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of Professor Milan Kolibiar

AXIOMATIZATION AND UNDECIDABILITY RESULTS FOR LINEAR BETWEENNESS RELATIONS

ROBERT MENDRIS* — PAVOL ZLATOŠ**

(*Communicated by Tibor Katriňák*)

ABSTRACT. Let $\mathbf{V}$ be a vector space over an ordered field $\mathbf{F}$. The ternary betweenness relation $T_{\mathbf{V}}$ on V, induced by the linear structure of $\mathbf{V}$ and the ordering of $\mathbf{F}$, is defined by

$$T_{\mathbf{V}}(x,y,z) \iff (\exists t \in F)\big(0 \leq t \leq 1 \ \&\ y - x = t(z-x)\big)$$

for $x, y, z \in V$. We will prove that the class $\mathcal{L}$ of all linear ternary structures, i.e., the class of all structures (A,T) with a single ternary relation T which can be embedded into $(V, T_{\mathbf{V}})$ for some vector space $\mathbf{V}$ over an arbitrary ordered field $\mathbf{F}$ (not just the real numbers), is an elementary class which can be axiomatized by a set of universal sentences. Further, we will show that the first-order theory of $\mathcal{L}$ is recursively axiomatizable, and its universal part is decidable. On the other hand, the theory of $\mathcal{L}$ is not finitely axiomatizable, and the theory of finite members of $\mathcal{L}$ is hereditarily undecidable.

Introduction

In our previous paper [MnZl], we have been examining metrizable betweenness spaces, i.e., structures of the form (A, T_d), where d is a metric on A with values in some ordered Abelian group $\mathbf{G}$, and T_d is the ternary betweenness relation on A induced by d, defined by

$$T_d(x,y,z) \iff d(x,z) = d(x,y) + d(y,z)$$

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for $x, y, z \in A$. We have proved that the class $\mathcal{M}$ of all metrizable betweenness spaces, regarded as a class of structures of the first-order language with equality $\equiv$ and a single ternary predicate T, is an elementary class with a recursive set of universal Horn axioms and decidable universal part of its theory. On the other hand, $\mathrm{Th}\,\mathcal{M}$ is not finitely axiomatizable, and the theory of the finite members of $\mathcal{M}$ is hereditarily undecidable.

In the present paper, we are going to prove, using rather similar methods, analogous results for the class $\mathcal{L}$ of all linear betweenness spaces which has been defined in the abstract.

It is just a matter of skill to verify that for $T = T_{\mathbf{V}}$ in every vector space $\mathbf{V}$ over an ordered field, and consequently in any linear betweenness space (A, T), the following eight axioms are satisfied:

(B0) $\quad T(x,y,x) \implies x \equiv y$,
(B1) $\quad T(x,x,y)$,
(B2) $\quad T(x,y,z) \implies T(z,y,x)$,
(B3) $\quad T(x,y,z) \ \&\ T(x,z,u) \implies T(x,y,u)$,
(B4) $\quad T(x,y,z) \ \&\ T(x,z,u) \implies T(y,z,u)$,
(B5) $\quad T(x,y,z) \ \&\ T(x,u,z) \implies T(x,y,u) \vee T(x,u,y)$,
(B6) $\quad T(x,y,z) \ \&\ T(x,y,u) \ \&\ x \not\equiv y \implies T(y,z,u) \vee T(y,u,z)$,
(B7) $\quad T(x,y,z) \ \&\ T(y,z,u) \ \&\ y \not\equiv z \implies T(x,y,u)$.

Note that they are all universal sentences involving at most four variables. They can also be regarded as "transitivities" in the sense of [PtSm].

Ternary structures (A, T) satisfying (B0) – (B4) have been called betweenness spaces in [MnZl]. It can easily be seen that these conditions are true in any metrizable betweenness space, justifying our terminology. On the other hand, none of the remaining conditions (B5) – (B7) is valid in every metrizable betweenness space. Moreover, none of them is preserved under the formation of direct products of two linear betweenness spaces. This shows that $\mathcal{L}$, in contradistinction to $\mathcal{M}$, cannot be a Horn class.

