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ON THE (m, n)-BASIS OF A DIGRAPH

MATÚŠ HARMINC

In the presented paper there is introduced the notion of an (m, n)-basis of a digraph (where m and n are positive integers). There is investigated the existence of an (m, n)-basis for digraphs of certain types. Some results of Richardson [6] and von Neumann and Morgenstern [5] are generalized.

Let us recall some fundamental notions. A finite directed graph $D=(V, A)$ with the set of points V and with the set of lines $A \subseteq V \times V$ with no loops or multiple lines is called a digraph. The concepts of a path, a cycle, an indegree of a point v (denoted id (v)) are used like in [4]. A transmitter is a point whose indegree is 0. The n-th power of a given digraph D is the digraph $D^{(n)}$, which has the same point set as D and a line uv is in $D^{(n)}$ if and only if there is a path in D from u to v of length $d \leqq n$ (see [4]). Throughout the paper the symbols c, d, k, m, n denote positive integers. For each set $M \subseteq V$ of points of D we denote by $H(M)$ the set consisting of terminal points of those lines that have initial points in M.

A set $S \subseteq V$ is m-independent if for no two distinct points $u, v \in S$ there exists a path of length $d \leqq m$ from u to v. A set $S \subseteq V$ is an n-cover in D if for each $v \in V-S$ there exists at least one $u \in S$ such that there exists a path of length $d \leqq n$ from u to v. A set S is an n-basis for D if it is n-independent and an n-cover for D (see Harary, Norman, Cartwright [4]). This concept is a generalization of the concept of a 1-basis (a solution) of a digraph ([1], [6]). (Some authors study the dual concept — the kernel of a digraph [2].)

Definition. *A subset S of V in a digraph $D=(V, A)$ is called an (m, n)-basis of D if*

(i) *S is m-independent, and*

(ii) *S is an n-cover in D.*

By definition of the (m, n)-basis it is clear that an (m, n)-basis is a (k, c)-basis of the same digraph for all $k \leqq m$ and $c \geqq n$.

For each positive integer n the notion of an (n, n)-basis coincides with the notion of an n-basis. In [4] it is established that, instead of studying the existence of an (n, n)-basis of a digraph D it suffices to study the existence of a 1-basis of $D^{(n)}$. We note that the situation with an (m, n)-basis in the case $m \neq n$ is rather different. Further we remark that the problem of the existence of a 1-basis for an arbitrary digraph is not solved in general (see [1]).

Theorem 1. a) *Every digraph has an* (m, n)*-basis for* $n \geqq 2m$.

b) *For each pair* (m, n), $n < 2m$, *there exists a digraph without an* (m, n)*-basis.*

Proof. The proof of a) will be established in two steps.

1) Using mathematical induction with respect to the number of points of a digraph we shall prove the theorem for $m = 1$. The digraph with one point and digraphs with two points have a $(1, 2)$-basis. Let each digraph with k points have a $(1, 2)$-basis for each $k < c$ and let a digraph D have c points. Let us take a point v and construct a digraph G generated in D by a set of points $V - \{v\} - H(\{v\})$. Let S be a $(1, 2)$-basis for G. If there exists $u \in S$ such that $v \in H(\{u\})$, then S is a $(1, 2)$-basis for D too. In the opposite case we can easily verify that $S \cup \{v\}$ is a $(1, 2)$-basis for D.

2) Now let us construct the digraph $D^{(m)}$ and denote by $S^{(m)}$ a $(1, 2)$-basis for $D^{(m)}$. We have $S^{(m)} \subseteq V$; the 1-independence in $D^{(m)}$ is equivalent to the m-independence in D and similarly the 2-cover for $D^{(m)}$ is the $2m$-cover for D. The set $S^{(m)}$ is an $(m, 2m)$-basis for D, i.e. an (m, n)-basis for D for each $n \geqq 2m$.

b) If $n < 2m$, then a digraph that consists of two cycles of length $2m + 1$ having a unique common line, has no $(m, 2m - 1)$-basis and therefore no (m, n)-basis for $n < 2m$.

Corollary 1 (Landau [4]). *In every tournament there exists a point* v *such that every point different from* v *is reachable from* v *by a path of length one or two.*

Corollary 2. *If in a digraph* D *there is no path of length* $n + 1$, *then* D *has an* (m, n)*-basis for each* m.

To prove this it is sufficient to take an $(m, 2m)$-basis S for D (such a basis exists according to Theorem 1). Since every path is of length at most n, the set S is an (m, n)-basis for D, too.

It is possible to establish stronger results than Theorem 1 for some special classes of digraphs. A digraph $D = (V, A)$ is called:

transitive, if $uv \in A$, $vw \in A$ implies $uw \in A$ for each triple of distinct points u, v, w;

acyclic, if D has no cycle;

symmetric, if for each pair of distinct point u, v the condition $uv \in A$ is equivalent to $vu \in A$;

asymmetric, if $uv \in A$ implies $vu \notin A$ for each pair of distinct points u, v.

