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RADICALS IN NON-COMMUTATIVE GENERALIZATIONS OF MV-ALGEBRAS

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ABSTRACT. GMV-algebras are a non-commutative generalization of MV-algebras. In the paper, structure properties of GMV-algebras are studied using a duality between GMV-algebras and unital ℓ-groups. Three kinds of radicals of GMV-algebras are investigated in general and, particularly, in the cases of finite valued and normal-valued GMV-algebras. A class of GMV-algebras possessing infinitely many state-morphisms is described.

1. Introduction

MV-algebras were introduced by C. C. Chang in [8] and [9] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic. Non-commutative logics that have been recently studied (see e.g. [1], [23], [24], [25], [30], [32]) correspond to non-commutative reasoning which can be observed in the theoretical computer science (there are even non-commutative programming languages, e.g. [3]) and also in the human reasoning.

GMV-algebras, introduced by G. Georgescu and A. Iorgulescu in [16] and [17] and independently by the author in [27], are a non-commutative generalization of MV-algebras and they can be taken as an algebraic semantics for a non-commutative generalization of a multiple valued reasoning.

DEFINITION. Let $A = (A, \oplus, \neg, \sim, 0, 1)$ be an algebra of type $\langle 2, 1, 1, 0, 0 \rangle$. Set $x \odot y = \sim(\neg x \oplus \neg y)$ for any $x, y \in A$. Then A is called a *generalized MV-algebra* (briefly: *GMV-algebra*) if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = x = 0 \oplus x$;

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(A3) $x \oplus 1 = 1 = 1 \oplus x$;
(A4) $\neg 1 = 0 = \sim 1$;
(A5) $\neg(\sim x \oplus \sim y) = \sim(\neg x \oplus \neg y)$;
(A6) $x \oplus (y \odot \sim x) = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = (\neg x \odot y) \oplus x$;
(A7) $(\neg x \oplus y) \odot x = y \odot (x \oplus \sim y)$;
(A8) $\sim\neg x = x$.

If we put $x \leq y$ if and only if $\neg x \oplus y = 1$, then $\leq$ is an order on A. Moreover, $(A, \leq)$ is a bounded distributive lattice in which $x \vee y = x \oplus (y \odot \sim x)$ and $x \wedge y = x \odot (y \oplus \sim x)$ for each $x, y \in A$, and 0 is the least and 1 is the greatest element in A, respectively.

(The above definition is that introduced by Georgescu and Iorgulescu in [16] and [17], where they use the name a *pseudo-MV algebra*.)

GMV-algebras are in a close connection with unital ℓ-groups. (Recall that a *unital ℓ-group* is a pair (G, u) where G is an ℓ-group and u is a strong order unit of G.) If G is an ℓ-group and $0 \leq u \in G$, then $\Gamma(G, u) = ([0, u], \oplus, \neg, \sim, 0, 1)$, where $[0, u] = \{x \in G : 0 \leq x \leq u\}$, and for any $x, y \in [0, u]$, $x \oplus y = (x + y) \wedge u$, $\neg x = u - x$, $\sim x = -x + u$, is a GMV-algebra. Conversely, A. Dvurečenskij in [13] proved that every GMV-algebra is isomorphic to $\Gamma(G, u)$ for an appropriate unital ℓ-group (G, u). Moreover, the categories of GMV-algebras and unital ℓ-groups are by [13] equivalent. (These results of Dvurečenskij generalize an analogous representation theorem proved for MV-algebras by D. Mundici in [26].) GMV-algebras are also equivalent to pseudo-Wajsberg algebras defined and studied by R. Ceterchi in [5], [6] and [7], and to certain dually residuated ℓ-monoids ([27]).

In the paper we use the duality between GMV-algebras and unital ℓ-groups to study structure properties of GMV-algebras. For any GMV-algebra we define three its radicals (the intersection of maximal ideals, essential ideals and closed prime ideals) and describe their mutual connections, especially for finite valued and normal-valued GMV-algebras. Moreover, a class of GMV-algebras possessing infinitely many state-morphisms is described.