As a direct consequence of conditions (B0), (B1), (B3), for every ternary structure (A, T) satisfying them and for each $x \in A$ the relation

$$y \le_x z \iff T(x,y,z)$$

is a partial order on A. Now, introducing the segments

$$(xy)_T = \{z \in A;\ T(x,z,y)\}$$

for $x, y \in A$, the intuitive meaning of the remaining axioms can be summed up as follows:

(B2): The partial order $\leq_x$ is reversed to the partial order $\leq_z$ on the segment $(xz)_T$.
(B4): Both the partial orders $\leq_x$ and $\leq_y$ coincide on the set $\{z \in A\,;\ y \leq_x z\}$.
(B5): All the segments $(xz)_T$ are linearly ordered with respect to $\leq_x$.
(B6): Segments do not split, in other words, "snake tongues" do not occur.
(B7): Two segments $(xz)_T$, $(yu)_T$ with a nontrivial overlap $(yz)_T$ can be put together and extended into the segment $(xu)_T$.

In addition to $\mathcal{L}$, let us also introduce the class $\mathcal{L}_0$ of all members of $\mathcal{L}$ which can embedded into a vector space over the ordered field $\mathbb{R}$ of real numbers. Similarly as in [MnZl], for the class $\mathcal{M}_0$ of all betweenness spaces metrizable by a real-valued metric, one can easily show that $\mathcal{L}_0$, being not closed under elementary extensions, is not an elementary class. Unfortunately, this analogy goes even further, as both the next questions remain open:

Is the smallest elementary class containing $\mathcal{M}_0$ equal to $\mathcal{M}$?

Is the smallest elementary class containing $\mathcal{L}_0$ equal to $\mathcal{L}$?

In what follows, we will use freely the usual terminology and notation common in model theory, as well as some results belonging to model-theoretical folklore. The standard references are the monographs [ChK], [H] and [Sh], where also the necessary information on ordered fields can be found. For the needed facts concerning (un)decidability of first-order theories, the reader is referred either to [BrSn] or to [ELTT].

1. Linear betweenness spaces

The connection between the linear betweenness relation $T_{\mathbf{V}}$ on a normed vector space $\mathbf{V}$ and the metric betweenness relation T_d on $\mathbf{V}$, where d is the metric induced by the norm, was already studied by M. F. Smiley in [Sm]. He observed the inclusion $T_{\mathbf{V}} \subseteq T_d$ and proved the equivalence of the notion of strict convexity, introduced by J. A. Clarkson [Cl], to the equality $T_{\mathbf{V}} = T_d$. Though Smiley stated the mentioned results for vector spaces over $\mathbb{R}$ and real-valued norms, only, a brief inspection shows that the same arguments work for an arbitrary ordered field $\mathbf{F}$ and any vector space $\mathbf{V}$ over $\mathbf{F}$, endowed with a norm taking values in any ordered field $\widehat{\mathbf{F}}$ extending $\mathbf{F}$.

Following up the just mentioned work, let us quote the next simple fact.

PROPOSITION 1. *All linear betweenness spaces are metrizable, i.e., $\mathcal{L} \subseteq \mathcal{M}$.*

Proof. It suffices to show that given any vector space $\mathbf{V}$ over an ordered field $\mathbf{F}$ and denoting $\widehat{\mathbf{F}}$ the real closure of $\mathbf{F}$, there is an $\widehat{\mathbf{F}}$-valued metric d such that $T_d = T_{\mathbf{V}}$. Using an arbitrary Hamel basis H of $\mathbf{V}$, $\mathbf{V}$ can be endowed with an $\widehat{\mathbf{F}}$-valued norm

$$\|x\| = \Big(\sum_{h \in H} |x_h|^2 \Big)^{1/2},$$

where $x_h \in F$ are the co-ordinates of $x \in V$ with respect to H, i.e., $x = \sum_{h \in H} x_h h$ and $\{h \in H;\ x_h \neq 0\}$ is finite. The argument from [Cl] can be used to show that $\mathbf{V}$ with the norm $\|x\|$ is strictly convex (Clarkson even proved the uniform convexity which is a stronger property). Hence, for the $\widehat{\mathbf{F}}$-valued metric $d(x,y) = \|x - y\|$ we have $T_{\mathbf{V}} = T_d$. □

