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FORCED OSCILLATIONS OF A CLASS OF NONLINEAR DELAY HYPERBOLIC EQUATIONS

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ABSTRACT. In this paper, we discuss a class of nonlinear hyperbolic equations with deviating arguments, and obtain sufficient conditions for the oscillation of such equations subject to two kinds of boundary value problems.

1. Introduction

Owing to delay partial differential equations arise frequently in many fields of natural science, the oscillation theory of such equations is of growing interest. For the oscillation of hyperbolic equations, we have a few results. For example, we can refer to the contributions by D. Gorgiou & K. Kreith [1], D. Mishev [2], Yoshida [3], B. S. Lalli, Y. H. Yu & B. T. Cui [4], [5], [6], Y. K. Li [7] and the references cited therein. But the corresponding theory is as yet not well developed. In this paper, we consider the forced oscillation of nonlinear partial differential equations of the form

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) = a(t)\Delta u(x,t) &+ \sum_{i=1}^{m} a_i(t)\Delta u\big(x,\rho_i(t)\big)\\ &- \sum_{j=1}^{n} p_j(x,t) f_j\big(u(x,\sigma_j(t))\big) + F(x,t) \qquad \text{(E)}\\ (x,t) \in \Omega\times\big[0,+\infty\big) &= G\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$. $\mathbb{R}_+ = [0,+\infty)$, Δu is the Laplacian in $\mathbb{R}^n$.

Suppose that the following conditions (H) hold.

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(H$_1$) $a, a_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, \ldots, m$.

(H$_2$) $\sigma_j, \rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing in t respectively; $\sigma_j \le t$, $\rho_i \le t$ and
$$\lim_{t\to+\infty} \rho_i(t) = \lim_{t\to+\infty} \sigma_j(t) = +\infty; \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$$

(H$_3$) $p_j \in C(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}_+)$; $F \in C(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}_+)$; $j = 1, 2, \ldots, n$.

(H$_4$) $f_i \in C(\mathbb{R}, \mathbb{R})$ and $f_i(u)$ is convex in $\mathbb{R}_+$; $uf_i(u) > 0$, $u \ne 0$, $i = 1, 2, \ldots, m$.

We consider two kinds of boundary value conditions:

(B$_1$) $u = \varphi(x, t)$ on $(x, t) \in \partial\Omega \times \mathbb{R}_+$.

(B$_2$) $\frac{\partial u}{\partial n} + \mu(x, t)u = 0$ on $(x, t) \in \partial\Omega \times \mathbb{R}_+$.

In which n is the unit exterior normal vector to $\partial\Omega$, μ is a nonegative continuous function on $\partial\Omega \times \mathbb{R}_+$.

The objective of this paper is to study the oscillatory properties of solutions of equation (E) subject to boundary conditions (B$_1$) and (B$_2$) respectively. The results generalize and improve the some results in [4], [5] and [6].

A solution $u(x, t)$ of equation (E) satisfying certain boundary conditions is called *oscillatory* in the domain G if for each positive number τ there exists a point $(x_0, t_0) \in G$, $t_0 \ge \tau$ such that $u(x_0, t_0) = 0$.

2. Oscillation criteria for problem (E), (B$_1$)

We consider the following problem
$$\begin{aligned} \Delta u + \alpha u &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u &= 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \end{aligned}$$
where α is a constant. It is well known ([7]) that the smallest eigenvalue α_0 is positive and the corresponding eigenfunction $\Phi(x)$ is also positive.

Lemma 2.1. ([4]) *Suppose that* $y \in C^2([t_0, +\infty), \mathbb{R})$, *and*
$$y(t) > 0, \quad y'(t) > 0 \quad \text{and} \quad y''(t) \le 0, \qquad t \ge t_0 > 0. \tag{2.1}$$
Then for any $\lambda_0 \in (0, 1)$, *there exists a number* $t_1 > t_0$ *such that*
$$y(t) \ge \lambda_0 t y'(t) \qquad \text{for} \quad t \ge t_1. \tag{2.2}$$

We define the function $p_j(t)$ by $p_j(t) = \min_{x\in\Omega}\{p_j(x, t)\}$, $j = 1, 2, \ldots, n$.