Necessary results concerning the theory of MV-algebras can be found in [10], [15], [20] and [31]. Moreover, the book [15] also contains the foundations of the theory of GMV-algebras. The corresponding notions and facts of the theory of ℓ-groups are contained in [2], [4], [18] and [22].

2. Ideals of GMV-algebras

Let us recall the notion of an ideal of a GMV-algebra. (See [17].) Let A be a GMV-algebra and $\emptyset \neq H \subseteq A$. Then H is called an *ideal* of A if

(i) $x \oplus y \in H$ for any $x, y \in H$;

(ii) $y \leq x$ implies $y \in H$ for any $x \in H$ and $y \in A$.

An ideal I of a GMV-algebra A is called *normal* if

(iii) $\neg x \odot y \in I$ if and only if $y \odot \sim x \in I$ for each $x, y \in A$.

If A is a GMV-algebra, denote by $\mathcal{C}(A)$ the set of ideals and by $\mathcal{I}(A)$ the set of normal ideals of A. Then $\mathcal{C}(A)$ and $\mathcal{I}(A)$ ordered by set inclusion are complete lattices. Analogously, for any ℓ-group G denote by $\mathcal{C}(G)$ and $\mathcal{I}(G)$ the complete lattices of convex ℓ-subgroups and ℓ-ideals, respectively.

In the sequel we will frequently use the following theorem. (The first part of the theorem was proved in [29] and the second, in [14].)

THEOREM 1.

a) *Let (G, u) be a unital ℓ-group and $A = \Gamma(G, u)$. Then the correspondence φ which to each ideal H of the GMV-algebra A assigns the set $\{x \in G : |x| \wedge u \in H\}$ is an isomorphism of the lattice $\mathcal{C}(A)$ onto $\mathcal{C}(G)$. The inverse isomorphism to φ is the mapping ψ such that $\psi(K) = K \cap [0, u]$ for each $K \in \mathcal{C}(G)$.*

b) *The restriction of φ to $\mathcal{I}(A)$ gives an isomorphism between the lattices $\mathcal{I}(A)$ and $\mathcal{I}(G)$.*

Remark. Theorem 1 generalizes a similar result proved in [11] for MV-algebras and abelian ℓ-groups.

An ideal H of a GMV-algebra A is called *prime* (see [17]) if H is a finitely meet-irreducible element in the lattice $\mathcal{C}(A)$. It is known that, analogously, *prime subgroups* of an ℓ-group G are defined as finitely meet-irreducible elements in $\mathcal{C}(G)$. In the sequel, we will suppose that $A = \Gamma(G, u)$. Denote by $\mathcal{P}(A)$ and $\mathcal{P}(G)$ the set of prime ideals of A and the set of prime subgroups of G, respectively. Then by Theorem 1 we immediately obtain that the restriction of φ on $\mathcal{P}(A)$ is an order isomorphism of $\mathcal{P}(A)$ onto $\mathcal{P}(G)$.

Let $0 \neq a \in A$ and $H \in \mathcal{C}(A)$. Then H is called a *value* of a if it is maximal with respect to the property "not containing a". Denote by $\mathrm{val}_A(a)$ the set of values of a. Further, $H \in \mathcal{C}(A)$ is called a *regular ideal* of A if H is meet-irreducible in $\mathcal{C}(A)$. By [17], $H \in \mathcal{C}(A)$ is regular if and only if $H \in \mathrm{val}_A(a)$ for some $0 \neq a \in A$. If G is an ℓ-group, then *regular subgroups* and *values* of $0 \neq a \in G$ are defined in a similar way and they are mutually connected as in GMV-algebras.

An ideal H of A is called *essential* if it contains all values of some $0 \neq a \in A$. Further, $H \in \mathcal{C}(A)$ is said to be *special* if H is the unique value of some $0 \neq a \in A$. The corresponding terms for ℓ-groups are *essential and special subgroups*.