THEOREM 1. *The class $\mathcal{L}$ of all linear betweenness spaces is an elementary class which can be axiomatized by a set of universal axioms.*

Proof. The class $\widetilde{\mathcal{L}}$ of all ternary structures which are isomorphic to $(V, T_{\mathbf{V}})$ for some vector space $\mathbf{V}$ over some ordered field $\mathbf{F}$ is in fact the class of all (isomorphic copies of) reducts of two-sorted structures of the form $(F, V, +, \cdot, 0, 1, <, \oplus, \odot, \mathbf{0}, T)$, such that $\mathbf{F} = (F, +, \cdot, 0, 1, <)$ is an ordered field, $\mathbf{V} = (V, \oplus, \odot, \mathbf{0})$ is a vector space over $\mathbf{F}$ (the signs $\oplus$, $\odot$ and $\mathbf{0}$ are used just in this place in order to distinguish the operations in $\mathbf{F}$ and $\mathbf{V}$) and $T = T_{\mathbf{V}}$, which obviously form an elementary class. Thus $\mathcal{L}$, being the class of all structures (A, T) embeddable into some structure from the pseudoelementary class $\widetilde{\mathcal{L}}$, is a universal elementary class. □

Let us denote $\mathcal{L}_{fin}$ the class of all finite members of $\mathcal{L}$. Obviously, each $(A, T) \in \mathcal{L}_{fin}$ can be embedded into some *finite-dimensional* vector space over some ordered field $\mathbf{F}$, i.e., into $\mathbf{F}^n$ for some $n \in \mathbb{N}$. Now, using the facts that

(a) the first-order theory of real-closed ordered fields is complete;
(b) $\mathbb{R}$ is real-closed;
(c) every ordered field can be embedded into a real-closed one;
(d) by the Feferman-Vaught theorem, direct products of structures preserve elementary equivalence;

(see, e.g., [Sh; §5.5] and [H; §8.4 and §9.5]), one can easily see that $\mathcal{L}_{fin}$ coincides with the class of all finite ternary structures embeddable into $(\mathbb{R}^n, T_n)$ for some n, where T_n is the linear betweenness relation on the vector space $\mathbb{R}^n$.

PROPOSITION 2. *$\mathcal{L}_{fin}$ coincides with the class of all finite ternary structures embeddable into $(\mathbb{R}^2, T_2)$.*

Proof. We will show that for each $n \geq 2$ and every finite set $A \subseteq \mathbb{R}^{n+1}$, the ternary structure $(A, T_{n+1} \cap A^3)$ can be embedded into $(\mathbb{R}^n, T_n)$. Then the needed conclusion follows by an induction argument.

Let us identify $\mathbb{R}^n$ with the subspace $\{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1};\ x_0 = 0\}$ of $\mathbb{R}^{n+1}$. Then T_{n+1} induces T_n on $\mathbb{R}^n$. For each element p of the projective space $\mathbb{R}P^n$, with homogeneous co-ordinates $[p_0, p_1, \ldots, p_n]$ such that $p_0 \neq 0$, and $x \in \mathbb{R}^{n+1}$ let us denote $p(x)$ the parallel projection of x into the hyperplane $\mathbb{R}^n$ along the direction p. We put $p(A) = \{p(x);\ x \in A\}$. As A is finite, there are only finitely many p's such that the corresponding projection of A onto $p(A)$ is not one-to-one. Projections mapping some pairs of distinct lines with endpoints in A into one line are represented by p's from a union of finitely many one-dimensional projective subspaces of $\mathbb{R}P^n$. As $n \geq 2$, there is a p determining a bijective projection $A \to p(A)$, such that $T_{n+1}(x, y, z)$ if and only if $T_n(p(x), p(y), p(z))$ for all $x, y, z \in A$. $\square$

As it follows from the last Proposition, the only information concerning the dimension of a vector space $\mathbf{V}$ over an ordered field $\mathbf{F}$ which can be deduced from the finite betweenness subspaces of $(V, T_{\mathbf{V}})$ is the answer to the question whether $\dim \mathbf{V} = 0$ or $\dim \mathbf{V} = 1$ or $\dim \mathbf{V} \geq 2$.

