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# REGULARITY AND EXTENSION OF MAPS

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ABSTRACT. We investigate the notion of regularity for convergence spaces and $\mathcal{L}$-spaces. We show that regularity can be characterized in terms of certain extensions of continuous maps. This generalizes a characterization of regularity for topological spaces.

## Introduction

There are some topological properties which can be conveniently characterized by asserting the existence of extensions of certain types of functions. The Tietze Extension Theorem gives such a characterization of normality. A topological space is completely regular if and only if, for each subspace $B = A \cup \{x\}$, where $A$ is closed and $x \in X - A$, the function $f \colon B \to \mathbb{R}$ defined by $f(A) = 0$ and $f(x) = 1$ has a continuous extension $g \colon X \to \mathbb{R}$.

In this context, apparently less well-known is the role of regularity. A characterization of regular topological spaces in terms of extensions and maps is stated in [BOUR] and extended to pretopological spaces in [CECH]. In order to state this theorem concisely, we introduce some simplifying terminology. If $X_0$ is a subspace of $X$, $f \colon X_0 \to Y$ is continuous and there is a family $\mathcal{S} = \{f_x : x \in X - X_0\}$ such that, for each $x \in X - X_0$, $f_x \colon X_0 \cup \{x\} \to Y$ is a continuous extension of $f$, then we say that $f$ is *pointwise extendable to* $X$ *relative to* $\mathcal{S}$. Assuming that $f \colon X_0 \to Y$ is pointwise extendable to $X$ relative to $\mathcal{S}$, let $f_{\mathcal{S}} \colon X \to Y$ be the extension of $f$ such that $f_{\mathcal{S}}(x) = f_x(x)$, for all $x \in X - X_0$. If $f_{\mathcal{S}} \colon X \to Y$ is continuous, we say that $f$ is *fully extendable to* $X$ *relative to* $\mathcal{S}$.

**THEOREM 0.1.** [CECH] *A pretopological space $Y$ is a regular if and only if whenever $X_0$ is a dense subspace of a topological space $X$ and a continuous*

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*function* $f\colon X_0 \to Y$ *is pointwise extendable to* $X$ *relative to* $S = \{f_x : x \in X - X_0\}$, *then* $f$ *is fully extendable to* $X$ *relative to* $\mathcal{S}$.

We shall refer to the preceding theorem (and modified versions of it appropriate to other types of topological structures) as a "regularity extension theorem".

In [FRIC], it is proved that in the realm of sequential topological spaces the regularity of $Y$ can be replaced by a weaker notion of $c$-regularity. In [CECH], it is pointed out that this theorem is not valid if $X$ is a pretopological space. However, in Section 1 we prove the regularity extension theorem for pretopological spaces (indeed, more generally, for convergence spaces) by adding the assumption that $X_0$ is a *strict* subspace of $X$. An alternate version of the regularity extension theorem for $T_1$ convergence spaces is also established. In Section 2, we use the regularity extension theorem as the definition of "regularity" for $\mathcal{L}$-spaces, and then prove a version of this theorem which yields a simpler characterization for regular $\mathcal{L}$-spaces.

We present a simple universal example showing that the assumption of $X_0$ being a strict subspace of $X$ cannot be avoided (see Example 2.3).

## 1. Regular convergence spaces

Let $X$ be a set, $\mathbb{F}(X)$ the set of all (proper) filters on $X$. For each $x \in X$, $\dot{x}$ denotes the ultrafilter generated by $\{x\}$.

A *convergence structure* $q$ on $X$ is a relation between $\mathbb{F}(X)$ and $X$ such that:

(1) $(\dot{x}, x) \in q$, for all $x \in X$;
(2) $(\mathcal{F}, x) \in q$ and $\mathcal{F} \subseteq \mathcal{G}$ implies $(\mathcal{G}, x) \in q$;
(3) $(\mathcal{F}, x) \in q$ implies $(\mathcal{F} \cap \dot{x}, x) \in q$.

