{x^{s+1}}\,\mathrm{d}x\\&\ll\int_{1-}^{t^2}x^{-\sigma}\,\mathrm{d}(P^*+P)(x)+\frac{1}{|t|}+|t|\int_{t^2}^{\infty}x^{-\sigma-1/2}\,\mathrm{d}x\\&=\int_{1-}^{t^2}x^{-\sigma}\,\mathrm{d}(P^*+P)(x)+O(1)\,.\end{aligned}$$

To justify the $O(1)$ term in the last expression notice that

$$|t|\cdot|t|^{2(\frac{1}{2}-\sigma)}\le|t|\cdot|t|^{-1+2/\log(|t|+8)}\ll 1\,.$$

Because of Proposition 3 we now have

$$\begin{aligned}\int_{1^-}^{t^2} x^{-\sigma}\,\mathrm{d}P(x) &\ll \int_{1^-}^{t^2} x^{-1}\,\mathrm{d}P(x)\\ &= \frac{P(t^2)}{t^2} + \int_1^{t^2} \frac{P(x)}{x^2}\,\mathrm{d}x\\ &\ll \frac{1}{\log(|t|+3)} + \int_1^{t^2} \frac{\mathrm{d}x}{x\log x}\\ &\ll \log\log\big(|t|+8\big)\,.\end{aligned}$$

By Lemma 10 the same estimate holds for $P^*(x)$ in place of $P(x)$. This finishes the proof of the lemma. □

The following proposition gives an asymptotic evaluation of $N^*(x)$. We state it without proof.

PROPOSITION 12. *Let N^* be as defined at the beginning of* §6. *Then there exists a constant $c^* > 0$ such that*

$$N^*(x) \sim c^* x\,. \tag{19}$$

§7. The Möbius sum function

Once we have a discrete number system (as constructed in Section §6) we are in a position to define analogs of the Möbius function. Thus, if $n_j = p_{j_1}^{a_1}\cdots p_{j_k}^{a_k}$, we let

$$\mu(n_j) = \begin{cases} 1 & \text{if } n_j = 1\,, \text{ i.e., all } a_k = 0\,,\\ (-1)^k & \text{if } a_1 = \cdots = a_k = 1\,,\\ 0 & \text{otherwise.}\end{cases}$$

We also let $M(x) = \sum\limits_{n_j \le x} \mu(n_j)$.

In this section, we use Helson's Method (see [5]) to show that the relation $M(x) = o(x)$ holds for these examples and in the next section, we obtain an asymptotic expression for $P^*(x)$.

More specifically, we will prove that $M(x) = o(x)$, for the discrete versions of the analogs of Hall's examples as well as the discrete version of the examples considered in §5. However, for the latter, we will require that $0 < |\alpha| < 1/2$; for technical reasons we exclude the case $\alpha = -1/2$.

We start with the identity

$$\frac{1}{\zeta^*(s)} = \int_{1^-}^{\infty} x^{-s}\, \mathrm{d}M(x).$$

The usual integration by parts yields

$$\begin{aligned}\frac{1}{s\zeta^*(s)} &= \int_1^{\infty} x^{-s-1} M(x)\, \mathrm{d}x \\ &= \int_0^{\infty} \mathrm{e}^{-sx}\, M(\mathrm{e}^x)\, \mathrm{d}x = \int_{-\infty}^{+\infty} \mathrm{e}^{-\sigma x}\, M(\mathrm{e}^x)\, \mathrm{e}^{-\mathrm{i}tx}\, \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} F_\sigma(x)\, \mathrm{e}^{-\mathrm{i}tx}\, \mathrm{d}x,\end{aligned}$$

where we have set $F_\sigma(x) = \mathrm{e}^{-\sigma x}\, M(\mathrm{e}^x)$. From Plancherel's theorem we get

$$\int_{-\infty}^{+\infty} \left|\frac{1}{(\sigma+\mathrm{i}t)\zeta^*(\sigma+\mathrm{i}t)}\right|^2 \mathrm{d}t = 2\pi \int_{-\infty}^{+\infty} |F_\sigma(x)|^2\, \mathrm{d}x. \tag{20}$$

Our job now is to show that the integral on the right hand side is finite when $\sigma = 1$. Since $|F_\sigma(x)|$ increases as $\sigma \to 1^+$, it suffices to show that the integral on the left hand side of (20) is finite whenever $\sigma \geq 1$. That this is the case will follow from the fact that the singularities of $\zeta^*(s)$ on the line $\sigma = 1$ are of sufficiently small order: $-\alpha_j < 1/2$.