By [14], the isomorphism $\varphi\colon \mathcal{C}(A) \to \mathcal{C}(G)$ induces also bijections between the sets of regular, essential and special ideals of A and regular, essential and special subgroups of G, respectively. (Using the isomorphism φ, some properties of prime and regular ideals of GMV-algebras were derived also in [21].)

By the definition of a regular ideal of A it is obvious that every regular ideal H of A has a unique cover H^* in the lattice $\mathcal{C}(A)$. A GMV-algebra A is called *normal-valued* if for any regular ideal H of A and any $x \in H^*$, $x \oplus H = H \oplus x$. Analogously, for any regular subgroup K of G there exists its unique cover K^* in $\mathcal{C}(G)$, and G is *normal-valued* if every regular subgroup K of G is normal in K^*. Recall that if $A = (A, \oplus, \neg, \sim, 0, 1)$ is a GMV-algebra, then it is possible to introduce a partial binary operation $+$ via: $x + y$ is defined if and only if $x \le \neg y$ (if and only if $y \le {\sim} x$), and in this case $x + y = x \oplus y$. (See e.g. [15; 6.4.1].) If $H \in \mathcal{C}(A)$ and $x \in A$, set $x + H = \{x + h : h \in H\}$ and $H + x = \{h + x : h \in H\}$. A. Dvurečenskij in [14] defined the notion of a normal-valued GMV-algebra using the mentioned partial operation $+$ as follows: A GMV-algebra A is normal-valued if for any regular ideal H of A and $x \in H^*$, $x + H = H + x$. One can easily see that both definitions are equivalent, because if $x \in A$ and $h \in H$, then by [15; Remark 6.4.5], $x \oplus h = x + \big((x \oplus h) \odot {\sim} x\big) = x + ({\sim} x \wedge h)$, and ${\sim} x \wedge h \in H$. Similarly $h \oplus x = h_1 \in H$, hence $x \oplus H = H \oplus x$ if and only if $x + H = H + x$. Therefore, by [14; Proposition 6.2, the proof of Proposition 6.1], A is normal-valued if and only if G is normal-valued.

Finitely, a GMV-algebra A is called *finite valued* ([14]) if $|\operatorname{val}_A(a)| < \infty$ for every $0 \ne a \in A$. A *finite valued ℓ-group* is defined analogously. It holds ([14; Proposition 6.2]) that A is finite valued if and only if G is finite valued.

Regular, essential and special ideals of MV-algebras, as well as finite valued MV-algebras, were studied by A. Di Nola, G. Georgescu and S. Sessa in [12]. Here we generalize, among others, some of their results to the theory of GMV-algebras.

3. Three kinds of radicals of GMV-algebras

Let A be a GMV-algebra. An element $0 \ne a \in A$ is called *infinitesimal* if a satisfies condition

$$n \odot a \le \neg a \qquad \text{for each} \quad n \in \mathbb{N}.$$

Let us recall that if G is an ℓ-group and $a, b \in G$, then a is said to be *infinitary small with respect to* b (notation $a \ll b$) if $na \le b$ for each $n \in \mathbb{Z}$.

Lemma 2. *If $A = \Gamma(G, u)$ is a GMV-algebra and $a \in A$, then a is infinitesimal in A if and only if $a \ll u$ in G.*

Proof. Let $n \odot a \leq \neg a$ for each $n \in \mathbb{N}$. Then $na \leq u - a$ for each $n \in \mathbb{N}$, thus $a \ll u$ in G.

Conversely, let $a \ll u$. Then $na \leq u$ for all $n \in \mathbb{N}$, hence $na \leq u - a$ for each $n \in \mathbb{N}$, and therefore $n \odot a \leq \neg a$ for each $n \in \mathbb{N}$. □

COROLLARY 3. *Let A be a GMV-algebra and $a \in A$. Then the following conditions are equivalent:*

a) *a is infinitesimal.*

b) *$n \odot a \leq \sim a$ for each $n \in \mathbb{N}$.*

Let A be any GMV-algebra. Let us denote by $\mathrm{Infinit}(A)$ the set of all infinitesimal elements in A and by $\mathrm{Rad}(A)$ the intersection of all maximal ideals of A. Analogously, $\mathrm{Rad}(G)$ will denote the intersection of all maximal convex ℓ-subgroups of a unital ℓ-group G.