THEOREM 2. *The first-order theory of $\mathcal{L}$ is recursively axiomatizable, and the universal part of* $\mathrm{Th}\,\mathcal{L}$ *is decidable.*

Proof. A formula φ will be called a basic formula if it is of the form $\theta_1 \vee \cdots \vee \theta_m$, where each θ_l is an atomic formula or negation of an atomic formula. We will say that a formula ψ occurs in such a basic formula φ if ψ is among the formulas $\theta_1, \ldots, \theta_m$. According to Theorem 1, it suffices to describe an algorithm deciding for each basic formula φ in the language of $\mathcal{L}$ whether $\mathcal{L} \models \varphi$ or not. Then (the universal closures of) the basic formulas φ which are true in $\mathcal{L}$ will form a recursive set of axioms for $\mathrm{Th}\,\mathcal{L}$. Furthermore, the decidability question for any universal formula (after putting it into prenex form with matrix consisting of a conjunction of basic formulas) can be reduced to the same question for its maximal basic subformulas.

Now, let φ be a basic formula of the form $\theta_1 \vee \cdots \vee \theta_m$, where each θ_l is an atomic formula or negation of an atomic formula with variables included in the list $z_1, \ldots, z_n$. Let us denote Σ_φ the system of polynomial equations and inequalities in the unknowns x_i, y_i, t_{ijk}, where $i, j, k = 1, \ldots, n$, ((x_i, y_i) are the co-ordinates of z_i in $\mathbb{R}^2$), constructed as follows:

$$\begin{aligned}
t_{ijk} &\geq 0 && \text{for all } i,\ j,\ k,\\
t_{iij} &= 0 && \text{for all } i,\ j,\\
t_{ijk} + t_{kji} &= 1 && \text{for all } i,\ j,\ k,\\
(x_j - x_i, y_j - y_i) &= t_{ijk}(x_k - x_i, y_k - y_i) && \text{if } \neg T(z_i, z_j, z_k) \text{ occurs in } \varphi,\\
(x_j - x_i, y_j - y_i) &\neq t_{ijk}(x_k - x_i, y_k - y_i) && \text{if } T(z_i, z_j, z_k) \text{ occurs in } \varphi,\\
(x_i, y_i) &= (x_j, y_j) && \text{if } z_i \not\equiv z_j \text{ occurs in } \varphi,\\
(x_i, y_i) &\neq (x_j, y_j) && \text{if } z_i \equiv z_j \text{ occurs in } \varphi.
\end{aligned}$$

We use the vector (or ordered pairs) notation for abbreviation's sake. Note that the inequality of the form $(a,b) \neq (c,d)$, i.e., the disjunction $(a \neq c) \vee (b \neq d)$, is equivalent to the single inequality $(a-c)^2 + (b-d)^2 > 0$. Also note that, in general, Σ_φ may contain many redundant equations and inequalities, which can be omitted in concrete situations.

From Proposition 2, it follows that the condition $\mathcal{L} \models \varphi$ holds if and only if Σ_φ has no solution in $\mathbb{R}$. The decidability of the last problem is clear, according to the decidability of the first-order theory of the ordered field $\mathbb{R}$, based on quantifier elimination, due to A. Tarski [T] (see also [ELTT]). ☐

In the proof of the next theorem, we shall need the following slight and straightforward generalization of a result from [Vt].

LEMMA. *Let $\mathcal{K}$ be a universal class of structures of a first-order language containing only finitely many relational symbols and without any constant and functional symbols. Then for a class $\mathcal{J} \subseteq K$ the following conditions are equivalent:*

(i) *There is a single universal sentence φ such that*

$$\mathcal{J} = \{\mathbf{A} \in \mathcal{K};\ \ \mathbf{A} \models \varphi\}.$$

(ii) *There is a natural number $n > 0$ such that for every $\mathbf{A} \in \mathcal{K}$ we have $\mathbf{A} \in \mathcal{J}$ if and only if each n-element substructure $\mathbf{B}$ of $\mathbf{A}$ can be embedded into a suitable structure from $\mathcal{J}$.*