Let $\sigma = 1$. From Proposition 9 we see that

$$\left|\frac{1}{\zeta^*(1+\mathrm{i}t)}\right| = \left|\mathrm{e}^{-\varphi(1+\mathrm{i}t)}\, \frac{t}{t-\mathrm{i}\rho}\right| \prod_{0<|j|<n} \left|\frac{t-j}{t-j-\mathrm{i}\rho}\right|^{\alpha_j}.$$

Recall from Lemma 11 that there is a constant K such that

$$\left|\frac{1}{\zeta^*(1+\mathrm{i}t)}\right| \ll \log^K\big(|t|+3\big).$$

Because of the factor $(1+\mathrm{i}t)^{-2}$ on the left hand side of (20) and because $\int_n^{\infty} \frac{\log^{2K} t}{t^2}\, \mathrm{d}t$ converges, we have only to show that

$$\int_{-n}^{+n} \left|\frac{1}{\zeta^*(1+\mathrm{i}t)}\right|^2 \mathrm{d}t \ll \int_{-n}^{+n} \prod_{0<|j|<n} |t-j|^{2\alpha_j}\, \mathrm{d}t < \infty.$$

Since $2\alpha_j > -1$, the last integral is indeed finite. Therefore, by the Monotone Convergence Theorem, we conclude that the right hand side of (20) is finite when $\sigma = 1$, i.e.,

$$\int_0^\infty |\,\mathrm{e}^{-x} M(\mathrm{e}^x)|^2 \,\mathrm{d}x = \int_{-\infty}^{+\infty} |F_1(x)|^2 \,\mathrm{d}x < \infty\,.$$

Let $g(x) = M(x)/x$. From the fact $\int_1^\infty g^2(x)\frac{\mathrm{d}x}{x} = \int_0^\infty g^2(\mathrm{e}^x)\,\mathrm{d}x < \infty$, we would like to conclude that $g(x) = o(1)$. To show this, we take advantage of the fact that $g(x)$ varies in a slow fashion. Indeed, assume for a generalized integer n_j, that $|g(n_j)| \geq \eta > 0$. Then we will show that

$$\left|g(n_{j+k})\right| \geq \frac{\eta}{4} \qquad \text{for} \quad 0 \leq k \leq \frac{c\eta}{2c+\eta} n_j\,, \tag{21}$$

where $c = c^*$ is as in (19). To prove (21), we need two lemmas.

Lemma 13. *Consider the set N^* of discrete generalized integers: $N^* = \{n_1, n_2, \dots\}$. Assume that the counting function $N^*(x)$ satisfies $N^*(x) = cx + o(x)$. Let $k \geq 0$. Then for $j \to \infty$,*

$$n_{j+k} = n_j + k/c + o(n_{j+k})\,.$$

Proof.

$$k = N^*(n_{j+k}) - N^*(n_j) = cn_{j+k} - cn_j + o(n_{j+k})\,.$$

□

Lemma 14. *We have $|M(y) - M(x)| \leq |N^*(y) - N^*(x)|$.*

Proof. Assume $x < y$. Then

$$|M(y) - M(x)| = \left|\int_x^y \mathrm{e}^{-\,\mathrm{d}P^*}\right| \leq \int_x^y \mathrm{e}^{\mathrm{d}P^*} = N^*(y) - N^*(x)\,.$$

□

Now we prove (21). Assume n_j is large and that $|g(n_j)| \geq \eta$. Then

$$\begin{aligned}|g(n_{j+k})| &= \frac{|M(n_{j+k})|}{n_{j+k}} \geq \frac{|M(n_j)| - k}{n_{j+k}} \geq \frac{\eta n_j - k}{n_{j+k}} \geq \frac{1}{2}\,\frac{\eta n_j - k}{n_j + k/c} \\ &\geq \frac{1}{2}\,\frac{\eta n_j - \frac{c\eta}{2c+\eta} n_j}{n_j + \frac{\eta}{2c+\eta} n_j} = \frac{\eta}{2}\,\frac{1 - \frac{c}{2c+\eta}}{1 + \frac{\eta}{2c+\eta}} = \frac{\eta}{4}\,.\end{aligned}$$