THEOREM 4.

a) *Let A be any GMV-algebra. Then* $\mathrm{Rad}(A) \subseteq \mathrm{Infinit}(A)$.

b) *If A is normal-valued, then* $\mathrm{Rad}(A) = \mathrm{Infinit}(A)$.

Proof.

a) Let $A = \Gamma(G, u)$. Then by Theorem 1 we have $\mathrm{Rad}(A) = \psi(\mathrm{Rad}(G))$, and $x \ll u$ for any $x \in \mathrm{Rad}(G)$ by [4; Proposition 4.3.9]. Hence, if $x \in \mathrm{Rad}(G) \cap G^+$, then $x \in A$ and by Lemma 2 $x \in \mathrm{Infinit}(A)$.

b) If A is normal-valued, then by [14; Proposition 6.2], G is also normal-valued, and thus all its maximal convex ℓ-subgroups are ℓ-ideals. Hence by [4; Proposition 4.3.9], $\mathrm{Rad}(G) = \{x \in G : x \ll u\}$, therefore $\mathrm{Rad}(A) = \mathrm{Infinit}(A)$. □

A GMV-algebra A is called *archimedean* (see e.g. [15]) if $\mathrm{Infinit}(A) = \{0\}$.

THEOREM 5. *If a GMV-algebra A is normal-valued, then* $\mathrm{Rad}(A) = \{0\}$ *if and only if A is an archimedean GMV-algebra (and hence A is an archimedean (= semisimple) MV-algebra).*

Proof. Let $\mathrm{Rad}(A) = \{0\}$. Then by Theorem 4, if $a \in A$ is such that $n \odot a \leq \neg a$ for each $n \in \mathbb{N}$, then $a = 0$. Hence A is an archimedean GMV-algebra and thus by [13; Theorem 4.2], A is an archimedean MV-algebra.

This also implies, conversely, that for any archimedean GMV-algebra A, $\mathrm{Rad}(A) = \{0\}$. □

THEOREM 6. *A GMV-algebra A is finite valued if and only if each regular ideal of A is special.*

Proof. If A is finite valued, then every its regular ideal is special by [14; Proposition 6.4].

Conversely, let every regular ideal of A be special. Let $A = \Gamma(G, u)$. If K is a regular subgroup of G, then $K \in \mathrm{val}_G(g)$ for some $0 \neq g \in G$, and hence by [14; Proposition 6.2], $\psi(K) \in \mathrm{val}_A(|g| \wedge u)$. Thus there exists $0 \neq a \in A$ such that $\mathrm{val}_A(a) = \{\psi(K)\}$. Hence, again by [14; Proposition 6.2], $\mathrm{val}_G(a) = \{K\}$, and so K is special in G. Therefore by [4; Théorème 6.4.3], G is finite valued, thus A is by [14; Proposition 6.2] finite valued, too. □

Remark. In [12; Remark 5.2], it is noted that if an MV-algebra A is finite valued, then A has a finite number of maximal ideals (i.e. A is semilocal). The converse implication is in [12; Proposition 5.2], proved under the assumption that A is semisimple. Nevertheless, by [14; Proposition 6.4], the converse implication is valid even for arbitrary GMV-algebra A. Therefore a GMV-algebra is finite valued if and only if it has a finite number of maximal ideals.

Let A be a GMV-algebra and $B \subseteq A$. Then B is called *closed* if for any subset $C \subseteq B$ such that $c = \sup C$ in A exists, the element c belongs to B. (See [29].)

(Recall that closed ideals of MV-algebras were investigated in [12].)