Let $\mathcal{J}$, $\mathcal{K}$ be two classes of structures of the same first-order language. We will say that $\mathcal{J}$ is (finitely) axiomatizable with respect to $\mathcal{K}$ if there is a (finite) set of sentences S such that

$$\mathcal{J} = \{\mathbf{A} \in \mathcal{K};\ \ \mathbf{A} \models S\}.$$

THEOREM 3. *The class $\mathcal{L}$ of linear betweenness spaces is not finitely axiomatizable with respect to the class $\mathcal{M}$ of all metrizable betweenness spaces.*

Proof. As $\mathcal{L}$ is a universal class as well, according to the Lemma, it is enough to show that for each sufficiently large $n \in \mathbb{N}$ there is a ternary structure $(A,T) \in \mathcal{M}$ such that $(A,T) \notin \mathcal{L}$, and each n-element substructure of (A,T) is in $\mathcal{L}$.

For each $n > 1$ let us introduce the $(4n+2)$-element set

$$A_n = \{a_i;\ \ 1 \leq i \leq n\} \cup \{b_i;\ \ 0 \leq i \leq 2n\} \cup \{c_i;\ \ 0 \leq i \leq n\}.$$

Let T_n be the ternary relation on A_n consisting of the triples

(c_0, b_0, a_1), (c_0, b_1, c_n),

all the triples of the form

(b_i, b_j, b_k) for $0 \leq i < j < k \leq 2n$,
(a_i, b_{2i}, c_i) for $1 \leq i \leq n$,
(c_i, b_{2i+1}, a_{i+1}) for $1 \leq i \leq n-1$,

and, of course, all the triples which one has to add in order to satisfy the conditions (B1), (B2).

The structures (A_1, T_1) and (A_3, T_3) can be seen on the following diagram (for three distinct vertices x, y, z there is a (maybe broken) line from x to z passing through y if and only if $T(x, y, z)$ holds):

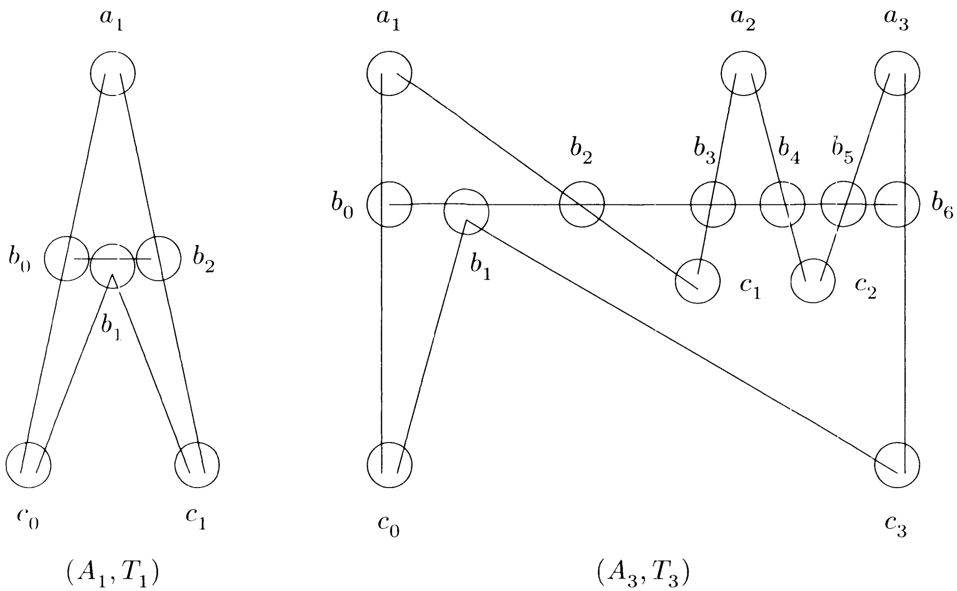
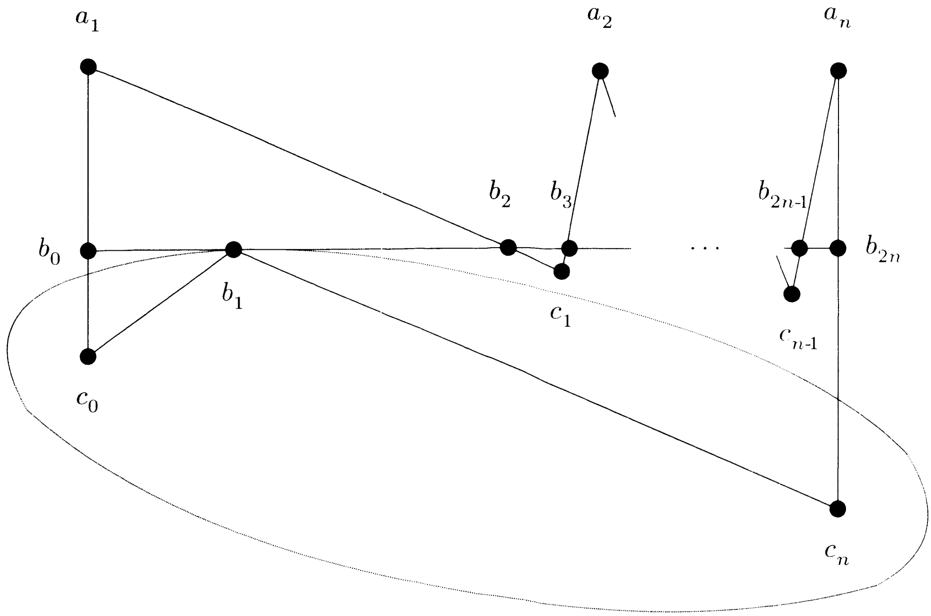


