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THE NON-NORMAL QUARTIC CM-FIELDS AND THE DIHEDRAL OCTIC CM-FIELDS WITH IDEAL CLASS GROUPS OF EXPONENT ≤ 2

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ABSTRACT. After having proved that there are only finitely many of them, we determine all the non-normal quartic CM-fields whose ideal class groups have exponent ≤ 2. There are 678 non isomorphic such quartic CM-fields and 37 out of them have class number 1, 205 out of them have class number 2, 284 out of them have class number 4, 140 out of them have class number 8 and 12 out of them have class number 16. We then deduce that there are 116 dihedral octic CM-fields whose ideal class groups have exponent ≤ 2 and 17 out of them have class number 1, 7 out of them have class number 2, 50 out of them have class number 4, 31 out of them have class number 8, 3 out of them have class number 16, 3 out of them have class number 32, and 5 out of them have class number 64.

1. Introduction

Lately, the class number one and two problems for non-normal quartic CM-fields and dihedral octic CM-fields have been solved (see [L01] and [YK]). The aim of this paper is more general: we determine all the non-normal quartic CM-fields and all the dihedral octic CM-fields whose ideal class groups have exponent ≤ 2. To this end, we first carefully explain in Section 3 how one can use class field theory to efficiently construct all the non-normal quartic CM-fields $\mathbf{K}$ whose discriminants are less than or equal to a prescribed large upper bound. We will reduce their construction to the construction of primitive quadratic modular characters on the rings of algebraic integers of their real quadratic subfields $\mathbf{k}$ (but we also explain in Section 3.5 how to compute totally positive algebraic integers $\beta_{\mathbf{K}/\mathbf{k}} \in \mathbf{k}$ such that $\mathbf{K} = \mathbf{k}\left(\sqrt{-\beta_{\mathbf{K}/\mathbf{k}}}\right)$). However, we will contend

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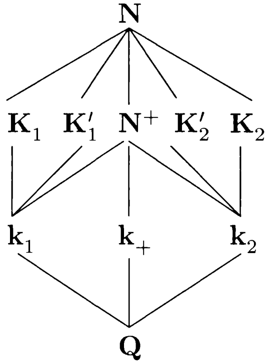
ourselves with this construction in the case that the exponents of the narrow ideal class groups of these $\mathbf{k}$ are ≤ 2. Indeed, in that situation the construction is easier and we will prove that this restriction is natural when dealing with our determination (see Points 7. and 8. of Proposition 1). We then remind the reader in Section 4 of the efficient method exposed in [Lou3] for computing relative class numbers of non-normal quartic CM-fields. We are now in a position to compute the class numbers of all the non-normal quartic CM-fields and of all the dihedral octic CM-fields of discriminants less than or equal to any reasonable upper bound. According to such computations, there are a lot of non-normal quartic CM-fields and a lot of dihedral octic CM-fields whose ideal class groups have exponent ≤ 2. However, our next aim is to prove in Section 5 that there are only finitely many such CM-fields and to obtain an explicit bound on their discriminants. Finally, using necessary conditions for the exponent of the ideal class groups of such CM-fields to be ≤ 2 to drastically alleviate the amount of required relative class number computation, we will finally determine all the non-normal quartic CM-fields and all the dihedral octic CM-fields whose ideal class groups have exponent ≤ 2. There are $678 + 116 = 794$ such CM-fields and we will provide the reader with the list of these CM-fields at the end of the paper. Of course, we spent quite a lot of time checking that we did not make any misprint while filling this list. We are rather confident that we did not make any mistake in our determinations for the required computations were done separately in Caen (France) by the first author and in Seoul (Korea) by the second and third authors and in the end our results dovetailed perfectly.

2. Non-normal quartic fields and dihedral octic fields

If $\mathbf{E}$ is a number field, we let $d_{\mathbf{E}}$ denote the absolute value of its discriminant. If $\mathbf{E}/\mathbf{F}$ is an extension of degree $m \geq 1$ of number fields, we let $\mathcal{D}_{\mathbf{E}/\mathbf{F}}$ denote the different of the extension $\mathbf{E}/\mathbf{F}$ and set $d_{\mathbf{E}/\mathbf{F}} = N_{\mathbf{E}/\mathbf{Q}}(\mathcal{D}_{\mathbf{E}/\mathbf{F}}) = d_{\mathbf{E}}/d_{\mathbf{F}}^m$. If $\mathbf{E}/\mathbf{F}$ is abelian, we let $\mathcal{F}_0$ denote the finite part of the conductor $\mathcal{F}_{\mathbf{E}/\mathbf{F}}$ of the extension $\mathbf{E}/\mathbf{F}$.