A *convergence space* $(X, q)$ is a set $X$ equipped with a convergence structure $q$. If $(\mathcal{F}, x) \in q$, we say that $\mathcal{F}$ *$q$-converges to* $x$. For $A \subset X$, $c\ell_q A = \{x \in X\colon$ there is $\mathcal{F} \in \mathbb{F}(X)$ such that $A \in \mathcal{F}$ and $(\mathcal{F}, x) \in q\}$ is called the *$q$-closure of* $A$. If $\mathcal{F} \in \mathbb{F}(X)$, $c\ell_q\mathcal{F}$ is the filter on $X$ generated by $\{c\ell_q F\colon F \in \mathcal{F}\}$. A convergence space $(X, q)$ is *regular* if $c\ell_q\mathcal{F}$ $q$-converges to $x$ whenever $\mathcal{F}$ $q$-converges to $x$. A convergence space is *pretopological* (and $q$ is a *pretopology*) if, for each $x \in X$, there is a filter $\mathcal{V}_q(x)$ (called the $q$-neighbourhood filter at $x$) such that $\mathcal{F}$ $q$-converges to $x$ if and only if $\mathcal{V}_q(x) \subseteq \mathcal{F}$. It is well known, that each closure operator on $X$ (in the sense of Č e c h [CECH]) defines a unique pretopology on $X$ and vice-versa; thus the pretopological spaces and essentially the closure spaces of Čech.

The notion of " strict compactification" of a convergence space was introduced in [KERI]. More generally, a subspace $(X_0, q_0)$ of a convergence space $(X, q)$ is defined to be a *strict subspace* of $(X, q)$ if $(X_0, q_0)$ is dense in $(X, q)$ (i.e.,

$c\ell_q X_0 = X$ ) and, if $\mathcal{F}$ $q$-converges to $x$, there is $\mathcal{G} \in \mathbb{F}(X)$ such that $X_0 \in \mathcal{G}$, $c\ell_q\mathcal{G} \subseteq \mathcal{F}$, and $\mathcal{G}$ $q$-converges to $x$. If $(X, q)$ is a topological space, then every dense subspace is strict, but dense subspaces of pretopological spaces need not be strict. Strict extensions for convergence spaces and groups are considered in [FKČG] and [FKNC], and for $\mathcal{L}$-spaces and $\mathcal{L}$-groups in [FRZA].

**THEOREM 1.1.** *A convergence space $(Y, p)$ is regular if and only if, whenever $(X_0, q_0)$ is a strict subspace of a convergence space $(X, q)$ and a continuous function $f\colon (X_0, q_0) \to (Y, p)$ is pointwise extendable to $(X, q)$ relative to $\mathcal{S} = \{f_x\colon x \in X - X_0\}$, then $f$ is fully extendable to $(X, q)$ relative to $\mathcal{S}$.*

Proof. Assume that $(Y, p)$ is regular, and let $\mathcal{F}$ $q$-converge to $x$ in $(X, q)$. Assume that $f\colon (X_0, q_0) \to (Y, p)$ is continuous and pointwise extendable to $(X, q)$ relative to $\mathcal{S}$. Let $g = f_{\mathcal{S}}$. If $A \subseteq X_0$, one can verify that $g(c\ell_q A) \subseteq c\ell_p f(A)$. Since $(X_0, q_0)$ is a strict subspace, there is $\mathcal{G}$ $q$-converging to $x$ such that $X_0 \in \mathcal{G}$ and $c\ell_q\mathcal{G} \subset \mathcal{F}$. Note that $c\ell_p f(\mathcal{G}) \subseteq g(c\ell_q\mathcal{G}) \subseteq g(\mathcal{F})$. If $x \in X_0$, then $\mathcal{G}$ $p$-converges to $x$, and $f(\mathcal{G})$ $p$-converges to $f(x) = g(x)$ by continuity of $f$. Also $c\ell_p f(\mathcal{G})$ $p$-converges to $g(x)$ by regularity of $(Y, p)$, and so $g(\mathcal{F})$ $p$-converges to $g(x)$. If $x \in X - X_0$, and $\mathcal{G}'$ denotes the filter on $X_0 \cup \{x\}$ generated by $\mathcal{G}$, then $\mathcal{G}'$ $q_x$-converges to $x$, where $g_x$ is the restriction of $q$ to $X_0 \cup \{x\}$. Thus $f(\mathcal{G}) = f_x(\mathcal{G}')$ $p$-converges to $f_x(x) = g(x)$ by continuity of $f_x$. Therefore, $c\ell_p f(\mathcal{G})$ $p$-converges to $g(x)$, which implies $g(\mathcal{F})$ $p$-converges to $g(x)$. Thus $g\colon (X, q) \to (Y, p)$ is continuous.

