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DIRECT VARIATIONAL METHODS IN NONREFLEXIVE SPACES

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In the present paper we investigate the variational problems of the type of a nonparametric minimal surface by means of direct methods.

Let us consider the nonparametric minimal surface problem. We are looking for the minimum of the functional

$$J(u, \Omega) = \int_\Omega (1 + |\nabla u|^2)^{1/2}\, dx$$

on the set of functions with a prescribed boundary condition $u = \varphi$ on $\partial\Omega$.

It is natural to look in the case of this problem for a weak solution in the Sobolev space $W_1^1(\Omega)$ and to take the boundary condition φ from $L_1(\partial\Omega)$. But it is clear that we cannot use here the direct methods because of the nonreflexivity of $W_1^1(\Omega)$. This difficulty can be removed if we extend the functional J to the larger space $W_\mu^1 \supset W_1^1$, which has a compact ball in some weak topology.

The following conditions must be satisfied if we want to use the direct methods for a larger space $W_\mu^1 \supset W_1^1$:

i) it is possible to choose a convergent subsequence from the minimizing sequence in some topology (weak topology in W_μ^1);
ii) the limit element of this subsequence satisfies the boundary condition;
iii) the extended functional is lower semicontinuous with respect to the topology mentioned in point i).

If $u_n \in W_1^1$, $n = 1, 2, \ldots$ is a minimizing sequence of our problem, then $\|u_n\|_{W_1^1}$ is uniformly bounded (this is the only a priori information) and hence the derivative $\frac{\partial u_n}{\partial x_i}$ is a bounded sequence in $L_1(\Omega)$. A space with a weakly compact ball, which contains the space $L_1(\Omega)$, is the space $L_\mu(\bar\Omega)$ of all bounded Borel measures on $\bar\Omega$ with a usually weak topology. We define a weak convergence in W_μ^1 as the weak convergence of the functions together with their derivatives in $L_\mu(\bar\Omega)$ and we define W_μ^1 as the closure of $W_1^1(\Omega)$ in this convergence. But there is a problem to guarantee point ii) concerning the notion of the traces for the elements of W^1_μ. Consider the following example. Let $\{u_n\}$ be the sequence

$$u_n(x)=\begin{cases} 0 & \text{for } 0\leq x\leq 1-\dfrac{1}{n} \\ 1-n(1-x) & \text{for } 1-\dfrac{1}{n}\leq x\leq 1 \end{cases}$$

Thus $u_n(0)=0$ and $u_n(1)=1$ *for* $n=1, 2, \ldots$. The function $u\equiv 0$ is the limit of $\{u_n\}$ (in the weak convergence as well as in each reasonable convergence). The trace of the function $u\equiv 0$ is zero (in each reasonable definition of the trace). In what sense can point ii) be satisfied? Let us consider the derivatives of the functions u_n. It is easy to see that the sequence $\left\{\dfrac{\mathrm{d}u_n}{\mathrm{d}x}\right\}$ convereges to the Dirac measure in the point 1 (in a weak convergence in $L_{\mu^{\cdot}}(\langle 0, 1\rangle)$). Thus, the Dirac measure forms some side (in the point 1) of the element $u\equiv 0$. This fact allows us to define the trace in point 1 (of the limit element) as the usual trace of the function $u\equiv 0$ (the socalled inner trace) plus the magnitude of the side in point 1. Now it is clear that the „function" from W^1_μ is a generalized function; to determine this "function" we must define the values of this function in Ω and at the same time its derivative $\dfrac{\partial u}{\partial x_i}$ (which are generally measures) on $\bar{\Omega}$. In the domain Ω the derivatives of $\hat{u}\in W^1_\mu$ are determined as the distributive derivatives of u but on $\partial\Omega$ these derivative are independent of u. The trace of an element of W^1_μ can then be defined by Green's theorem. This trace clearly depends also on the derivatives of u.

The space W^1_μ (and its generalization W^k_μ) is defined and investigated in [1].

To guarantee point iii) we assume that the functional J on W^1_1 is weakly lower semicontinuous and coercive. We extend the functional J on W^1_μ so that the condition iii) will be satisfied. The extension F (for $u\in W^1_\mu$) of J is defined by

$$F(u)=\inf \lim_{n\to\infty} J(u_n)$$

where we take the infimum over all sequence $u_n\in W^1_1$ such that $u_n\rightharpoonup u$ in W^1_μ. It will be proved that this functional is weakly lower semicontinuous and coercive on the space W^1_μ. (see § 1). In § 2 we prove the solvability of our variational problem (with the functional F) in the space W^1_μ. In § 3 we consider the functional J in the form

$$J(u)=\int_\Omega f(x, D^i u)\,\mathrm{d}x\,, \qquad |i|\leq k,\ u\in W^k_1\,,$$

where $f(x, \xi)$ is convex in the vector ξ, $\Omega\subset E^N$ is a bounded domain with the boundary $\partial\Omega$ from the class C^1.

In § 4 we study the uniqueness (in some special cases) of the generalized solution of our variational problem. More detailed results can be found in [2], but in special cases of the function f and for $k=1$.

In paper [3] it is proved that in some special cases of the functional J there exists no weak solution in W_1^1 of our variational problem.

Thus, the definition of the spaces W_μ^k and the extension F of the functional J is substantial.

For the case of minimal surfaces stronger results have been obtained in [5], [6], see also [7].

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§ 1. The spaces W_μ^k

In this paragraf we recapitulate for the reader some basic definitions and theorem from [1], which will be used latter.

Let $L_\mu(E)$ be the space of all σ-additive bounded Borel measures defined on a compact set $E \subset E_N$ with the norm

$$\|\alpha\|_{L_\mu(E)} = |\alpha|(E),$$

($|\alpha|$ is the total variation of the measure α).

Asually, we denote

$$\alpha_n \rightharpoonup \alpha \text{ in } L_\mu(E), \text{ iff } \int_E \varphi \, d\alpha_n \to \int_E \varphi \, d\alpha \text{ for all } \varphi \in C(E).$$

We shall identify each absolutely continuous measure $\alpha \in L_\mu(\bar{\Omega})$ (with respect to the N-dimensional Lebesgue measure) with its density and each absolutely continuous measure $\beta \in L_\mu(\partial\Omega)$ (with respect to the $(N-1)$-dimensional Hausdorff measure dS on $\partial\Omega$) with its density.