LEMMA 7. *If $A = \Gamma(G, u)$ is a GMV-algebra and $H \in \mathcal{C}(A)$, then H is an essential ideal of A if and only if $\varphi(H)$ is an essential subgroup of G.*

Proof. The assertion follows from [14; Proposition 6.2(1)]. □

PROPOSITION 8.

a) *Every essential ideal of any GMV-algebra A is closed.*

b) *Every special ideal of A is closed.*

Proof.

a) Let $A = \Gamma(G, u)$ be a GMV-algebra. If H is an essential ideal of A, then by Lemma 7, $\varphi(H)$ is an essential subgroup of G, thus $\varphi(H)$ is closed in G by [4; 6.1.3], and hence H is a closed ideal of A by [29; Proposition 13].

b) The second assertion is a particular case of the first one. □

In the sequel, we will compare the radical $\mathrm{Rad}(A)$ of a GMV-algebra A with further two kinds of radicals of A.

If A is any GMV-algebra, then the intersection $D(A)$ of all closed prime ideals of A will be called the *distributive radical of* A. (See also [29].) Let us recall that analogously it is defined the *distributive radical* $D(G)$ of any ℓ-group G, as the intersection of all closed prime subgroups of G.

Further, let denote by $R(A)$ the intersection of all essential ideals of any GMV-algebra A. Analogously, for any ℓ-group G, $R(G)$ is defined as the intersection of all essential subgroups of G.

THEOREM 9.

a) *For any GMV-algebra A, $D(A) \subseteq R(A)$.*
b) *If A is normal-valued, then $D(A) = R(A)$.*

Proof.
a) Let H be an essential ideal of A. Then H is a prime ideal which is closed by Proposition 8. Hence $D(A) \subseteq R(A)$.
b) Let $A = \Gamma(G, u)$. Then by the proof of [29; Proposition 20], $D(G) = \varphi(D(A))$. By [4; 6.2.3], $D(G)$ is equal to the intersection of all closed regular subgroups of G. Thus by [29; Proposition 13], [14; Proposition 6.2] and Theorem 1, $D(A)$ is equal to the intersection of all closed regular ideals of A.

Let A be normal-valued. Then by [14; Proposition 6.2], G is also normal-valued, hence each its closed regular subgroup is essential by [4; 6.1.14], therefore $D(G) = R(G)$. From this, using once more [14; Proposition 6.2], we get $D(A) = R(A)$. □

THEOREM 10. *If a GMV-algebra A is finite valued, then $R(A) \subseteq \mathrm{Rad}(A)$.*

Proof. If A is finite valued, then each maximal ideal of A is essential by Theorem 6. This gives the assertion. □

COROLLARY 11. *If a GMV-algebra A is finite valued, then every element in the radical $R(A)$ (and thus also in $D(A)$) is infinitesimal.*

Proof. By [14; Proposition 6.4], every finite valued GMV-algebra is normal-valued. Hence $R(A) \subseteq \mathrm{Rad}(A) = \mathrm{Infinit}(A)$ by Theorems 4 and 10. □

PROPOSITION 12. *If A is a finite valued GMV-algebra and $H \in C(A)$, then H is closed.*

Proof. By Theorem 6, every regular ideal of A is special, and thus, by Proposition 8, also closed. The assertion now follows from the fact that every $H \in C(A)$ is an intersection of regular ideals. □

COROLLARY 13. ([29; Proposition 16]) *If A is a linearly ordered GMV-algebra, then every ideal of A is closed.*

Let us recall that a lattice is called *completely distributive* if the equality

$$\bigwedge_{\alpha\in\Gamma}\bigvee_{\beta\in\Delta} x_{\alpha\beta} = \bigvee_{f\in\Delta^\Gamma}\bigwedge_{\alpha\in\Gamma} x_{\alpha f(\alpha)}$$

holds for any elements $x_{\alpha\beta} \in L$, $\alpha \in \Gamma$, $\beta \in \Delta$, if the above $\bigwedge$ and $\bigvee$ exist.

A GMV-algebra A is called *completely distributive* (see also [29]) if the lattice $(A, \vee, \wedge)$ is completely distributive.