Conversely, assume $(Y, p)$ is not regular. Then there is $\mathcal{F} \in \mathbb{F}(Y)$ and $y \in Y$ such that $\mathcal{F}$ $p$-converges to $y$, but $c\ell_p\mathcal{F}$ does not $p$-converge to $y$. We shall construct a convergence space $(X, q)$ with strict subspace $(X_0, q_0)$ such that, for each $x \in X - X_0$, there is a continuous extension $f_x\colon X_0 \cup \{x\} \to Y$, but $f$ is not fully extendable to $(X, q)$ via this set of pointwise extensions.

Let $A = \left(\bigcap c\ell_p\mathcal{F}\right) - \left[\left(\bigcap \mathcal{F}\right) \cup \{y\}\right] = \{a_\alpha\colon \alpha \in I\}$. For each $\alpha \in I$, let $A_\alpha = \{a_{\alpha i}\colon i \in \mathbb{N}\}$ be a countable infinite set such that $A_\alpha \cap A_\beta = \emptyset$, for $\alpha \neq \beta$ in $I$, and $A_\alpha \cap Y = \emptyset$, for all $\alpha \in I$. Let $X_2 = \bigcup\{A_\alpha\colon \alpha \in I\}$.

Next, let $X' = \{x \in Y\colon x \text{ is not an isolated point in } (Y, p)\} - \left(A \cap \{y\}\right)$. For each $x \in X'$, let $\varphi(x)$ be a point such that $\varphi(x) \notin Y \cup X_2$, and, for $x, z \in X'$, $\varphi(x) = \varphi(z)$ if and only if $x = z$. Let $X_1 = \{\varphi(x)\ x \in X'\}$; then $X_1$ is a copy of $X'$ which is disjoint from $Y \cup X_2$.

Let $X = X_0 \cup X_1 \cup X_2$, where $X_0 = Y$. Note that $X_1$ or $X_2$ (but not both) could be empty. Let $\mathcal{F}'$ be the filter on $X$ generated by $\mathcal{F}$. Also, observe that $c\ell_p\mathcal{F}$ necessarily has a trace on $X' \cup A$. If $c\ell_p\mathcal{F}$ has a trace on $X'$, let $\mathcal{H}$ be this trace, and let $\mathcal{H}'$ be the filter on $X$ generated by $\varphi(\mathcal{H})$. If $A \neq \emptyset$, let $\mathcal{K}'$ be the filter on $X$ generated by $\{X_2 - F\colon F \text{ a finite subset of } X_2\}$. If $c\ell_p\mathcal{F}$ has a trace on $X'$ and $A \neq \emptyset$, we define $\mathcal{G} = \mathcal{F}' \cap \mathcal{H}' \cap \mathcal{K}'$. If $c\ell_p\mathcal{F}$ does not have a trace on $X'$, then $A \neq \emptyset$, and in this case we define $\mathcal{G} = \mathcal{F}' \cap \mathcal{K}'$. Finally, if $A = \emptyset$, then $c\ell_p\mathcal{F}$ must have a trace on $X'$, and in this case we let $\mathcal{G} = \mathcal{F}' \cap \mathcal{K}'$.