Definition A. *$W_\mu^1(\Omega)$ is the space of all $(N+1)$-tuples $(u, \alpha_1, \dots, \alpha_N)$ for which*

i) $u \in L_1(\Omega)$, $\alpha_1, \dots, \alpha_N \in L_\mu(\bar{\Omega})$,

ii) *there exists a measure $\beta \in L_\mu(\partial\Omega)$ such that*

$$\int_{\partial\Omega} \varphi v_i \, d\beta = \int_\Omega u\varphi_{x_i} \, dx + \int_\Omega \varphi \, d\alpha_i, \qquad i = 1, \dots, N$$

for all $\varphi \in C^1(\bar{\Omega})$ (where $v = (v_1, \dots, v_N)$ *is the exterior normal to* $\partial\Omega$).

The measure β which is clearly uniquely determined by (u, α_i) will be called the trace of the element (u, α_i).

The space $\mathring{W}_\mu^1(\bar{\Omega})$ is the space of all elements from $W_\mu^1(\bar{\Omega})$ with the trace $\beta = 0$.

Definition B. *Let $e_m = (0, \dots, 0, 1, 0, \dots 0)$ be the N-dimensional vector with* 1 *on the m-th place and let $\varkappa$ be the number of all multiindices i for which $|i| = k$.*

$W_\mu^k(\bar\Omega)$ *is the space of all* $(\varkappa+1)$*-tuples* $(u, \alpha_i)_{|i|=k}$ *such that*

i) $u \in W_1^{k-1}(\Omega)$, $\alpha_i \in L_\mu(\bar\Omega)$, $|i|=k$,

ii) *the element*

$$(D^i u, \alpha_{i+e_1}, \ldots, \alpha_{i+e_m})$$

belongs to the space $W_\mu^1(\bar\Omega)$ *for each mutliindex* i, $|i|=k-1$. $\mathring{W}_\mu^k(\bar\Omega)$ *is the space of all* $(u, \alpha_i) \in W_\mu^k(\bar\Omega)$ *for which* $u \in \mathring{W}_1^{k-1}(\Omega)$ *and*

$$(D^i u, \alpha_{i+e_m}) \in \mathring{W}_\mu^1(\bar\Omega)$$

for all i, $|i|=k-1$ *and* $m=1, \ldots, N$.

The norm in W_μ^k is defined by

$$\|(u, \alpha_i)\|_{W_\mu{}^k} = \|u\|_{W_1{}^{k-1}} + \sum_{|i|=k} \|\alpha_i\|_{L_\mu}.$$

We shall define in $\mathring{W}_\mu^k$ the pseudonorm as well:

$$\|(u, \alpha_i)\|'_{\mathring{W}_\mu{}^k} = \sum_{|i|=k} \|\alpha_i\|_{L_\mu}.$$

The elements from $W_\mu^k(\bar\Omega)$ will be called the functions and we shall denote $\hat u = (u, \alpha_i) = (u, \alpha_i)_{|i|=k}$. We shall define the weak convergence in $W_\mu^k(\bar\Omega)$ by

$$(u_n, \alpha_{ni}) \rightharpoonup (u, \alpha_i) \quad \text{in } W_\mu^k(\bar\Omega) \text{ if}$$

$$D^i u_n \rightharpoonup D^i u \quad \text{in } L_\mu(\bar\Omega), \quad \text{for all } |i| \le k-1 \text{ and}$$

$$\alpha_{ni} \rightharpoonup \alpha_i \quad \text{in } L_\mu(\bar\Omega), \quad \text{for all } |i|=k$$

Theorem A. *The unit ball in* $W_\mu^k(\bar\Omega)$ *is weakly compact. The space* $\mathring{W}_\mu^k(\bar\Omega)$ *is weakly closed in* $W_\mu^k(\bar\Omega)$ *and hence the unit ball in* $\mathring{W}_\mu^k(\bar\Omega)$ *is weakly compact as well.*

Theorem B. *The imbedding* $W_\mu^k(\bar\Omega)$ *into* $W_q^{k-1}(\Omega)$:

$$(u, \alpha_i)_{|i|=k} \in W_\mu^k \to u \in W_q^{k-1}$$

is continuous for $q \le \dfrac{N}{N-1}$ *and compact for* $q < \dfrac{N}{N-1}$.

Theorem C. *For all* $\hat u \in W_\mu^k$ *(respectively* $\hat u \in \mathring{W}_\mu^k$*) there exists a sequence* $u_n \in W_1^k$ $(u_n \in \mathring{W}_1^k)$ *such that*

$$u_n \rightharpoonup u \text{ in } W_\mu^k, \; \|u_n\|_{W_1{}^k} \le \|\hat u\|_{W_\mu{}^k} \qquad n=1, 2, \ldots.$$

w_1^k

Remark. The assertion of this theorem is proved in [1] only for $k=1$ but the proof for $k>1$ is exactly the same as for $k=1$.

Theorem D. *The pseudonorm* $\|u\|'_{W_\mu^k}$ *is the equivalent norm in the space* $\mathring{W}_\mu^k$.

In §4. some further theorems about the space $W_\mu^k(\bar{\Omega})$, will be needed.

Definition C. *Let* $E \subset E_N$ *be a compact subset,* $\alpha \in L_\mu(E)$, $\varphi \in C(E)$, *then we define* $\varphi . \alpha$ *usually:*

$$\int_E \psi \, \mathrm{d}(\varphi\alpha) = \int_E \psi\varphi \, \mathrm{d}\alpha, \quad \text{for all } \psi \in C(E)$$

Let $(u, \alpha_i) \in W_\mu^k(\bar{\Omega})$. *The side of* (u, α_i) *is the measure*

$$\alpha_\nu = \nu_1 \alpha_1|_{\partial\Omega} + \ldots + \nu_N \alpha_N|_{\partial\Omega} \in L_\mu(\partial\Omega),$$

where $\alpha_i|_{\partial\Omega}$ *is the restriction* α_i *on* $\partial\Omega$.