PROPOSITION 14. *Every finite valued GMV-algebra is completely distributive.*

P r o o f. If A is any GMV-algebra, then by [29; Theorem 21], A is completely distributive if and only if $D(A) = \{0\}$. Since in the case of a finite valued GMV-algebra A (by Proposition 12) $\{0\}$ is a closed ideal, $D(A) = \{0\}$, and therefore we obtain the assertion. □

COROLLARY 15. *If a GMV-algebra A is not completely distributive, then A has infinitely many maximal ideals.*

P r o o f. Let a GMV-algebra A be not completely distributive. Then there exists $0 \neq a \in A$ which have infinitely many values, therefore also the greatest element 1 has by [14; Proposition 6.4] infinitely many values. □

D v u r e č e n s k i j introduced in [14] the notion of a state-morphism on any GMV-algebra A as a homomorphism of A into the standard MV-algebra $[0,1] = \Gamma(\mathbb{R}, 1)$. In the same time he showed ([14; Propositions 4.3, 4.5, 4.6] that there is a one-to-one correspondence between the state-morphisms on A and the maximal ideals of A which are normal. Hence we get, as a consequence, the following assertion.

COROLLARY 16. *If a GMV-algebra A is not completely distributive and is normal-valued, then A possesses infinitely many state-morphisms.*

Remark. Let G be an ℓ-group which possesses a strong unit u. By [29; Theorem 21], G is completely distributive if and only if the GMV-algebra $\Gamma(G, u$ is completely distributive. Thus by Theorem 1 and Corollary 15, if an ℓ-group G which has a strong unit is not completely distributive, then G has infinitely many maximal convex ℓ-subgroups.

Moreover, let us recall that a *state* on a unital ℓ-group (G, u) is any mapping $s\colon G \to \mathbb{R}$ such that $s(g_1 + g_2) = s(g_1) + s(g_2)$, $g_1 \in G^+ \implies s(g_1) \in \mathbb{R}^+$ for every $g_1, g_2 \in G$, and $s(u) = 1$. (See [14], and for abelian unital ℓ-groups [19]. By [14], the states on G are in a one-to one correspondence with the so-called states on the GMV-algebra $\Gamma(G, u)$. Since the state-morphisms on $\Gamma(G, u)$ are special cases of the states on $\Gamma(G, u)$, Corollary 16 with Theorem 1 imply the following assertion: If a normal-valued ℓ-group G has a strong unit and is not completely distributive, then G possesses infinitely many states.

As one is known, the finite valued ℓ-groups can be characterized by means of properties of the lattices of their convex ℓ-subgroups. Let us show that a similar characterization exists also for GMV-algebras.

THEOREM 17. *Let A be a GMV-algebra. Then the following conditions are equivalent.*

a) *A is finite valued.*

b) *The lattice* $\mathcal{C}(A)$ *is completely distributive.*
c) *The lattice* $\mathcal{C}(A)$ *is dually Brouwerian.*
d) *The lattice* $\mathcal{C}(A)$ *is freely generated by the root system of regular ideals.*

Proof. Let $A = \Gamma(G, u)$. Then by Theorem 1, the lattices $\mathcal{C}(A)$ and $\mathcal{C}(G)$ are isomorphic and by [14; Proposition 6.2], A is finite valued if and only if G is finite valued. Hence all conditions a), b) and c) are equivalent by [4; Théorème 6.4.8].

Conditions a) and d) are equivalent by Theorem 1 and [2; Theorem 10.12]. □

THEOREM 18. *A GMV-algebra A is finite valued if and only if any element $0 \neq a \in A$ is a supremum of a finite number of pairwise orthogonal special elements in A.*

Proof. Let $A = \Gamma(G, u)$ and let $0 \neq a \in A$. By [4; Théorème 6.4.1], $|\operatorname{val}_G(a)| < \infty$ if and only if there exist pairwise orthogonal special elements $a_1, \dots, a_n \in G^+$ such that $a = a_1 + \dots + a_n$ in G. The sum of pairwise orthogonal elements in G^+ is equal to their join, hence $a = a_1 \vee \dots \vee a_n$. At the same time, every a_i belongs to A and by [14; Proposition 6.2], a_i is special also in A. □

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