For a solution $u(x,t)$ of equation (E) satisfying boundary condition (B_1), we define

$$U(t) = \frac{\int\limits_\Omega u(x,t)\Phi(x)\,\mathrm{d}x}{\int\limits_\Omega \Phi(x)\,\mathrm{d}x}\,. \tag{2.3}$$

LEMMA 2.2. *Suppose that conditions* (H) *hold, and*

(A_1) *There exists a positive constant* ε *such that*

$$f_i(u) \ge \varepsilon u\,. \tag{2.4}$$

(A_2) *There exists an oscillatory function* $H \in C^2(\mathbb{R},\mathbb{R})$ *with* $\lim\limits_{t\to+\infty} H(t) = 0$, *and*

$$H''(t) = \left(\int\limits_\Omega \Phi(x)\,\mathrm{d}x\right)^{-1}\left\{\int\limits_\Omega F(x,t)\Phi(x)\,\mathrm{d}x - \int\limits_{\partial\Omega}\left[a(t)\varphi(x,t) + \sum_{i=1}^m a_i(t)\varphi\big(x,\rho_i(t)\big)\right]\frac{\partial\Phi}{\partial n}\,\mathrm{d}\omega\right\}. \tag{2.5}$$

If $u(x,t)$ *is a positive solution of problem* (E), (B_1) *on* $\Omega\times[t_0,+\infty)$, *then the delay differential inequality*

$$y''(t) + \lambda_0\left\{\alpha_0\left[a(t)y(t) + \sum_{i=1}^m a_i(t)y\big(\rho_i(t)\big)\right] + \varepsilon\sum_{j=1}^n p_j(t)y\big(\sigma_j(t)\big)\right\} \le 0 \tag{2.6}$$

have eventually positive solutions

$$y(t) = U(t) - H(t)\,. \tag{2.7}$$

Proof. Suppose that $u(x,t)$ is a positive solution of problem (E), (B_1) on $\Omega\times[t_0,+\infty)$. In view of (H_2) there is a number $t_1 \ge t_0$ such that

$$u\big(x,\rho_i(t)\big) > 0\,, \qquad u\big(x,\sigma_j(t)\big) > 0\,.$$

Multiplying both sides of (E) by the $\Phi(x)$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int\limits_\Omega u(x,t)\Phi(x)\,\mathrm{d}x\right]\\ &= a(t)\int\limits_\Omega \Delta u(x,t)\Phi(x)\,\mathrm{d}x + \sum_{i=1}^m a_i(t)\int\limits_\Omega \Delta u\big(x,\rho_i(t)\big)\Phi(x)\,\mathrm{d}x\\ &\quad - \sum_{j=1}^n \int\limits_\Omega p_j(x,t)f_j\Big(u\big(x,\sigma_j(t)\big)\Big)\Phi(x)\,\mathrm{d}x + \int\limits_\Omega F(x,t)\Phi(x)\,\mathrm{d}x\,, \qquad t \ge t_1\,.\end{aligned} \tag{2.8}$$

Using Green's theorem, we have

$$\int_{\Omega} \Delta u(x,t)\Phi(x)\ \mathrm{d}x = -\int_{\partial\Omega} \varphi(x,t)\frac{\partial\Phi}{\partial n}\ \mathrm{d}\omega - \alpha_0 \int_{\Omega} u(x,t)\Phi(x)\ \mathrm{d}x\,, \qquad t \geq t_1\,, \tag{2.9}$$

$$\int_{\Omega} \Delta u\big(x,\rho_i(t)\big)\Phi(x)\ \mathrm{d}x = -\int_{\partial\Omega} \varphi\big(x,\rho_i(t)\big)\frac{\partial\Phi}{\partial n}\ \mathrm{d}\omega - \alpha_0 \int_{\Omega} u\big(x,\rho_i(t)\big)\Phi(x)\ \mathrm{d}x\,, \qquad t \geq t_1\,. \tag{2.10}$$

Noticing the definition of $p_j(t)$, using (H_4) and Jensen's inequality, we have

$$\begin{aligned} &\int_{\Omega} p_j(x,t) f_j\big(u(x,\sigma_j(t))\big)\Phi(x)\ \mathrm{d}x \\ &\geq p_j(t)\int_{\Omega} f_j\big(u(x,\sigma_j(t))\big)\Phi(x)\ \mathrm{d}x \\ &\geq p_j(t) f_j\left(\frac{\int_{\Omega} u\big(x,\sigma_j(t)\big)\Phi(x)\ \mathrm{d}x}{\int_{\Omega}\Phi(x)\ \mathrm{d}x}\right)\int_{\Omega}\Phi(x)\ \mathrm{d}x\,, \qquad t \geq t_1\,. \end{aligned} \tag{2.11}$$