DIAGRAM 1.

It can easily be verified that any of the ternary structures (A_n, T_n) even satisfies the axioms (B0)–(B7) and (after a suitable embedding) is metrizable by the metric d inherited from the Euclidean plane, except for the distance of c_0 and c_n for which we put

$$d(c_0, c_n) = d(c_0, b_1) + d(b_1, c_n) .$$

However, in order to guarantee the strict inequalities $d(c_0, c_n) < d(c_0, x) + d(x, c_n)$ for each $x \in A_n \setminus \{b_1, c_0, c_n\}$, one has to arrange the situation in such a way that b_1 lies on the ellipse with foci c_0, c_n, and all the remaining points of A_n lie outside of the ellipse. This obviously can always be achieved – see Diagram 2.



(A_n, T_n)

DIAGRAM 2.

On the other hand, none of the betweenness spaces (A_n, T_n), $n \geq 1$, is linear. To see this, it suffices to realize that, in such a case, the members of the three couples of points (a_1, c_0), (a_1, c_1) and (c_0, c_n) have to lie in different half-planes with respect to the line $b_0 b_n$. However, c_1, c_n lie in the same half-plane. This is a contradiction. (Consult any of the Diagrams 1, 2.)

The proof will be complete once we show that omitting any point x from A_n, the ternary structure $(A_n \setminus \{x\}, T_n \cap (A_n \setminus \{x\})^3)$ thus obtained, already is embeddable into the Euclidean plane. The more, every n-element substructure of (A_n, T_n) is then embeddable into the plane. This is, however, clear, since omitting any point will break the circle $a_1 b_0 c_0 b_1 c_n b_{2n} a_n b_{2n-1} c_{n-1} \dots a_2 b_3 c_1 b_2 a_1$ which we have used for deriving the contradiction in proving the nonlinearity of the structures (A_n, T_n). □

COROLLARY. *The theory* $\mathrm{Th}\,\mathcal{L}$ *is not finitely axiomatizable.*

The structures (A_n, T_n), for $n \geq 2$, can be regarded as a kind of "forbidden patterns" for the class $\mathcal{L}$. Obviously, their occurrence as substructures can be excluded by satisfaction of some universal (even basic) formulas. Unfortunately,

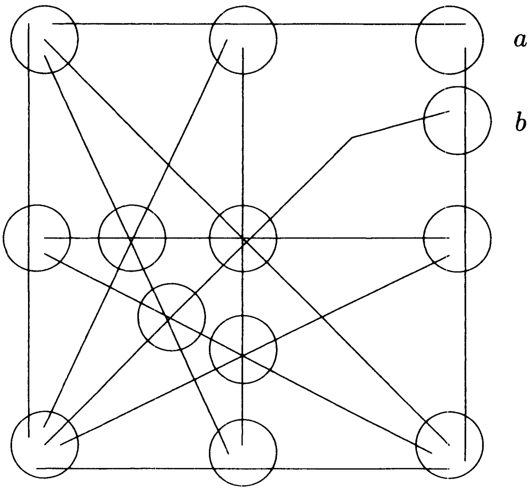


DIAGRAM 3.