The convergence structure $q$ on $X$ is defined to be the finest convergence structure subject to the following conditions:

(1) $\mathcal{G} \cap \dot{y}$ $q$-converges to $y$;
(2) If $\mathcal{L}$ is a filter on $X$ such that $X_0 \in \mathcal{L}$ and $\mathcal{L}|_Y$ $p$-converges to $x \in X'$, then $\mathcal{L} \cap \varphi(\dot{x})$ $q$-converges to $\varphi(x)$ in $X_1$;
(3) If $\mathcal{L}$ is a filter on $X$ such that $X_0 \in \mathcal{L}$ and $\mathcal{L}|_Y$ $p$-converges to $a_\alpha \in A$, then $\mathcal{L} \cap \dot{a}_{\alpha i}$ $q$-converges to $a_{\alpha i}$, for all $i \in \mathbb{N}$.

It is obvious from this construction that the subspace $(X_0, q_0)$ is dense in $(X, q)$, and one can easily verify that $c\ell_p\mathcal{F}' \subseteq \mathcal{G}$, which implies that $(X_0, q_0)$ is a strict subspace of $(X, q)$. Let $f\colon (X_0, q_0) \to (Y, p)$ be the identity map; it is obvious from our construction that $f$ is a continuous. If $x \in X_1$ we define $f_x(x) = z$, where $z$ is the unique element of $X'$ such that $\varphi(z) = x$; if $x \in X_2$, then $x = a_{\alpha i}$ for some $\alpha \in I$ and $i \in \mathbb{N}$ and we define $\varphi(x) = a_\alpha$. It follows easily that $f_x$ is a continuous extension of $f$ for all $x \in X - X_0$. Let $g$ be the extension of $f$ to $X$ which coincides with $f_x$ for each $x \in X - X_0$; $g$ is not continuous because $\mathcal{G}$ $q$-converges to $y$, but $g(\mathcal{G}) = c\ell_p\mathcal{F}$ does not $p$-converge to $g(y) = y$. Then, $f$ is not extendable to $(X, q)$ via the set $\{f_x\colon\ x \in X - X_0\}$ of pointwise extensions, and the proof is complete. □

Another version of the regularity extension theorem for $T_1$ convergence spaces can be stated as follows.

**THEOREM 1.2.** *Let $(Y, p)$ be a $T_1$ convergence space. Then $(Y, p)$ is regular if and only if, whenever $(X_0, q_0)$ is a strict subspace of a convergence space $(X, q)$, and a continuous function $f\colon (X_0, q_0) \to (Y, p)$ is pointwise extendable to $(X, q)$, then $f$ has a continuous extension $h\colon (X, q) \to (Y, p)$.*

Proof. If $(Y, p)$ is regular, $h$ can be taken to be the function $g$ of the preceding proof. Conversely, if $(Y, p)$ is not regular, the converse part of the preceding proof can be repeated, and one can use the $T_1$ property of $(Y, p)$ to show that any continuous extension of $f$ to $(X, q)$ must necessarily coincide with $f_x$ for each $x \in X - X_0$. Thus the function $g$ of the preceding proof is the only possible choice for $h$ and since $g$ is discontinuous, no continuous extension of $f$ to $(X, q)$ can exist. □

If the space $(Y, p)$ of Theorem 1.1 is pretopological, then the space $(X, q)$ constructed in the proof of this theorem is also pretopological. Thus Theorems 1.1 and 1.2 remain valid if "convergence space" is everywhere replaced by "pretopological space". If *limit space* and *pseudotopological space* are defined as in

[PREU], Theorems 1.1. and 1.2 are also valid when "convergence space" is replaced by "limit space" or " pseudotopological space", with no alteration required in the proof of either theorem. One can also show that these theorems hold when "convergence space" is everywhere replaced by "Cauchy space" and "continuous function" by "Cauchy continuous function"; in this case, of course, the proofs must be altered. These results suggest that the regularity extension theorem could serve as a suitable definition for regularity in a wide variety of topological categories, an idea which we pursue further in the next section.