Combing (2.8) – (2.11), yields

$$\begin{aligned} U''(t) \leq &-\alpha_0\left[a(t)U(t) + \sum_{i=1}^{m} a_i(t)U\big(\rho_i(t)\big)\right] - \sum_{j=1}^{n} p_j(t) f_j\big(U(\sigma_j(t))\big) \\ &+ \left(\int_{\Omega}\Phi(x)\ \mathrm{d}x\right)^{-1}\left\{\int_{\Omega} F(x,t)\Phi(x)\ \mathrm{d}x \right. \\ &\left. - \int_{\partial\Omega}\left[a(t)\varphi(x,t) + \sum_{i=1}^{m} a_i(t)\varphi\big(x,\rho_i(t)\big)\right]\frac{\partial\Phi}{\partial n}\ \mathrm{d}\omega\right\}. \end{aligned}$$

Let

$$y(t) = U(t) - H(t)\,. \tag{2.12}$$

Using the condition (A_1) and (A_2), we get

$$y''(t) + \alpha_0\left[a(t)U(t) + \sum_{i=1}^{m} a_i(t)U\big(\rho_i(t)\big)\right] + \varepsilon\sum_{j=1}^{n} p_j(t)U\big(\sigma_j(t)\big) \leq 0\,, \qquad t \geq t_1\,, \tag{2.13}$$

from (2.13), we have $y''(t) < 0$ for $t \geq t_1$, and we can prove that there exists a $t_2 \geq t_1$, such that $y(t) > 0$. In fact, if $y(t) \leq 0$ then $U(t) \leq H(t)$, which is impossible in view of the fact that $U(t) > 0$ and $H(t)$ is oscillatory.

From $y(t) > 0$ and $y''(t) < 0$, we have $y'(t) > 0$, $t \geq t_2$.

Furthermore, since $y(t)$ is an increasing function and $\lim\limits_{t\to+\infty} H(t) = 0$, it follows that there is a number $t_3 \geq t_2$ and $\lambda_0 \in (0,1)$ such that

$$U(t) \geq \lambda_0 y(t)\,, \quad U\big(\rho_i(t)\big) \geq \lambda_0 y\big(\rho_i(t)\big)\,, \quad U\big(\sigma_j(t)\big) \geq \lambda_0 y\big(\sigma_j(t)\big)\,, \qquad t \geq t_1\,. \tag{2.14}$$

Then from (2.14) it follows that the function $U(t)$ defined by (2.3) is a positive solution of the delay inequality (2.6). □

THEOREM 2.1. *Suppose that conditions* (H), (A_1) *and* (A_2) *hold. If the delay inequality* (2.6) *have no eventually positive solutions, then every solution of the problem* (E), (B_1) *is oscillatory on* $\Omega \times \mathbb{R}_+$.

P r o o f . Suppose that there is a nonoscillatory solution $u(x,t)$ of the problem, we may assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0, +\infty)$, by Lemma 2.2, we get

$$y(t) = U(t) - H(t)$$

is a eventually positive solution, which is a contradiction.

If $u(x,t) < 0$ then set $\overline{u}(x,t) = -u(x,t)$, using the condition (H_4), it is easy to check that $\overline{u}(x,t)$ is a positive solution of the problem (E), (B_1), defining

$$\overline{U}(t) = \frac{\int\limits_\Omega \overline{u}(x,t)\Phi(x)\,\mathrm{d}x}{\int\limits_\Omega \Phi(x)\,\mathrm{d}x} \tag{2.15}$$

then by Lemma 2.2

$$\overline{y}(t) = \overline{U}(t) - H(t) \tag{2.16}$$

is a eventually positive solution, which is also a contradiction. This completes the proof of the Theorem 2.1. □

Remark 1. Theorem 2.1 generalize and improve Theorem 2.2 in [4], Theorem 3.1 in [5] and Lemma 2.1 in [6].