Proposition 1.

1. *Let $\mathbf{K}$ be a non-normal quartic field. Assume that $\mathbf{K}$ contains a quadratic subfield $\mathbf{k}$. Then, the normal closure $\mathbf{N}$ of $\mathbf{K}$ is a dihedral octic field. Moreover, $\mathbf{K}$ is a CM-field if and only if $\mathbf{N}$ is a CM-field.*
2. *Let $\mathbf{N}$ be a dihedral octic field. We have the following lattice of subfields*



$$\mathrm{Gal}(\mathbf{N}/\mathbf{Q}) = D_8$$

$$\mathrm{Gal}(\mathbf{N}/\mathbf{k}_+) = C_4$$

$$\mathrm{Gal}(\mathbf{N}/\mathbf{k}_i) = C_2 \times C_2$$

$$\mathrm{Gal}(\mathbf{N}^+/\mathbf{Q}) = C_2 \times C_2$$

where $\mathbf{K} = \mathbf{K}_1$ *and* $\mathbf{K}_1'$ *are isomorphic non-normal quartic fields with the same quadratic subfield* $\mathbf{k} = \mathbf{k}_1$ *(we will say that* $\mathbf{K}_1'$ *is the conjugate field of* $\mathbf{K}_1$*),* $\mathbf{K}_2$ *and* $\mathbf{K}_2'$ *are isomorphic non-normal quartic fields with the same quadratic subfield* $\mathbf{k}_2$ *and* $\mathbf{N}^+$ *is the maximal abelian subfield of* $\mathbf{N}$*. In this situation,* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *are called dual non-normal quartic fields, and* $\mathbf{k}_1$ *and* $\mathbf{k}_2$ *are called dual quadratic fields. Moreover, if* $\mathbf{N}$ *is a CM-field, then the* $\mathbf{K}_i$ *and* $\mathbf{K}_i'$*'s are non-normal quartic CM-fields and* $\mathbf{N}^+$ *is the maximal totally real subfield of* $\mathbf{N}$*.*