We shall define the measures $\bar{\alpha}_1, \ldots, \bar{\alpha}_N$ by

$$\left.\begin{array}{l} \bar{\alpha}_i = 0 \quad \text{on } \partial\Omega \\ \bar{\alpha}_i = \alpha_i \quad \text{on } \Omega \end{array}\right\} \quad i = 1, \ldots, N.$$

The inner trace β^0 of (u, α_i) is defined as the trace of the function $(u, \bar{\alpha}_i)$. (In [1] it is proved that $(u, \bar{\alpha}_i)$ belongs really to W_μ^1).

Remark. It is easy to see that $\alpha_i|_\Omega$ and hence also β^0 are uniquely determined by the function u. On the other hand the side α_ν of (u, α_i) and the trace β are independent of u. The measures $\alpha_i|_{\partial\Omega}$ are independent of u but they must satisfy the following conditions:

Theorem E. *Let* β, β^0 *and* α_ν *be the trace, the inner trace and the side of* (u, α_i). *Then there holds*

$$\alpha_i|_{\partial\Omega} = \nu_i \alpha_\nu, \qquad i = 1, \ldots, N,$$

$$\beta = \beta^0 + \alpha_\nu$$

and the inner trace β^0 *belongs to* $L_1(\partial\Omega)$.

Let

$$K^h(x) = \frac{1}{\varkappa h^N} \mathrm{e}^{\frac{|x|^2}{|x|^2 - h^2}} \quad \text{for } |x| < h$$

$$K^h(x) = 0 \quad \text{for } |x| \geqslant h$$

where $\varkappa$ is chosen in order that $\int_{E_N} K^h(x) \, \mathrm{d}x = 1$ holds.

Theorem F. *Suppose* $(u, \alpha_i) \in W_\mu^1(\Omega)$ *such that the trace* β *of* (u, α_i) *belongs to* $L_1(\partial\Omega)$. *Let* $\Omega^* \supset \bar{\Omega}$ *be a bounded domain with a sufficiently smooth boundary.*

Then there exists the function $(u^*, \alpha_i^*) \in W_\mu^1(\Omega^*)$, *which is „the extension" of* (u, α_i) *in the following sense:*

$$\begin{aligned} u^* &= u && \text{on } \Omega \\ \alpha_i^* &= \alpha_i && \text{on } \Omega \\ \alpha_i^* &= 2\alpha_i && \text{on } \partial\Omega \end{aligned} \Bigg\} \quad i = 1, \ldots, N$$

Then for the function (for $h > 0$ *small)*

$$u_h(x) = \int_{\Omega^*} K^h(x - y) u^*(y) \, dy, \qquad x \in \Omega$$

there holds $u_h \xrightarrow[h \to 0]{} (u, \alpha_i)$ *in* $W_\mu^1(\bar{\Omega})$ *and*

$$\|u_h\|_{W_1^1(\Omega)} \xrightarrow[h \to 0]{} \|(u, \alpha_i)\|_{W_\mu^1(\bar{\Omega})}.$$

Very often we shall write $\|u\|$, $\|u\|'$ istead of $\|u\|_{W_\mu^k}$, $\|u\|_{\mathring{W}_\mu^k}$ respectively, for $u \in W_\mu^k$ (also for $u \in W_1^k(\Omega)$).

§ 2. The abstract calculus of variations over the space W_1^k and W_μ^k

Let $J(u)$ be a functional on the space $W_1^k(\Omega)$. We shall try to solve the following variational problem: to find the minimum of the functional J on the set of all the functions u satisfying the boundary condition $u - u_0 \in \mathring{W}_1^k$, where the function $u_0 \in W_1^k$ is fixed.

We shall consider only the functionals J satisfying the following conditions:

A) the functional J is weakly lower semicontinuous in this sense:

$$\text{if } u_n, u \in W_1^k, \text{ and } u_n \rightharpoonup u \text{ (in } W_\mu^k\text{), then } J(u) \leq \varliminf_{n \to \infty} J(u_n),$$

B) the functional J is coercive, i. e.

$$J(u) \geq \vartheta(\|u\|'_{W_1^k}),$$

where $\vartheta(t)$ is a nondecreasing function in $\langle 0, \infty)$,

$$\vartheta(t) \to \infty \text{ for } t \to \infty.$$

C) the functional J satisfies the condition

$$J(u) \leq C(1 + \|u\|_{W_1^k}).$$

The unit ball of the space W_1^k is not weakly compact, hence, if we want to use direct methods, we must consider our variational problem on the space $W_\mu^k(\bar{\Omega})$.

First, we must suitably extend the definition of the functional J on the space W_μ^k.

Definition 1. *If $\hat{u} \in W_\mu^k$ we define*

$$F(\hat{u}) = F(\hat{u}, \bar{\Omega}) = \inf \varliminf_{n\to\infty} J(u_n, \Omega),$$

where the infimum is taken over all the sequences $\{u_n\} \in W_1^k$ such that $u_n \rightharpoonup \hat{u}$ in W_μ^k.

Lemma 1.

i) *Suppose A),*
then $F(u) = J(u)$ for all $u \in W_1^k$.
ii) *Suppose B),*
then $F(\hat{u}) \geq \vartheta(\|u\|'_{W_\mu^k})$ for all $\hat{u} \in W_\mu^k$.
iii) *Suppose C),*
then $F(\hat{u}) \leq C(1 + \|\hat{u}\|_{W_\mu^k})$ for all $\hat{u} \in W_\mu^k$.

Proof. Assertion i) is evident.
ii) Suppose $u_n \in W_1^k$, $u_n \rightharpoonup \hat{u}$,

then $\|\hat{u}\|' \leq \varliminf_{n\to\infty} \|u_n\|'$

and $\varliminf_{n\to\infty} J(u_n) \geq \varliminf_{n\to\infty} \vartheta(\|\hat{u}\|')$,

hence the desired inequality must hold for $F(\hat{u})$.
iii) From Theorem C it follows that there exist functions $u_n \in W_1^k$ such that

$u_n \rightharpoonup \hat{u}$, $\|u_n\| \leq C\|\hat{u}\|$,

hence $F(\hat{u}) \leq \varliminf J(u_n) \leq | C(1 + \varliminf \|u_n\|) \leq C(1 + \|\hat{u}\|)$.