Theorem 2. a) *Every transitive digraph has an* (m, n)*-basis for each pair* (m, n).

b) *Every acyclic digraph has an* (m, n)*-basis for* $m \leqq n$. *Let* $m > n$; *there exists an acyclic digraph having no* (m, n)*-basis.*

c) *Every symmetric digraph has an* (m, n)*-basis for each* $m \leqq n$. *Let* $m > n$; *there exists a symmetric digraph having no* (m, n)*-basis.*

Proof. a) In a transitive digraph the existence of a path from u to v is equivalent to the existence of the line uv. According to this fact and to the definition of an (m, n)-basis it is evident that the following assertion holds: A set S is an (m, n)-basis for a transitive digraph if and only if it is a 1-basis for this digraph. As a transitive digraph has a 1-basis (see [2], [4]), part a) is proved.

b) It is known (cf. [4]) that an acyclic digraph has an m-basis for every m, therefore it has an (m, n)-basis for each $n \geqq m$. The digraph consisting of points u_0, $u_1, \ldots, u_m$ and of lines $u_0u_1, u_1u_2, \ldots, u_{m-1}u_m$ has no $(m, m-1)$-basis and therefore no (m, n)-basis for $n < m$. Moreover, the following holds: Assume that D is an acyclic digraph, $n < m$ and let W be the set of transmitters of D. Then D has an (m, n)-basis iff $V = W \cup H(W) \cup H(H(W)) \cup \ldots \cup H^n(W)$. (And in this case W is the (m, n)-basis of D.)

c) If a digraph D is symmetric, the digraph $D^{(m)}$ is symmetric, too. We denote by S a 1-basis of $D^{(m)}$ (in a symmetric digraph such a basis exists, cf. Berge [2]). The set S is an m-basis for D; it is also an (m, n)-basis for D for $n \geqq m$. The digraph which consists of the points $u_0, u_1, \ldots, u_{2m}$ and of the lines $u_0u_1, u_1u_0, u_1u_2, u_2u_1, \ldots,$ $u_{2m-1}u_{2m}, u_{2m}u_{2m-1}, u_{2m}u_0, u_0u_{2m}$ has no $(m, m-1)$-basis.

Theorem 3. *For any asymmetric digraph D the following statements are equivalent.*

(i) *$V = W \cup H(W)$, where W is the set of transmitters.*

(ii) *For any pair (m, n) the digraph D has an (m, n)-basis.*

(iii) *The digraph D has a $(2, 1)$-basis.*

Proof. (i) implies (ii): W is an m-independent set for any m and it is a 1-cover. (ii) implies (iii) immediately. Let S be a $(2, 1)$-basis for the digraph D, $s \in V - W - H(W)$. If $s \notin S$, there exists $w \in S$ such that $ws \in A$. If $s \in S$, we take $w = s$. Then there exists $v \in V - S$ such that $vw \in A$. Because the set S is a 1-cover, there is $u \in V$ in S such that $uv \in A$; since D is asymmetric we have $u \neq w$. This is a contradiction, since S is a 2-independent set.

Corollary. *Every asymmetric digraph with no transmitter has no $(m, 1)$-basis for each $m > 1$.*

Proof. If $m > 1$, then every $(m, 1)$-basis is a $(2, 1)$-basis. In an asymmetric digraph we have a contradiction between (i) of Theorem 3 and the assumption of the corollary.

Theorem 4. *Let D have an (m, n)-basis. Let C be a cycle such that* $\operatorname{id}(v) = 1$ *for each $v \in C$. Let d be the length of C. Then*

$$d \in \langle 2; n+1 \rangle \cup \bigcup_{c \geqq 2} \langle c(m+1);\ c(n+1) \rangle .$$

Proof. The cycle C as described in the theorem must have the property that each point of C is covered only by points from C. We denote by c the number of those points of the cycle, which are contained in an (m, n)-basis S. In the case $c = 1$ we obtain that for the length d of the cycle C we have $d \in \langle 2; n+1 \rangle$. Let $c \geqq 2$. If $d < c(m+1)$, the set S is not m-independent. If $d > c(n+1)$, it is not an n-cover. Thus $d \in \langle c(m+1); c(n+1) \rangle$.

The opposite assertion is not valid in general: A digraph consisting of the points u_1, u_2, u_3, v_1, v_2, v_3 and of the lines u_1u_2, u_2u_3, u_3u_1, u_1v_1, u_2v_2, u_3v_3 has $d \in \langle 2; n+1 \rangle$ for $m = 3$, $n = 2$, but has no (3, 2)-basis.

After this paper has been submitted I have found that the step 1 (in the part a) of Theorem 1 was proved already by V. Chvatal and L. Lovasz (Hypergraphs Seminar, Lecture Notes in Mathematics, 411, Springer-Verlag, Berlin 1974),

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ОБ (m, n)-БАЗЕ ОРИЕНТИРОВАННОГО ГРАФА

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Резюме

В работе определяется понятие (m, n)-базы ориентированного графа. Изучается существование (m, n)-базы для всех пар натуральных чисел m, n. Доказаны теоремы о необходимых и достаточных условиях существованиа (m, n)-базы графов определенных классов.