3. *Let* $\mathbf{K}/\mathbf{k}$ *be a quadratic extension of number fields. For any algebraic integer* α *of* $\mathbf{k}$ *such that* $\mathbf{K} = \mathbf{k}(\sqrt{\alpha})$ *there exists an integral ideal* $\mathcal{I}$ *of* $\mathbf{k}$ *such that* $(4\alpha) = \mathcal{I}^2\mathcal{F}_0$*, where* $\mathcal{F}_0$ *denotes the finite part of the conductor* $\mathcal{F}_{\mathbf{K}/\mathbf{k}}$ *of the quadratic extension* $\mathbf{K}/\mathbf{k}$*.*
4. *Let* $\mathbf{K} = \mathbf{k}(\sqrt{-\alpha})$ *be a quartic CM-field, where* α *is a totally positive algebraic integer of the real quadratic subfield* $\mathbf{k}$ *of* $\mathbf{K}$*. Then* $\mathbf{K}$ *is normal if and only if* $d_{\mathbf{K}/\mathbf{k}}$ *is a square in* $\mathbf{k}$*, which is equivalent to* $N_{\mathbf{k}/\mathbf{Q}}(\alpha)$ *being a square in* $\mathbf{k}$*. Moreover, if* $\mathbf{K}$ *is not normal, then* $\mathbf{k}' = \mathbf{Q}\left(\sqrt{N_{\mathbf{k}/\mathbf{Q}}(\alpha)}\right) = \mathbf{Q}\left(\sqrt{d_{\mathbf{K}/\mathbf{k}}}\right)$ *is the dual quadratic field of* $\mathbf{k}$ *and* $d_{\mathbf{k}'}$ *divides* $d_{\mathbf{K}/\mathbf{k}}$*.*
5. *Let* $\mathbf{N}$ *be a dihedral octic field and let the notation be as in Point 2.. We have* $d_{\mathbf{N}^+} = d_{\mathbf{k}_1}d_{\mathbf{k}_2}d_{\mathbf{k}_+}$*,* $d_{\mathbf{N}/\mathbf{K}_i} = d_{\mathbf{N}^+/\mathbf{k}_i}$ *and* $d_{\mathbf{K}_2/\mathbf{k}_2} = d_{\mathbf{k}_1}d_{\mathbf{K}_1/\mathbf{k}_1}/d_{\mathbf{k}_2}$*. Since* $\mathbf{k}_2 = \mathbf{Q}\left(\sqrt{d_{\mathbf{K}_1/\mathbf{k}_1}}\right)$*, we can easily deduce the values of* $d_{\mathbf{N}^+}$*,* $d_{\mathbf{k}_2}$ *and* $d_{\mathbf{K}_2/\mathbf{k}_2}$ *from* $d_{\mathbf{k}_1}$ *and* $d_{\mathbf{K}_1/\mathbf{k}_1}$*.*
6. $\mathbf{N}/\mathbf{K}_i$ *is unramified at all the finite places if and only if* $d_{\mathbf{N}^+} = d_{\mathbf{k}_i}^2$*. In particular, at least one of the two extensions* $\mathbf{N}/\mathbf{K}_1$ *or* $\mathbf{N}/\mathbf{K}_2$ *is ramified at some finite place.*
7. *Assume that the exponent of the ideal class group of a dihedral octic CM-field* $\mathbf{N}$ *is* ≤ 2*. Then, the exponent of the ideal class group of at least one of its four non-normal quartic CM-subfields* $\mathbf{K}$ *is* ≤ 2*. Moreover, if*

$d_{\mathbf{N}^+} \neq d_{\mathbf{k}_1}^2$ and $d_{\mathbf{N}^+} \neq d_{\mathbf{k}_2}^2$, then the exponents of the ideal class groups of $\mathbf{K}_1$ and $\mathbf{K}_2$ are both ≤ 2.

8. *Assume that the exponent of the ideal class group $\mathrm{Cl}_{\mathbf{K}}$ of a non-normal quartic CM-field $\mathbf{K}$ is ≤ 2. Then, the exponent of the narrow ideal class group $\mathrm{Cl}_{\mathbf{k}}^+$ of its real quadratic subfield $\mathbf{k}$ is also ≤ 2.*

Proof.