## 2. Regular $\mathcal{L}$-spaces

In this section, we use the regularity extension theorem to extend the notion of regularity to $\mathcal{L}$-spaces. We then give an internal characterization of regular $\mathcal{L}$-spaces; a deeper study of such spaces and their relationship to regular filter convergence spaces will be developed in another paper [FRKE].

Let $X$ be a set, $\mathbb{N}$ the set of natural numbers. Then $X^{\mathbb{N}}$ denotes the set of all sequences on $X$. If $S \in X^{\mathbb{N}}$, let $\mathcal{F}(S)$ be the filter generated by $S$. For a "double sequence" $\zeta = \langle S_n \rangle \in (X^{\mathbb{N}})^{\mathbb{N}}$ on $X$, let $\mathcal{F}(\zeta)$ be the filter with base consisting of sets in the form $\left\{ \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \{S_k(\ell)\} \right\}$, for $n \in \mathbb{N}$. Furthermore, if $f \in \mathbb{N}^{\mathbb{N}}$, let $\zeta_f = \langle S_n(f(n)) \rangle$ be the "diagonal" of $\zeta$ determined by $f$. Let MON denote the set of all monotone functions in $\mathbb{N}^{\mathbb{N}}$; if $S \in X^{\mathbb{N}}$, then all subsequences of $S$ are of the form $S \circ s$, for some $s \in$ MON.

An $\mathcal{L}$-structure $\mathbb{L}$ on $X$ is a relation between $X^{\mathbb{N}}$ and $X$ such that:

(1) $(S, x) \in \mathbb{L}$ whenever $x \in X$ and $S$ is the constant sequence $S(n) = x$, for all $n \in \mathbb{N}$;

(2) If $(S, x) \in \mathbb{L}$, then $(S \circ s, x) \in \mathbb{L}$, for each $s \in$ MON. If $\mathbb{L}$ is an $\mathcal{L}$-structure on $X$, then $(X, \mathbb{L})$ is called an $\mathcal{L}$-space. If $(S, x) \in \mathbb{L}$, we say that "$S$ $\mathbb{L}$-converges to $x$". $(X, \mathbb{L})$ is called an $\mathcal{L}_0$-space if each sequence $\mathbb{L}$-converges to at most one point; it is an $\mathcal{L}^*$-space if the Urysohn axiom is satisfied:

(*) $(S, x) \in \mathbb{L}$ if, for each $s \in$ MON, there is a $t \in$ MON such that $(S \circ s \circ t, x) \in \mathbb{L}$.

Let $(X, \mathbb{L})$ be an $\mathcal{L}$-space. Let $S \in X^{\mathbb{N}}$, and $x \in X$. Then $S$ and $x$ are said to be *linked* if there is a double sequence $\zeta = \langle S_n \rangle \in (X^{\mathbb{N}})^{\mathbb{N}}$ such that, for each $k \in \mathbb{N}$, the sequence $S_k$ $\mathbb{L}$-converges to $S(k)$, and, if $T \in X^{\mathbb{N}}$ has the property $\mathcal{F}(T) \supseteq \mathcal{F}(\zeta)$, then $T$ $\mathbb{L}$-converges to $x$. In this case, we say that *$\zeta$ links $S$ and $x$*.

If $(X, d)$ is a pseudo-metric space and $\mathbb{M}$ denotes the usual convergence of sequences in $(X, d)$, then one can easily verify that $S$ and $x$ are linked if and only if $S$ $\mathbb{M}$-converges to $x$. We omit the straightforward proof of the next proposition.