Remark. It is clear now that the value of $F(\hat{u})$ is finite for all $\hat{u} \in W_\mu^k$.

Lemma 2. *Let us denote $K_R = \{\hat{u} \in W_\mu^k;\ \|\hat{u}\| \leq R\}$. There exists a functional $\varrho(\hat{u})$ on $W_\mu^k(\varrho^*(x, y) = \varrho(x - y)$ is a metric) such that for $\hat{u}, \hat{u}_1, \hat{u}_2, \ldots \in K_R$ there holds*

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } W_\mu^k \text{ iff } \varrho(u_n - u) \to 0.$$

Proof. For $\hat{u} = (u, \alpha_i)_{|i|=k} \in W_\mu^k$ and for $\varphi = \{\varphi_i;\ |i| \leq k, \varphi_i \in C(\bar{\Omega})\}$ let us denote

$$\langle \hat{u}, \varphi \rangle = \sum_{|i| \leq k-1} \int_\Omega D^i u.\ \varphi_i \,\mathrm{d}x + \sum_{|i|=k} \int_\Omega \varphi_i \,\mathrm{d}\alpha_i.$$

Suppose that $\{\varphi^{(m)}\}_{m=1}^\infty$ is dense in $[C(\bar{\Omega})]^{\varkappa'}$ (where $\varkappa'$ is the number of all multiindices i, $|i| \leq k$).

Let us denote

$$\varrho(\hat{u}) = \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{|\langle \hat{u}, \varphi^{(m)}\rangle}{1 + |\langle u, \varphi^{(m)}\rangle|}.$$

If $\varrho(\hat{u}_n) \to 0$, then $\langle \hat{u}_n, \varphi^{(m)}\rangle \xrightarrow[n\to\infty]{} 0$ for all m and because $\hat{u}_n \in K_R$, we have $\hat{u}_n \rightharpoonup 0$.

Suppose $\hat{u}_n \rightharpoonup 0$, $\hat{u}_n \in K_R$, then $\langle \hat{u}_n, \varphi^{(m)}\rangle \xrightarrow[n\to\infty]{} 0$ for all m and by the limit process we obtain $\varrho(\hat{u}_n) \to 0$.

Lemma 3. *Suppose B). Then for all $\hat{u} \in W_\mu^k$ there exists a sequence of functions $u_n \in W_1^k$, $n = 1, 2, \ldots$, such that*

$$u_n \rightharpoonup \hat{u} \text{ in } W_\mu^k \text{ and } J(u_n) \to F(\hat{u}).$$

Proof. By Definition 1 there exists functions $w_k^n \in W_1^k$, $n, k = 1, 2, \ldots$ such that $w_k^n \xrightarrow[k\to\infty]{} \hat{u}$ in W_μ^k and

$$J(w_k^n) < F(\hat{u}) + \frac{1}{n} \quad \text{for all } n, k.$$

From B) we obtain

$$\|w_k^n\|' \leq R_1 \quad \text{for all } n, k.$$

From $w_k^n \xrightarrow[k\to\infty]{} \hat{u}$, by Theorem B on imbedding, we have

$$w_k^n \xrightarrow[k\to\infty]{} u \text{ in } W_1^{k-1} \quad \text{for all } n,$$

where $\hat{u} = (u, \alpha_i)$.

Hence there exists a sequence of integers $k_1, k_2, \ldots$ such that

$$\|w_k^n\|_{W_1^{k-1}} \leq \|u\|_{W_1^{k-1}} + 1 \quad \text{for all } n, k \geq k_n.$$

From this we obtain the estimate

$$(1) \qquad \|w_k^n\|_{W_1^k} \leq R \quad \text{for all } n, k \geq k_n.$$

Hence there holds

$$\varrho(w_k^n - u) \xrightarrow[k\to\infty]{} 0 \quad \text{for all } n,$$

where ϱ is the functional from Lemma 2.

For all n there exists $l(n)$ such that for $u_n = w_{l(n)}^n$ there holds

$$\varrho(u_n - u) < \frac{1}{n}, \; J(u_n) < F(\hat{u}) + \frac{1}{n}.$$

From (1) we obtain

$$u_n \rightharpoonup \hat{u}, \ \overline{\lim}\, J(u_n) \leq F(\hat{u}).$$

The opposite inequality $\underline{\lim}\, J(u_n) \geq F(\hat{u})$ follows immediately from Definition 1.

Theorem 1. *Suppose B). Then F is weakly lower semicontinuous on the space W^k_μ, i. e., if*

$$\hat{u}_n, \hat{u} \in W^k_\mu,\ \hat{u}_n \rightharpoonup \hat{u} \text{ in } W^k_\mu \text{ then } F(\hat{u}) \leq \lim F(\hat{u}_n).$$

Proof. Suppose $\hat{u}$, $\hat{u}_n \in W^k_\mu$, $\hat{u}_n \rightharpoonup \hat{u}$.

By Definition 1 there exist functions $w^n_k \in W^k_1$ $(n, k = 1, 2, \dots)$ such that

$$w^n_k \xrightarrow[k\to\infty]{} u_n, \quad J(w^n_k) < F(\hat{u}_n) + \frac{1}{n}, \quad \text{for all } n, k.$$

First, we must estimate the norms $\|w^n_k\|_{W_1{}^k}$.

There is no loss of generality in assuming that $F(\hat{u}_n) \leq R_1$ (for we want to prove the inequality $F(\hat{u}) \leq \underline{\lim}\, F(u_n)$). Hence $J(w^n_k) \leq R_1 + 1$ for all n, k.

From B) we obtain $\|w^n_k\|' \leq R_2$.