If $g(x) = o(1)$ is false, then we can assume that $|g(n_j)| > \eta$ is true for an infinity of indices j and take a subsequence $n'_k = n_{j_k}$, $k = 1, 2, \dots$, such that the above inequality holds for every n'_k. We can assume that $n'_{k+1} > n'_k\big(1 + c\eta/(2c+\eta)\big)$. From all this we get a contradiction:

$$\int_1^\infty g^2(x)\frac{\mathrm{d}x}{x} \geq \sum_{k=1}^\infty \int_{n'_k}^{n'_k\left(1+\frac{c\eta}{2c+\eta}\right)} \frac{\eta^2}{16}\frac{\mathrm{d}x}{x} = \sum_{k=1}^\infty \frac{\eta^2}{16}\log\left(1 + \frac{c\eta}{2c+\eta}\right) = +\infty\,.$$

Hence we conclude that $g(x) = o(1)$.

THEOREM 15. *Let N^* be the set of generalized integers as defined in §6. Let $M(x)$ be as defined at the beginning of §7. Then*

$$\lim_{x\to\infty}\frac{M(x)}{x} = 0\,.$$

§8. Asymptotics of $P^*(x)$

In this section we investigate the distribution of the set of generalized primes constructed in §6.

THEOREM 16. *Let*

$$\psi(x) := \sum_{\substack{n_j\leq x\\ n_j\in N^*}} \Lambda(n)\,, \qquad \text{where} \quad \Lambda(n_j) = \begin{cases} \log p_k & \text{if } n_j = p_k^a\,,\ p_k\in P^*\,,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\psi(x) = x + x\sum_{j=1}^{n-1}\frac{2\alpha_j}{1+j^2}\big[\cos(j\log x) + j\sin(j\log x)\big] + O\big(x^{1-\rho}\big)\,.$$

Proof. From Lemma 10 and because $0 < \rho < 1/2$ we have

$$\begin{aligned}
\psi(x) &= \int_1^x \log t \; \mathrm{d}P^*(t) \\
&= \int_1^x \log t \; \mathrm{d}P(t) + \big(P^*(x) - P(x)\big) \log x - \int_1^x \frac{P^*(t) - P(t)}{t} \; \mathrm{d}t \\
&= \int_1^x \log t \; \mathrm{d}P(t) + O\big(x^{1/2} \log x\big) \\
&= \int_1^x \log t \; \mathrm{d}P(t) + O\big(x^{1-\rho}\big) .
\end{aligned}$$

But

$$\begin{aligned}
\int_1^x \log t \; \mathrm{d}P(t) &= \int_1^x (1 - t^{-\rho}) G(\log t) \; \mathrm{d}t \\
&= \int_1^x G(\log t) \; \mathrm{d}t + O\big(x^{1-\rho}\big) \\
&= \int_1^x \Big(1 + 2 \sum_{j=1}^{n-1} \alpha_j \cos(j \log t)\Big) \; \mathrm{d}t + O\big(x^{1-\rho}\big) \\
&= x + 2 \sum_{j=1}^{n-1} \alpha_j \int_1^x \cos(j \log t) \; \mathrm{d}t + O\big(x^{1-\rho}\big) .
\end{aligned}$$

We consider now the above integrals:

$$\begin{aligned}
\int_1^x \cos(j \log t) \; \mathrm{d}t &= x \cos(j \log x) - 1 + j \int_1^x \sin(j \log t) \; \mathrm{d}t \\
&= x \cos(j \log x) - 1 + j x \sin(j \log x) - j^2 \int_1^x \cos(j \log t) \; \mathrm{d}t .
\end{aligned}$$

Hence,

$$\int_1^x \cos(j \log t) \; \mathrm{d}t = \frac{x}{1 + j^2} \big[\cos(j \log x) + j \sin(j \log x)\big] - \frac{1}{1 + j^2} .$$

Thus,

$$\psi(x) = x + x \sum_{j=1}^{n-1} \frac{2\alpha_j}{1+j^2} \big[\cos(j \log x) + j \sin(j \log x)\big] + O\big(x^{1-\rho}\big) .$$

□

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