**PROPOSITION 2.1.** *Let* $(X, \mathbb{L})$ *be an* $\mathcal{L}^*$*-space. Let* $\zeta = \langle S_n \rangle \in (X^{\mathbb{N}})^{\mathbb{N}}$, $S \in X^{\mathbb{N}}$, *and* $x \in X$. *Then* $\zeta$ *links* $S$ *and* $x$ *if and only if, for each* $k \in \mathbb{N}$, *the sequence* $S_k$ $\mathbb{L}$*-converges to* $S(k)$ *and, for each* $f \in \mathbb{N}^{\mathbb{N}}$, *the sequence* $\zeta_f$ $\mathbb{L}$*-converges to* $x$.

Let $(X, \mathbb{L})$ be an $\mathcal{L}$-space, and let $X_0$ be a dense subset of $X$ (i.e., $c\ell_{\mathbb{L}}(X_0) = X$). Then the $\mathcal{L}$-subspace $(X_0, \mathbb{L}_0)$ of $(X, \mathbb{L})$ is called a *strict subspace* if, for each $S \in X^{\mathbb{N}}$, and $x \in X$ such that $S$ $\mathbb{L}$-converges to $x$, there is $\zeta \in (X_0^{\mathbb{N}})^{\mathbb{N}}$ which links $S$ and $x$.

In the completion theory for $\mathcal{L}^*$-spaces, we usually begin by constructing an $\mathcal{L}$-convergence $\mathbb{L}$ and then pass to its Urysohn modification $\mathbb{L}^*$. To control an $\mathbb{L}^*$-convergent sequence, it suffices to control its $\mathbb{L}$-converging subsequences. Correspondingly, in [FRZA], the notion of a strict extension (precompletion, completion) is in terms of subsequences. However, for general $\mathcal{L}$-structures we have to adopt a stronger notion of strictness. Indeed, Proposition 3.4 in [FRZA] states that the Novak $\mathcal{L}_0^*$-completion $\nu Q$ of $Q$ is strict in our sense. Since the real line is a metric space, each of the $2^c$ $\mathcal{L}_0^*$-group completions of $Q$ constructed in [FRZA] are strict in our sense as well.

Recall that a function $f\colon (X, \mathbb{L}) \to (Y, \mathbb{M})$ between two $\mathcal{L}$-spaces is *continuous* if $f(S)$ $\mathbb{M}$-converges to $f(x)$ whenever $S$ $\mathbb{L}$-converges to $x$. The terminology in the paragraph preceding Theorem 0.1 may be applied to extension of functions between $\mathcal{L}$-spaces in the obvious way.

An $\mathcal{L}$-space $(Y, \mathbb{M})$ is defined to be *regular* if, whenever $(X_0, \mathbb{L}_0)$ is a strict subspace of $(X, \mathbb{L})$ and a continuous function $f\colon (X_0, \mathbb{L}) \to (Y, \mathbb{M})$ is pointwise extendable to $(X, \mathbb{L})$ relative to $\mathcal{S} = \{f_x : x \in X - X_0\}$, then $f$ is fully extendable to $(X, \mathbb{L})$ relative to $\mathcal{S}$.

It turns out that there is a simple characterization of regular $\mathcal{L}$-spaces which we state in the next theorem.

**THEOREM 2.2.** *An* $\mathcal{L}$*-space* $(Y, \mathbb{M})$ *is regular if and only if* $S$ $\mathbb{M}$*-converges to* $x$ *whenever* $S \in X^{\mathbb{N}}$, $x \in X$, *and* $S$ *and* $x$ *are linked.*