Since $\hat{u}_n \rightharpoonup \hat{u}$, we have by Theorem B that

$$u_n \rightarrow u \text{ in } W^{k-1}_1$$

(where $\hat{u}_n = (u_n, \alpha_{ni})$, $\hat{u} = (u, \alpha_i)$).

Hence $\|u_n\|_{W_1{}^{k-1}} \leq R_3$ for al n.

From $w^n_k \xrightarrow[k\to\infty]{} \hat{u}_n$ we obtain by Theorem B on imbedding

$$w^n_k \xrightarrow[k\to\infty]{} u_n \text{ in } W^{k-1}_1,$$

hence

$$\|w^n_k\|_{W_1{}^{k-1}} \leq R_3 + 1 \quad \text{for all } n, k \geq k_n,$$

where $\{k_n\}$ is a suitable sequence of positive integers.

Hence we have

$$\|w^n_k\|_{W_1{}^k} \leq R \quad \text{for all } n, k \geq k_n.$$

With the functional ϱ from Lemma 2 there holds

$$\varrho(\hat{u}_n - \hat{u}) \rightarrow 0, \quad \varrho(w^n_k - \hat{u}_n) \xrightarrow[k\to\infty]{} 0 \quad \text{for all } n.$$

For all n there exists $e(n)$ such that for $w_n = w^n_{e(n)}$ there holds

$$\varrho(w_n - \hat{u}_n) < \frac{1}{n}, \quad J(w_n) \leq F(\hat{u}_n) + \frac{1}{n}$$

and hence

$$\varrho(w_n - \hat{u}) \leq \varrho(\hat{u}_n - \hat{u}) + \varrho(w_n - \hat{u}_n) \xrightarrow[n \to \infty]{} 0.$$

From $\|w_n\|_{W_1^k} \leq R$ it follows $w_n \rightharpoonup \hat{u}$, and

$$F(\hat{u}) \leq \underline{\lim}\, J(w_n) \leq \underline{\lim} \left| F(\hat{u}_n) + \frac{1}{n} \right| = \underline{\lim}\, F(\hat{u}_n).$$

Theorem 2. *Suppose B). Let $\hat{u}_0 \in W^k_\mu$.*
Then there exists $\hat{u} \in W^k_\mu$ such that

$$F(\hat{u}) = \min_{\hat{v} \in \hat{u}_0 + \mathring{W}_k} F(\hat{v}), \quad \hat{u} \in \hat{u}_0 + \mathring{W}^k_\mu.$$

Proof. We can find a minimizing sequence $\{\hat{u}_n\}$. From B) we obtain $\|\hat{u}_n\|' \leq R_1$ for all n and by Theorem D it is clear that

$$\|\hat{u}_n\|_{W_\mu^k} \leq R.$$

The ball K_R is weakly compact by Theorem A and it is sufficient to use Theorem 1.

Remark. Choose a boundary condition $u_0 \in W^k_1$. Let $\hat{u}$ be the solution from Theorem 2. The question is whether

$$F(\hat{u}) = \inf_{u \in u_0 + \mathring{W}_k} J(u).$$

We must suppose A) (in order that $F(u) = J(u)$ on W^k_1), but it is not clear if it is sufficient. This is a consequence of the definition of F. In this definition we want only $u_n \in W^k_1$, $u_n \rightharpoonup \hat{u}$ and we do not require $u_n - \hat{u} \in \mathring{W}^k_\mu$. This is a reason why we shall define a new functional F_1.

Definition 2. *Let us denote*

$$\tilde{W}^k_\mu = \{\hat{u} \in W^k_\mu;\ \textit{there exists}\ v \in W^k_1,\ \hat{u} - v \in \mathring{W}^k_\mu\}.$$

The space $\tilde{W}^k_\mu$ is thus the space of functions from W^k_μ which have the same traces (i. e. the traces of derivatives up to the $(k-1)$-th order) as some function from W^k_1.

In the case of $k = 1$ the space $\tilde{W}^1_\mu$ is the space of functions, the traces of which are from $L_1(\partial\Omega)$.

Definition 3. *Let us define for* $\hat{u} \in \tilde{W}_\mu^k$

$$F_1(\hat{u}) = \inf \{\underline{\lim}_{n\to\infty} J(u_n);\ \text{for all } u_n \in W_1^k,\ u_n \rightharpoonup \hat{u},\ u_n - u \in \mathring{W}_\mu^k\}$$

Clearly $F_1(\hat{u}) \geq F(\hat{u})$. If $\hat{u} \notin \tilde{W}_\mu^k$, then there exists no sequence with the desired properties and the definition has no sense.

Lemma 4.

i) A) *implies* $F_1(u) = J(u)$ *for all* $u \in W_1^k$.

ii) B) *implies* $F_1(\hat{u}) \geq \vartheta(\|\hat{u}\|')$ *for all* $u \in \tilde{W}_\mu^k$.

iii) C) *implies* $F_1(\hat{u}) \leq C(1 + \|\hat{u}\|)$.

Lemma 5. *Suppose B). Then for all* $\hat{u} \in \tilde{W}_\mu^k$ *there exist functions* $u_n \in W_1^k$ *such that*

$$u_n - \hat{u} \in \mathring{W}_\mu^k,\ u_n \rightharpoonup \hat{u},\ J(u_n) \to F_1(\hat{u}).$$

Theorem 3. *Suppose B) and* $\hat{u}_n$, $\hat{u} \in \tilde{W}_\mu^k$, $\hat{u}_n - \hat{u} \in \mathring{W}_\mu^k$, $n = 1, 2, \ldots$ *Then from* $\hat{u}_n \rightharpoonup \hat{u}$ *it follows that*

$$F_1(\hat{u}) \leq \underline{\lim}_{n\to\infty} F_1(\hat{u}_n).$$

Theorem 4. *Suppose B) and let* $u_0 \in W_1^k$.
Then there exists $\hat{u} \in W_\mu^k$ *such that*

$$F_1(\hat{u}) = \min_{\hat{v} \in u_0 + \mathring{W}_\mu^k} F_1(\hat{v}),\ \hat{u} \in u_0 + \mathring{W}_\mu^k.$$

Moreover,

$$F_1(\hat{u}) = \inf_{v \in u_0 + \mathring{W}_k} J(v).$$

Proof. The proofs are the same as for the functional F. The only difference is that the boundary condition is satisfied for all the considered sequences. This allows us to simplify the parts of proofs in which we prove inequalities of the type $\|w_k^n\|_{W_1^k} \leq R$. Here it is sufficient to use Theorem D on equivalents norms. In the proof of iii) from Lemma 4 we must of course use Theorem C.