they are by far not all finite nonlinear (metrizable) betweenness spaces. Several sequences of structures of such a kind, as well as some isolated ones, can be constructed and even put together and combined in diverse manners. Thus there seems to be little hope for finding some reasonable axiom schemes for the class $\mathcal{L}$ (with respect to $\mathcal{M}$), consisting of such "forbidding" formulas. This is again similar to the situation with the metrizable betweenness spaces – cf. [MnZl].

An example of a finite nonlinear metrizable betweenness space satisfying all the axioms (B0) – (B7), different from the (A_n, T_n)'s, can be seen on Diagram 3, representing (a part of) a slightly modified square (it would become a square and a linear betweenness space under the identification of a and b).

By the term "graph" we understand a first-order structure (V, E), where E is an irreflexive and symmetric binary relation on V. It is known that the first-order theory of the class of all finite graphs is hereditarily undecidable – see, e.g., [BrSn] or [ELTT].

There is a rather natural interpretation of the theory of graphs in the theory of betweenness spaces (axioms (B0) – (B4)). Namely, a point x of a betweenness space (A, T) will be called a vertex if and only if it does not lie properly between any pair of points of (A, T), and two vertices x, y are connected by an edge if and only if there is a $z \in A$, $x \neq z \neq y$, between them. The formal definition can be found in [MnZl], where we have proved that every finite graph can be obtained from a finite metrizable betweenness space in this way. However, just a brief look at that proof shows that every finite graph can be obtained from a finite *linear* betweenness space using the described interpretation (the corresponding metric was obtained by embedding the graph into the Euclidean plane). Thus in fact we have constructed a semantic embedding of the class of all finite graphs into the class $\mathcal{L}_{fin}$ and proved the following theorem in [MnZl].

THEOREM 4. *The first-order theory of the class $\mathcal{L}_{fin}$ of all finite linear betweenness spaces is hereditarily undecidable.*

COROLLARY. *Both the classes $\mathcal{L}$ of all linear betweenness spaces and $\mathcal{L}_0$ of all betweenness spaces embeddable into a vector space over $\mathbb{R}$ have hereditarily undecidable first-order theories.*

However, as already mentioned in the final part of the Introduction, we do not know whether the theories $\mathrm{Th}\,\mathcal{L}$ and $\mathrm{Th}\,\mathcal{L}_0$ coincide or not.

REFERENCES

[BrSn] BURRIS, S.—SANKAPPANAVAR, H. P.: *A Course in Universal Algebra*, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[ChK] CHANG, C. C.—KEISLER, H. J.: *Model Theory* (3rd ed.), Elsevier, North-Holland, Amsterdam, 1990.

[Cl] CLARKSON, J. A.: *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[ELTT] ERSHOV, YU. L.—LAVROV, I. A.—TAĬMANOV, A. D.—TAĬCLIN, M. A.: *Elementary theories*, Uspekhi Mat. Nauk **20** (1965), 37–108. (Russian)

[H] HODGES, W.: *Model Theory*, Cambridge Univ. Press, Cambridge, 1993.

[MnZl] MENDRIS, R.—ZLATOŠ, P.: *Axiomatization and undecidability results for metrizable betweenness relations*, Proc. Amer. Math. Soc. (To appear).

[PtSm] PITCHER, E.—SMILEY, M. F.: *Transitivities of betweenness*, Trans. Amer. Math. Soc. **52** (1942), 95–114.

[Sh] SHOENFIELD, J. R.: *Mathematical Logic*, Addison-Wesley Reading, Mass., 1967.

[Sm] SMILEY, M. F.: *A comparison of algebraic, metric, and lattice betweenness*, Bull. Amer. Math. Soc. **49** (1943), 246–252.

[T] TARSKI, A.: *A Decision Method for Elementary Algebra and Geometry*, Univ. of California Press, Berkeley, 1951.

[Vt] VAUGHT, R.: *Remarks on universal classes of relational systems*, Indag. Math. **16** (1954), 589–591.

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