Proof. Assume that $(Y, \mathbb{M})$ is an $\mathcal{L}$-space with the property that $S$ $\mathbb{M}$-converges to $x$ whenever $S$ $x$ are linked. Let $(X, \mathbb{L})$ be an $\mathcal{L}$-space, let $(X_0, \mathbb{L}_0)$ be a strict subspace of $(X, \mathbb{L})$, and let $f\colon (X_0, \mathbb{L}) \to (Y, \mathbb{M})$ be a continuous map which is pointwise extendable to $X$ via a set $\{f_z : x \in X - X_0\}$ of pointwise extensions. Assume that $S$ $\mathbb{L}$-converges to $x$. Since $X_0$ is a strict subspace of $X$, there is a double sequence $\zeta = \langle S_n \rangle \in (X_0^{\mathbb{N}})^{\mathbb{N}}$ which links $S$ and $x$. Let $g\colon X \to Y$ be the extension of $f$ to $X$ defined by $g(x) = f_x(x)$ for all $x \in X - X_0$. For each $k \in \mathbb{N}$, let $T_k = \langle f(S_k(n)) \rangle$ and $\xi = \langle T_n \rangle$. Clearly $\xi$ links $g(S)$ and $g(x)$. Since $Y$ is regular, the sequence $g(S)$ $\mathbb{M}$-converges to $g(x)$, establishing continuity of $g$. Thus $(Y, \mathbb{M})$ is regular.

Conversely, assume that the specified condition fails. Thus there is a sequence $\langle x \rangle_n \in Y^{\mathbb{N}}$, a point $x \in Y$, and a double sequence $\zeta = \langle S_n \rangle \in (Y^{\mathbb{N}})^{\mathbb{N}}$ such that $\zeta$ links $\langle x_n \rangle$ and $x$, but $\langle x_n \rangle$ does not $\mathbb{M}$-convergence to $x$.

Let $X_0 = Y$, and let $X_1 = \{y_1, y_2, \dots\}$ be an infinite set disjoint from $Y$. Let $X = X_0 \cup X_1$, and let $\mathbb{L}$ be the finest $\mathcal{L}$-structure on $X$ such that:

(1) $(S_k, y_k) \in \mathbb{L}$, for all $k \in \mathbb{N}$;

(2) $(\langle y_n \rangle, x) \in \mathbb{L}$;

(3) $(S, x) \in \mathbb{L}$ for each $S \in X^{\mathbb{N}}$ such that $\mathcal{F}(\zeta) \subseteq \mathcal{F}(S)$.

Let $(X_0, \mathbb{L}_0)$ be the subspace of $(X, \mathbb{L})$ determined by $X_0$, and let $f\colon (X_0, \mathbb{L}_0) \to (Y, \mathbb{M})$ be the identity map, which is clearly continuous. Let $g\colon (X, \mathbb{L}) \to (Y, \mathbb{M})$ be the extension of $f$ defined by $g(y_k) = x_k$, for all $y_k \in X$. Each function $f_x$ obtained by restricting $g$ to $X_0 \cup \{x\}$ is continuous, for all $x \in X - X_0$. But $g\colon (X, \mathbb{L}) \to (Y, \mathbb{M})$ is not continuous, since $\langle y_n \rangle$ $\mathbb{L}$-converges to $x$, but $\langle g(y_n) \rangle = \langle x_n \rangle$ does not $\mathbb{L}$-converge to $g(x) = x$. Thus $(Y, \mathbb{M})$ is not regular and the proof is complete. □

We present a simple example showing that the assumption of $X_0$ being a strict subspace of $X$ cannot be avoided (see a remark preceding Theorem 1.1).