Remark. There is now a new problem if $F = F_1$ holds on $\tilde{W}_\mu^k$. In [2] it will be proved that $F = F_1$ on $\tilde{W}_\mu^1$ in the case of $k = 1$ if the functional J has the form

$$J(u) = \int_\Omega f\left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right) \mathrm{d}x,$$

where f is a continuous nonnegative convex function for which there holds $f(p) \leq C(1+|p|)$, $p \in E_N$.

Let us suppose A), B), C).

The assertion $F=F_1$ on $\tilde{W}^1_\mu$ has the following important consequences:

1) $\inf\limits_{u \in u_0 + \overset{o}{W}{}^1_\mu} F(u) = \inf\limits_{u \in u_0 + \overset{o}{W}{}^1_1}$, $u_0 \in W^1_1$ fixed.

2) If $u \in W^1_1$ is the solution of our variational problem over the space W^1_1:

$$J(u) = \min_{v \in u_0 + \overset{o}{W}{}^1_1} J(v),\ u \in u_0 + \overset{o}{W}{}^1_1,\ u_0 \in W^1_1 \text{ fixed},$$

then u is the solution of our variational problem over the space W^1_μ as well:

$$J(u) = \min_{\hat{v} \in u_0 + \overset{o}{W}{}^1_\mu} F(\hat{v}).$$

§ 3. The variational problem of the type of the minimal surface

Nov we shall consider the functional J in the form:

$$J(u) = \int_\Omega f(x, D^i u)\, dx,\ |i| \leq k,\ u \in W^k_1. \tag{2}$$

We want to find some sufficient conditions for the functional J to satisfy the conditions A), B), C) from the last paragraph.

A') 1) $f(x, \xi_i)$ ($|i| \leq k$) is a continuous function in all variables and a nonnegative one

2) for all x, ξ_0 the function f is convex in the variables ξ_i (for $0 < |i| \leq k$)

3) if $k > 1$, *suppose that* $f(x, \xi_i)$ is independent of ξ_0 and that there holds

$$|f(x, \xi_i) - f(y, \xi_i)| \leq \lambda(|x-y|)\,(1 + f(x, \xi_i)),$$

where $\lambda(t)$ is a continuous function at 0 and $\lim\limits_{t \to 0} \lambda(t) = 0$.

B') $f(x, \xi_i) \geq C_1 \sum\limits_{|i|=k} |\xi_i| - C_2$ for al x, ξ_i, where C_1, $C_2 > 0$.

C') $f(x, \xi_i) \leq C(1 + \sum\limits_{|i| \leq k} |\xi_i|)$ for all x, ξ_i.

It is clear that B') implies B) and C') implies C). We prove that A') implies A).

In the case of $k = 1$ the assertion is a consequence of the theorem from [4].

In the case of $k > 1$ it is possible (if we suppose A')) to prove the same Theorem as in [4] by the same method.

Now we can apply the results from §2 to the functional (2). Hence we have the existence Theorem for the extended functional F over the space W_μ^k for all domains Ω with a sufficiently smooth boundary and for all boundary conditions $u_0 \in W_1^k$.

§4. The uniqueness and the example

In this paragraph we whall assume that $k=1$ and that the functional J has the form

$$J(u, \Omega) = \int_\Omega f\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right) \mathrm{d}x,$$

where the function $f(x, \xi_0, \xi_1, \ldots, \xi_N)$ satisfy the conditions A′), B′), C′) from §3.

In [4] J. Serrin defined another extension $\bar{F}(u, \Omega)$ of the functional $J(u, \Omega)$.

Susspose that $\{\Omega_n\}$ is a sequence of open sets $\Omega_n \subset \Omega$ such that for all $K \subset \Omega$ compact there exist n_0 such that $\Omega_n \supset K$ for all $n \geqslant n_0$.

Suppose that $u \in L_{1,loc}(\Omega)$ (the space of local intergrable functions) is fixed and that $u_1, u_2, \ldots \in C^k(\Omega)$ are such functions that $u_n \to u$ in the space $L_{1,loc}(\Omega)$. Then we shall define

$$\bar{F}(u, \Omega) = \inf \varliminf_{n \to \infty} J(u_n, \Omega_n),$$

where the infimum is taken over all the sequences $\{u_n\}$, $\{\Omega_n\}$ with the properties required above.

From Theorem 1 it follows immediately that for $\hat{u} = (u, \alpha_i) \in W_\mu^1(\bar{\Omega})$ there holds $\bar{F}(u, \Omega) \leqslant F(\hat{u}, \bar{\Omega})$.

Theorem 5. *Let $\hat{u} = (u, \alpha_i)$ be a function from the space $W_\mu^1(\bar{\Omega})$ the side α_ν of which (see §1.) is equal to zero* (i. e. $|\alpha_\nu|(\partial\Omega) = 0$). *Then there holds*

$$F(\hat{u}, \bar{\Omega}) = \bar{F}(u, \Omega).$$

Proof. By Theorem E the inner trace β^0 of u is from $L_1(\partial\Omega)$ and from the formula $\beta = \beta^0 + \alpha_\nu$ we see that the trace β of $\hat{u}$ is from $L_1(\partial\Omega)$ as well. Hence we can apply Theorem F and for the functions

$$u_h(x) = \int_{\Omega^*} K^h(x-y) u^*(y) \, \mathrm{d}y, \; x \in \Omega$$

we have

$$u_h \xrightarrow[h \to 0]{} \hat{u} \text{ in } W_\mu^1(\bar{\Omega}),$$

where $(u^*, \alpha_i^*) \in W_\mu^1(\bar{\Omega}^*)$ is a suitable extension of Theorem F.