3. Oscillation criteria for problem (E), (B_2)

With a solution $u(x,t)$ of problem (E), (B_2), we define

$$V(t) = \frac{1}{|\Omega|}\int\limits_\Omega u(x,t)\,\mathrm{d}x\,, \quad t \geq 0\,, \qquad |\Omega| = \int\limits_\Omega \mathrm{d}x\,.$$

Lemma 3.1. *Suppose that conditions* (H), (A_1) *hold, and*

(A_3) *there exists an oscillatory function* $H \in C^2(\mathbb{R},\mathbb{R})$ *with* $\lim_{t\to+\infty} H(t) = 0$, *and*

$$H''(t) = \frac{1}{|\Omega|}\int_\Omega F(x,t)\,\mathrm{d}x \tag{3.1}$$

holds.

If $u(x,t)$ *is a positive solution of problem* (E), (B_2) *on* $\Omega\times[t_0,+\infty)$, *then the delay differential inequality*

$$y''(t) + \lambda\varepsilon\sum_{j=1}^{n} p_j(t)y\big(\sigma_j(t)\big) \le 0 \tag{3.2}$$

has eventually positive solutions

$$y(t) = V(t) - H(t)\,. \tag{3.3}$$

Proof. Suppose that $u(x,t)$ is a positive solution of problem (E), (B_2) on $\Omega\times[t_0,+\infty)$. In view of (H_2) there is a number $t_1 \ge t_0$ such that

$$u\big(x,\rho_i(t)\big) > 0\,, \qquad u\big(x,\sigma_j(t)\big) > 0\,.$$

Integrating (E) with respect to x over the domain Ω, we have

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_\Omega u(x,t)\,\mathrm{d}x\right]\\ &= a(t)\int_\Omega \Delta u(x,t)\,\mathrm{d}x + \sum_{i=1}^{m} a_i(t)\int_\Omega \Delta u\big(x,\rho_i(t)\big)\,\mathrm{d}x\\ &\quad -\sum_{j=1}^{n}\int_\Omega p_j(x,t)f_j\big(u(x,\sigma_j(t))\big)\,\mathrm{d}x + \int_\Omega F(x,t)\,\mathrm{d}x\,, \qquad t\ge t_1\,.\end{aligned} \tag{3.4}$$

Using the Green's theorem and (B_2), we have

$$\int_\Omega \Delta u(x,t)\,\mathrm{d}x = \int_{\partial\Omega}\frac{\partial u}{\partial n}\,\mathrm{d}\omega = -\int_{\partial\Omega}\mu(x,t)u(x,t)\,\mathrm{d}\omega \le 0\,, \tag{3.5}$$

$$\int_\Omega \Delta u\big(x,\rho_i(t)\big)\,\mathrm{d}x = \int_{\partial\Omega}\frac{\partial u\big(x,\rho_i(t)\big)}{\partial n}\,\mathrm{d}\omega = -\int_{\partial\Omega}\mu\big(x,\rho_i(t)\big)u\big(x,\rho_i(t)\big)\,\mathrm{d}\omega \le 0\,. \tag{3.6}$$

Noticing the definitions of $p_j(t)$ and (H_4), using Jensen's inequality, we have

$$\int_\Omega p_j(x,t)f_j\big(u(x,\sigma_j(t))\big)\,\mathrm{d}x \ge p_j(t)|\Omega|f_j\left(\frac{1}{|\Omega|}\int_\Omega u\big(x,\sigma_j(t)\big)\,\mathrm{d}x\right). \tag{3.7}$$

Combining (3.4) - (3.7), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\int_\Omega u(x,t)\,\mathrm{d}x\right] \le -p_j(t)|\Omega| f_j\left(\frac{1}{|\Omega|}\int_\Omega u(x,\sigma_j(t))\,\mathrm{d}x\right) + \int_\Omega F(x,t)\,\mathrm{d}x\,. \tag{3.8}$$

Let

$$y(t) = V(t) - H(t)\,. \tag{3.9}$$

Using conditions (A_1) and (A_3), we have

$$y''(t) + \lambda\varepsilon \sum_{j=1}^{n} p_j(t) V\big(\sigma_j(t)\big) \le 0\,, \qquad t \ge t_1\,, \tag{3.10}$$

the remainder of the proof is similar to that of Lemma 2.2, we omit it. □

Using Lemma 3.1, we have:

THEOREM 3.1. *Suppose that conditions* (H), (A_1) *and* (A_3) *hold. If the delay inequality* (3.2) *has no eventually positive solutions, then every solution of the problem* (E), (B_2) *is oscillatory on* $\Omega \times \mathbb{R}_+$.

Remark 2. Theorem 3.1 generalizes and improves Theorem 2.1 in [4], and Theorem 2.1 in [5].

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