Matúš Harminc

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SOLUTIONS AND KERNELS OF A DIRECTED GRAPH

MATÚŠ HARMINC

In this note the solutions and the kernels of directed graphs are dealt with. The following theorem will be proved: The number of solutions (kernels) of a directed graph is equal to the number of solutions (kernels) of its line graph. It will be shown how to construct the solutions of a line graph by means of the solutions of the original graph, and conversely.

Preliminaries

A directed graph $G=(V, A)$ with the set of points V and the set of lines $A \subseteq V \times V$ without loops and multiple lines is shortly called a graph. Concepts as a path, initial and terminal points of a line and others are used as in [3]. A point which is not an initial point of any line of G is called a receiver of G. We denote by $\mathcal{P}(M)$ the system of all subsets of a set M and the cardinality of M by card M. Now we define basic concepts: The line graph of $G=(V, A)$ is a graph $L(G)=(A, B)$, the point set of which is the set of lines of G, and for any $h, k \in A$ there is $hk \in B$ if and only if the corresponding lines h, k induce a path in G, i.e., the terminal point of h is the initial point of k. In what follows we denote the line $h=uv$ in G and the point h in $L(G)$ by the same symbol. If H is a set of lines of G, it is also a set of points of $L(G)$. If we want to emphasize our interest in H as the set of points of $L(G)$ we use the symbol H_L instead of H.

A subset R of V is a solution of $G=(V, A)$ if R is independent in G (i.e. if u, $v \in G$ implies $uv \notin A$) and if R is dominant in G (i.e. if for each $v \in V-R$ there exists $u \in R$ such that $uv \in A$). (See [1, 6, 7, 8].) In the literature this concept is known also as a 1-basis [3].

A subset J of V is a kernel of $G=(V, A)$ if J is independent in G and if J is absorbent in G (for each $v \in V-J$ there exists $u \in J$ such that $vu \in A$). (See [2].)

Results

Let $\mathscr{R}$ be the system of all solutions of a graph $G=(V, A)$ and let $\mathscr{S}$ be the system of all solutions of $L(G)$.

Theorem 1. *Card* $\mathscr{R}$ = *card* $\mathscr{S}$.

Before proving this theorem, we present some lemmas. Let us define a mapping f: $\mathscr{P}(V)\to\mathscr{P}(A)$ as follows: If $Z\subseteq V$, then $f(Z)$ is the set of all such lines, the initial point of which is in Z.

Lemma 1. *If* $R\in\mathscr{R}$, *then* $f(R)_L\in\mathscr{S}$.

Proof. $f(R)_l$ is independent: if $hk\in B$, then $\{h, k\}\nsubseteq f(R)_l$ since in the other case $h\in R\times R$, but this contradicts the independence of R. Now, let k be a point of $L(G)$, $k\in A_L$ $f(R)_L$. By the definition of $f(R)_L$ the initial point of k in G is not in R. From the dominance of R in G it is clear that there exists a line h in G with the initial point in R, the terminal point of which is identical with the initial point of k. Therefore $h\in f(R)_L$ and $hk\in B$ so that lemma is proved.

Lemma 2. *The mapping* f: $\mathscr{R}$ — $\mathscr{S}$ *is injective.*

Proof. Let $R, P\in\mathscr{R}$ and $R\neq P$. Let us suppose, e.g., that $R-P\neq\emptyset$, $v\in R$ P. Because P is a solution of G there is a point $u\in P$ such that $uv\in A$. Clearly $uv\in f(P)_L$. The independence of R in G implies $u\notin R$. Hence $uv\notin f(R)_L$ and the lemma is proved.

Define a mapping g: $\mathscr{P}(A)\to\mathscr{P}(V)$ as follows: If $H\subset A$, then $g(H)=X(H)\cup Y(H)$, where $X(H)$ is a set of all initial points of lines of H and $Y(H)$ is a set of all receivers r of G such that r is adjacent with no point of $X(H)$.

Lemma 3. *If* $H_L\in\mathscr{S}$, *then* $g(H)\in\mathscr{R}$.

Proof. In proving the independence of $g(H)$ let us assume that $u, v\in g(H)$, u, $v\in V$. We shall distinguish three cases:

(1) $u, v\in X(H)$,

(2) $u\in X(H)$, $v\in Y(H)$,

(3) $u\in Y(H)$.

In the case (1) u is the initial point of some line h and v is the initial point of some line k; $h, k\in H_L$. If $h=uv$, there is a line hk in G which is a contradiction with the independence of H_L. If $h=uw\neq uv=d$, then the independence of H_L implies $d\notin H_L$ and from the dominance of H_L it follows that there is $b\in H_L$ such that $bd\in B$. The terminal point of b and the initial point of h are identical with u; it follows that $bh\in B$ and this is a contradiction with the independence of H_L. In the cases (2) and (3) it follows immediately from the definitions of $X(H)$ and $Y(H)$ that $uv\notin A$. There will be proved the dominance of $g(H)$: Let $v\in V-g(H)$ $= V-X(H)$ $Y(H)$. For the point v we have one of the following two possibilities:

(a) v is an initial point of some line
(b) v is an initial point of no line and it is adjacent with some points of $X(H)$.
In the case (a) there exists $vt \in A$. Since $v \notin X(H)$, we obtain $vt \notin H_L$. The dominance of H_L in $L(G)$ implies the existence $uv \in H_L$; thus $u \in X(H)$. In the case (b) the proof of the dominance of $g(H)$ follows from the definitions of $X(H)$ and $Y(H)$ immediately.