Let us denote

$$S_h = \{x \in \Omega^*;\ \mathrm{dist}\,(x, \partial\Omega) \leq h\}.$$

From the Green theorem (see Def. A) we obtain that for $h > 0$ small there holds

$$u_{hx_i}(x) = -\int_{\Omega^*} K^h(x-y)\, \mathrm{d}\alpha_i^*(y).$$

We want to prove that $J(u_h, S_h) \to 0$. By the assumption C′) we have

$$J(u_h, S_h) = \int_{S_h} f(x, u_h, u_{hx_i})\, \mathrm{d}x \leq$$
$$\leq C\Big(\mathrm{meas}\, S_h + \int_{S_h} |u_h|\, \mathrm{d}x + \sum_{i=1}^{N} \int_{S_h} |u_{hx_i}|\, \mathrm{d}x\Big).$$

The first and second term clearly converge to zero for $h \to 0$. The third term is estimated as

$$\int_{S_h} |u_{hx_i}|\, \mathrm{d}x \leq \int_{S_h} \int_{S_{2h}} K^h(x-y)\, d|\alpha_i^*|(y)\, \mathrm{d}x \leq$$

$$\leq \int_{S_{2h}} 1 \cdot \mathrm{d}|\alpha_i^*|(y) = |\alpha_i|(S_{2h}) \underset{h\to 0}{\longrightarrow} 0$$

For $S_{2h} \to \partial\Omega$ and $|\alpha_i^*|(\partial\Omega) = 0$ (see Theorem E and F).

Hence

$$F(\hat{u}, \bar{\Omega}) \leq \underline{\lim}\, J(u_h, \Omega) \leq \underline{\lim}\, [J(u_h, \Omega - S_h) + J(u_h, S_h)] \leq \underline{\lim}\, J(u_h, \Omega - S_h).$$

But in Serrin's work [4] it is proved that

$$\bar{F}(u, \Omega) = \lim_{h \to 0} J(u_h, \Omega - S_h).$$

Now we are able to prove a partial uniqueness for the solution of our variational problem.

Theorem 6. *Suppose that the function $f(x, \xi_0, \xi_1, \ldots, \xi_N)$ satisfies the conditions A′), B′), C′) and in addition let f be strictly convex in the variables $\xi_1, \ldots, \xi_N$.*

Suppose that $u_0 \in W_1^1(\Omega)$ and $\hat{u}_1 \in W_\mu^1(\bar{\Omega})$ are two solutions of our problem, i. e. $u_0,\ \hat{u}_1 \in \tilde{u} + \mathring{W}_\mu^1,\ \tilde{u} \in W_1^1$,

$$F(u_0, \bar{\Omega}) = F(\hat{u}_1, \bar{\Omega}) = \min_{\hat{v} \in u + \mathring{W}_\mu^1} F(\hat{v}, \bar{\Omega}).$$

Suppose further that the side of $\hat{u}_1$ is equal to zero. Then $u_0 = \hat{u}_1$.

Proof. Let us set $\hat{u}_t=(1-t)u_0+t\hat{u}_1$, $0\leq t\leq 1$. The functions $\hat{u}_t$ have clearly the sides equal to zero and by preceding theorem we have ($\hat{u}_t=(u_t, \alpha_{ti})$)

$$\bar{F}(u_t, \Omega)=F(\hat{u}_t, \bar{\Omega}), \qquad 0\leq t\leq 1.$$

Suppose $(u, \alpha_i)\in W^1_\mu(\bar{\Omega})$. We can decompose (as in the work [2]) the measure α_i into absolutely continuous and singular parts with respect to Lebesgue measure. The density of absolutely continuous part of α_i we denote u_{x_i} and we set

$$J(u, \Omega)=\int_\Omega f(x, u, u_{x_i})\, dx$$

(it clear that this definition is correct for $u\in W^1_1$).

In the work [4] there are proved (with assumptions of Theorem 6) two following assertions:

I. Suppose that for some t, $0<t<1$ there holds

$$\bar{F}(u_t, \Omega)=(1-t)F(u_0, \Omega)+tF(u_1, \Omega).$$

Then $u_{0x_i}=u_{1x_i}$ a. e. in Ω, $i=1, \ldots, N$.

II. There holds

$$J(u_1, \Omega)\leq\bar{F}(u_1, \Omega)$$

with equality if and only if the derivatives $\alpha_{1i}|_\Omega$ of u_1 (in the sense of distributions) are absolutely continuous with respect to Lebesgue measure.

The assumptions from I are satisfied here for u_0 and $\hat{u}_1$ are the solutions of our variational problem, hence we have

$$u_{0x_i}=u_{1x_i} \text{ a. e. in } \Omega, \qquad i=1, \ldots, N$$

and from this it follows that

$$J(u_0, \Omega)=J(u_1, \Omega).$$

By II we have

$$F(u_0, \bar{\Omega})=\bar{F}(u_0, \Omega)=J(u_0, \Omega)=J(u_1, \Omega)\leq\bar{F}(u_1, \Omega)=F(\hat{u}_1, \bar{\Omega}),$$

but $F(u_0)=\bar{F}(\hat{u}_1)$, hence from II it follows that u_1 is from $W^1_1(\Omega)$.

As the side of $\hat{u}_1$ is equal to zero, we obtain by Theorem E $\alpha_{1i}(\breve{\Omega})=0$, hence (see § 1) $\alpha_{1i}(E)=\int_E u_{1x_i}\, dx$ for all E, which implies $\hat{u}_1\equiv u_1\in W^1_1(\Omega)$. Since $u_0|_{\partial\Omega}=\tilde{u}|_{\partial\Omega}$, $\beta=\beta^0=\tilde{u}|_{\partial\Omega}$ and $u_{0,x_i}=u_{1,x_i}$, we have $u_0=u_1=\hat{u}_1$ and the proof is complete.

Remark. In the following work [2] there will be proved a similar theorem on uniqueness, but without the assumption that the side of u_1 is equal to zero. Of course we will have to suppose more about the function f. A general theorem on uniqueness (i. e. without the assumption that one solution is a priori from W_1^1) cannot be proved in the usual way, because the functional F need not be strictly convex on W_μ^1 in the case when f is strictly convex. This assertion, with the example, will be introduced in [2].