Example 2.3. Consider a countable infinite set $X$ arranged into a double sequence $\langle S_n \rangle$, a sequence $S$ and a point $x$. Denote by $\mathbb{L}$ the finest $\mathcal{L}^*$-structure of $X$ such that $S$ converges to $x$ and each $S_k$ converges to $S(k)$, $k \in \mathbb{N}$. Clearly, $\mathbb{L}$ has unique sequential limits. Put $X_0 = X - \{S(k);\ k \in \mathbb{N}\}$ and $\mathbb{L}_0 = \mathbb{L}|X_0$. It is easy to see that $(X_0, \mathbb{L}_0)$ is a dense subspace of $(X, \mathbb{L})$, but it fails to be strict. Let $Y = X$. Denote by $\mathbb{M}$ the finest $\mathcal{L}_0^*$-structure on $Y$ such that each $S_k$ converges to $S(k)$, $k \in \mathbb{N}$. It follows from the construction that $(Y, \mathbb{M})$ is regular, and the identity map $\mathrm{id}_0$ of $X_0$ into $Y$ is continuous and can be continuously extended to each subspace $X_0 \cup \{S(k)\}$, $k \in \mathbb{N}$, but not to $X$. Consider the associated pretopological spaces $pX_0$, $pX$, $pY$. Then $pY$ is regular and $pX_0$ fails to be a strict subspace of $pX$. Also in this case $\mathrm{id}_0$ can be continuously extended over $X_0 \cup \{S(k)\}$, $k \in \mathbb{N}$, but not over $X$.

In [POCH], the following condition has been considered for an $\mathcal{L}$-space $(X, \mathbb{L})$:

(P) Let $S \in X^{\mathbb{N}}$, $x \in X$. If there is a double sequence $\zeta = \langle S_n \rangle \in X^{\mathbb{N}}$ such that, for each $k \in \mathbb{N}$, the sequence $S_k$ $\mathbb{L}$-converges to $S(k)$ and, for each $s, t \in \mathrm{MON}$, the sequence $\langle S_{s(n)}(t(s(n))) \rangle$ $\mathbb{L}$-converges to $x$, then $S$ $\mathbb{L}$-converges to $x$.

It follows from Proposition 2.1 and Theorem 2.2 that an $\mathcal{L}^*$-space $(X, \mathbb{L})$ is regular if and only if it satisfies condition (P).

In the case of $\mathcal{L}_0^*$-spaces, regularity can be characterized as follows: An $\mathcal{L}_0^*$-space $(Y, \mathbb{M})$ is regular if, whenever $(X_0, \mathbb{L}_0)$ is a strict subspace of $(X, \mathbb{L})$ and a continuous function $f: (X_0, \mathbb{L}_0) \to (Y, \mathbb{M})$ is pointwise extendable to $(X, \mathbb{L})$, then $f$ has a continuous extension $h: (X, \mathbb{L}) \to (Y, \mathbb{M})$. It would be of interest to describe the largest class of $\mathcal{L}$-spaces for which this characterization of regularity is valid.

It should be noted that our definition of regularity for $\mathcal{L}$-spaces depends on the definition of "strict subspace", which in turn depends on the definition of "linkage" between a sequence and point. By modifying the requirements imposed on the double sequence $\zeta$ in defining the statement "$\zeta$ links $S$ to $x$", one thereby alters the definition of regularity, but simultaneously one changes the condition in Theorem 2.1 characterizing regularity, so that Theorem 2.1 remains valid. We have chosen to define linkage between sequences and points so that the resulting definition of $\mathcal{L}$-space regularity translates to the usual (filter) convergence space definition of regularity under the modification functor $\gamma$ of Beattie and Butzmann (see [BEBU] and [FRKE]). Further studies involving regularity in the setting of $\mathcal{L}$-spaces may indicate that an alternate notion of linkage is more appropriate. Various diagonal conditions which might be considered to define a linkage can be found in [FRVO] and [NOBE].

Finally, it should be mentioned that we defined "$X_0$ is a strict subspace of $X$" with the requirement that $X$ be the (first) closure of $X_0$ in $X$. In applying regularity in the study of extensions and completions (both for convergence spaces and $L$-spaces), it may be desirable to broaden the definition of "strict" to include the case where $X$ is the topological (or iterated) closure of $X_0$. Such situations appear, e.g., in connection with extension of measures ([KRAT], [NOME]). The appropriate formulation of the regularity extension theorem under these circumstances will be left as a problem for further investigation.

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