Lemma 4. *The mapping* $g\colon \mathcal{S} \to \mathcal{R}$ *is injective.*

Proof. Let $S_L \neq T_L$; $S_L, T_L \in \mathcal{S}$. We suppose for example that $S_L - T_L \neq \emptyset$, $h \in S_L - T_L$. Let us denote by v the initial point of h. Thus $v \in g(S)$, since v is the initial point of a line of S. As $h \notin T_L$ and because T_L is dominant in $L(G)$, there exists a line k in G such that $k \in T_L$ and $kh \in B$. Let us denote by u the initial point of k; the terminal point of k is v. The point k belongs to T_L, hence $u \in g(T)$ and the independence of $g(T)$ in G implies $v \notin g(T)$. Thus the lemma is proved.

Proof of Theorem 1. According to Lemma 2 and Lemma 4 we obtain

$$\operatorname{card} \mathcal{R} \leq \operatorname{card} \mathcal{S} \leq \operatorname{card} \mathcal{R},$$

which implies

$$\operatorname{card} \mathcal{R} = \operatorname{card} \mathcal{S}.$$

Corollary 1. *The graph G has a solution iff its line graph L(G) has a solution.*

Corollary 2. *If there is an isomorphism between* $L(G_1)$ *and* $L(G_2)$, *then* G_1 *and* G_2 *have the same number of solutions.*

Remark 1. It is possible to verify that in the graph G each $R \in \mathcal{R}$ satisfies the identity $g(f(R)) = R$. Analogously, $f(g(S)) = S$ for each $S \in \mathcal{S}$.

Let G be a graph, $G = (V, A)$ and let $\operatorname{con} G$ be the graph with the point set V in which $uv \in \operatorname{con} G$ if and only if $vu \in A$. It is easy to see that the following propositions are equivalent:
(i) M is a solution of G.
(ii) M is a kernel of $\operatorname{con} G$.
We shall denote the system of all kernels of G by the symbol $\mathcal{K}$ and the system of all kernels of $L(G)$ by $\mathcal{L}$.

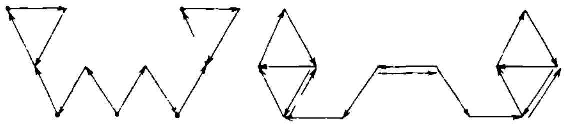
Theorem 2. *Card* $\mathcal{K} = \operatorname{card} \mathcal{L}$.

Proof. With respect to the equivalence of (i) to (ii) the system $\mathcal{K}$ consists of all solutions of $\operatorname{con} G$ and $\mathcal{L}$ is the system of all solutions of $\operatorname{con} L(G)$. The definitions of graphs $L(G)$ and $\operatorname{con} G$ imply immediately $\operatorname{con} L(G) = L(\operatorname{con} G)$. The systems of solutions of the graphs $\operatorname{con} G$ and $L(\operatorname{con} G)$ have the same cardinality (cf. Theorem 1), i.e. the systems of solutions of the graphs $\operatorname{con} G$ and $\operatorname{con} L(G)$ have the same cardinality, too. Thus $\operatorname{card} \mathcal{K} = \operatorname{card} \mathcal{L}$.

Corollary 3. *G has a kernel iff L(G) has a kernel.*

Corollary 4. *If there is an isomorphism between $L(G_1)$ and $L(G_2)$, then G_1 and G_2 have the same number of kernels.*

Remark 2 If we define the line graph $L(G)$ of a graph G in the sense of [5], then Theorem 1 and Theorem 2 are not valid.



Fig

According to [5] the line graph of $G = (V, A)$ is defined by $L(G) = (A, B)$, where $hk \in B$ for $h, k \in A$ if and only if the initial or the terminal points of h and k coincide or if the terminal point of h is the initial point of k (since, from our point of view, the multiplicity of lines is irrelevant, the original definition is modified here to suit our purpose).

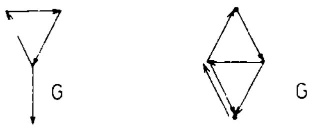


Fig. 2

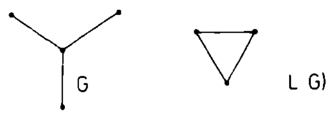


Fig 3

Examples. Figure 1 shows a graph G with a solution and its line graph $L(G)$ with no solution. The graph G of Figure 2 has no solution, but its line graph $L(G)$ has a solution

Remark 3 If we define the line graph $L(G)$ of an undirected graph G in the usual way (see [4]), then Theorem 1 and Theorem 2 are not valid.

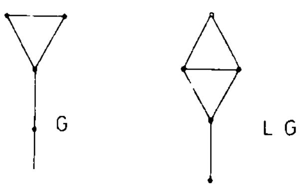


Fig. 4

Examples. The graph G of Figure 3 has two solutions and its line graph $L(G)$ has three solutions. On the other hand, Figure 4 shows a graph G with five solutions and its line graph $L(G)$ with four solutions.

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Dislokovane pracovisko v Košiciach
Karpatská 5
040 01 Košice

РЕШЕНИЯ И ЯДРА ОРГРАФА

Матуш Гарминц

Резюме

В работе доказана теорема: Мощность множества решений (ядер) графа равна мощности множества решений (ядер) его реберного графа. Показана конструкция решений реберного графа $L(G)$ с помощью решений графа G и наоборот.