Example. Suppose $\Omega=(0,1)\times(0,1)$, $g(x_1)\in C_0^\infty(0,1)$, $x=(x_1,x_2)$ and $L=\langle 0,1\rangle\times\{0\} = \{(x_1,x_2)\in\bar{\Omega}\,;\,x_2=0\}$. We define

$$u_n(x_1,x_2)=\begin{cases} g(x_1)\,(1-nx_2) & \text{for } 0\leq x_1\leq 1,\quad 0\leq x_2\leq \dfrac{1}{n}\\ 0 & \text{for } 0\leq x_1\leq 1,\quad \dfrac{1}{n}\leq x_2\leq 1\end{cases}$$

Clearly $u_n\in W_1^1$. Now we define the function $(u,\alpha_1,\alpha_2)\in W_\mu^1(\bar{\Omega})$ by the following way

$$u=0 \text{ on } \Omega,\quad \alpha_1=0 \text{ on } \bar{\Omega} \text{ and } \alpha_2=0 \text{ on } \bar{\Omega}-L,$$

$\alpha_2(E)=-\int_E g(x_1)\,\mathrm{d}x_1$ for all $E\subset L$, E is a Borel set. The measure α_2 can be equivalently defined by the formula

$$\int_{\bar{\Omega}}\varphi(x)\,\mathrm{d}\alpha_2=-\int_0^1\varphi(x_1,0)g(x_1)\,\mathrm{d}x_1 \quad \text{for all } \varphi\in C(\bar{\Omega}).$$

We prove that $u_n\rightharpoonup(u,\alpha_i)$ in $W_\mu^1(\bar{\Omega})$, which impliess (see § 1) $(u,\alpha_1,\alpha_2)\in W_\mu^1(\bar{\Omega})$.

For all $\varphi\in C(\bar{\Omega})$ we have

$$\int_\Omega u_n\varphi\ \mathrm{d}x \xrightarrow[n\to\infty]{} 0=\int_\Omega u\varphi\ \mathrm{d}x$$

$$\int_\Omega u_{nx_1}\varphi\ \mathrm{d}_x \xrightarrow[n\to\infty]{} 0=\int_{\bar{\Omega}}\varphi\ \mathrm{d}\alpha_1$$

$$\int_\Omega u_{nx_2}\varphi\ \mathrm{d}x \xrightarrow[n\to\infty]{} -\int_0^1 g(x_1)\varphi(x_1,0)\ \mathrm{d}x_1=\int_{\bar{\Omega}}\varphi\ \mathrm{d}\alpha_2,$$

which impliess $u_n\rightharpoonup(u,\alpha_1,\alpha_2)$. The function g is the density of the measure $\alpha_2|_L$ (with respect to the one-dimensional measure dx_1). The side of the function (u,α_1,α_2) equals $\alpha_\nu=-\alpha_2|_L=g$.

The inner trace β^0 (see § 1) of (u,α_1,α_2) equals zero since $u\equiv 0$. The trace β of (u,α_1,α_2) is obtained from the formula

$$\beta=\beta^0+\alpha_\nu,\quad \beta=g \text{ on } L,\quad \beta=0 \text{ on } \partial\Omega-L.$$

Now we investigate the functionals $J(u_n)$ and $F((u, \alpha_i))$, (J is defined in the introduction). We have the estimates

$$J(u_n) = 1 - \frac{1}{n} + \int_0^1 \int_0^{1/n} [1 + u_{nx_1}^2 + u_{nx_2}^2]^{1/2} \, dx \geq$$

$$\geq 1 - \frac{1}{n} + \int_0^1 \int_0^{1/n} |u_{nx_2}| \, dx = 1 - \frac{1}{n} + \int_0^1 g(x_1) \, dx_1 \xrightarrow[n \to \infty]{} 1 + \int_0^1 |g| \, dx_1 .$$

On the other hand we have

$$J(u_n) \leq 1 - \frac{1}{n} + \int_0^1 \int_0^{1/n} (1 + |u_{nx_1}| + |u_{nx_2}|) \, dx \leq$$

$$\leq 1 + \int_0^1 |g'| \, dx_1 \cdot \int_0^{1/n} |1 - nx_2| \, dx_2 + \int_0^1 |g| \, dx_1 \xrightarrow[n \to \infty]{} 1 + \int_0^1 |g| \, dx_1 .$$

Hence $J(u_n) \xrightarrow[n \to \infty]{} 1 + \int_0^1 |g| \, dx_1$. Since $u_n \rightharpoonup (u, \alpha_1, \alpha_2)$ in W_μ^1 and u_n have the same traces as (u, α_1, α_2) we conclude

$$1 \leq F((u, \alpha_i)) \leq F_1((u, \alpha_i)) \leq 1 + \int_0^1 |g| \, dx_1 .$$

(The first inequality follows from the estimate $J(u) \geq 1$). From the results proved in [2] we easily deduce

$$F((u, \alpha_1, \alpha_2)) = 1 + \int_0^1 |g| \, dx_1 .$$

In the formula for the surface of (u, α_1, α_2) the first term corresponds to the surface of Ω and the term $\int_0^1 |g| \, dx_1$ corresponds to the surface of the side.

Remark. The boundary of the domain Ω (from the example) is not of the class C^1, but all the considered functions are equal to zero in a neighbourhood of vertices (uniformly). Thus all the results remain true.

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ПРЯМЫЕ ВАРИЯЦИОННЫЕ МЕТОДЫ В НЕРЕФЛЕКСИВНЫХ ПРОСТРАНСТВАХ

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Резюме

Варияционные задачи типа непараметрических минимальных поверхностей исследованы при помощи прямых варияционных методов. Нерефлексивные пространства Соболева $W_1^1(\Omega)$ размирены на пространства функций $W_\mu^1(\bar{\Omega})$, производные которых меры. Функционалы типа минимальных поверхностей расширены на пространства $W_\mu^1(\bar{\Omega})$ в которых получается решение основной варияционной задачи.