1. See [Lou2].

2. See also [Lou2].

3. Let $\mathbf{A_k}$ and $\mathbf{A_K}$ be the rings of algebraic integers of $\mathbf{k}$ and $\mathbf{K}$, respectively. For $\alpha \in \mathbf{K}$, let α' be its conjugate in the quadratic extension $\mathbf{K}/\mathbf{k}$. Since $\mathcal{D}_{\mathbf{K}/\mathbf{k}} = \gcd\{(\alpha - \alpha') : \alpha \in \mathbf{A_K} \text{ and } \mathbf{K} = \mathbf{k}(\alpha)\}$ (see [Lan2; Chap. III, Proposition 8]), for any $\alpha \in \mathbf{A_K}$ such that $\mathbf{K} = \mathbf{k}(\alpha)$ there exists an integral ideal $\mathcal{I}_\alpha$ of $\mathbf{K}$ such that $(\alpha - \alpha') = \mathcal{I}_\alpha \mathcal{D}_{\mathbf{K}/\mathbf{k}}$. Since the ideals $(\alpha - \alpha')$ and $\mathcal{D}_{\mathbf{K}/\mathbf{k}}$ are invariant under the action of $\mathrm{Gal}(\mathbf{K}/\mathbf{k})$, so is $\mathcal{I}_\alpha$, and $\mathcal{I}_\alpha = \mathcal{R}\mathcal{J}_{\mathbf{k}}$ for some integral ideal $\mathcal{J}_{\mathbf{k}}$ of $\mathbf{k}$ and some integral ideal $\mathcal{R}$ of $\mathbf{K}$ which is a product of prime ideals of $\mathbf{K}$ ramified in the quadratic extension $\mathbf{K}/\mathbf{k}$. Fix a prime ideal $\mathcal{P}_{\mathbf{K}}$ of $\mathbf{K}$ ramified in the quadratic extension $\mathbf{K}/\mathbf{k}$. Let $\nu_{\mathcal{P}_{\mathbf{K}}}$ denote the associated valuation. Since $\nu_{\mathcal{P}_{\mathbf{K}}}\big((\alpha - \alpha')\mathcal{D}_{\mathbf{K}/\mathbf{k}}^{-1}\big) = \max\{\nu_{\mathcal{P}_{\mathbf{K}}}\big((\alpha - \alpha')/(\beta - \beta')\big) : \beta \in \mathbf{A_K}$ and $\mathbf{K} = \mathbf{k}(\beta)\}$ and since $(\alpha - \alpha')/(\beta - \beta') \in \mathbf{k}$ for any such β, we obtain $\nu_{\mathcal{P}_{\mathbf{K}}}(\mathcal{R}) \equiv \nu_{\mathcal{P}_{\mathbf{K}}}(\mathcal{R}\mathcal{J}_{\mathbf{k}}) = \nu_{\mathcal{P}_{\mathbf{K}}}(\mathcal{I}_\alpha) = \nu_{\mathcal{P}_{\mathbf{K}}}\big((\alpha - \alpha')\mathcal{D}_{\mathbf{K}/\mathbf{k}}^{-1}\big) \equiv 0 \pmod 2$. Hence $\mathcal{R}$ is the square of some integral ideal of $\mathbf{K}$ and $\mathcal{I}_\alpha = \mathcal{R}\mathcal{J}_{\mathbf{k}}$ is an integral ideal $\mathcal{I}_{\mathbf{k}}$ of $\mathbf{k}$, i.e.: if $\alpha \in \mathbf{A_K}$ is such that $\mathbf{K} = \mathbf{k}(\alpha)$, then there exists an integral ideal $\mathcal{I}_{\mathbf{k}}$ of $\mathbf{k}$ such that $(\alpha - \alpha') = \mathcal{I}_{\mathbf{k}}\mathcal{D}_{\mathbf{K}/\mathbf{k}}$. In particular, for any $\alpha \in \mathbf{A_k}$ such that $\mathbf{K} = \mathbf{k}(\sqrt{\alpha})$ there exists an integral ideal $\mathcal{I}_{\mathbf{k}}$ of $\mathbf{k}$ such that $(2\sqrt{\alpha}) = \mathcal{I}_{\mathbf{k}}\mathcal{D}_{\mathbf{K}/\mathbf{k}}$. Taking relative norms, we do get $(4\alpha) = \mathcal{I}_{\mathbf{k}}^2 N_{\mathbf{K}/\mathbf{k}}(\mathcal{D}_{\mathbf{K}/\mathbf{k}}) = \mathcal{I}_{\mathbf{k}}^2\mathcal{F}_0$.

4. Since $\mathbf{K}$ is normal if and only if $N_{\mathbf{k}/\mathbf{Q}}(\alpha)$ is a square in $\mathbf{k}$, Point 3. provides us with the first result. The second assertion is easily proved.

5. Follows from $\big(d_{\mathbf{K}_1}/d_{\mathbf{k}_1}\big)^2 = d_{\mathbf{N}}/d_{\mathbf{N}^+} = \big(d_{\mathbf{K}_2}/d_{\mathbf{k}_2}\big)^2$ (see [Lou2]).

6. Now $d_{\mathbf{N}^+} = d_{\mathbf{k}_1}^2$ and $d_{\mathbf{N}^+} = d_{\mathbf{k}_2}^2$ would imply $d_{\mathbf{k}_1}^2 d_{\mathbf{k}_2}^2 d_{\mathbf{k}_+}^2 = d_{\mathbf{N}^+}^2 = d_{\mathbf{k}_1}^2 d_{\mathbf{k}_2}^2$ and $d_{\mathbf{k}_+} = 1$, a contradiction.

7. If the quadratic extension $\mathbf{N}/\mathbf{K}$ is ramified at some finite place, then, according to class field theory, the norm map from the ideal class group of $\mathbf{N}$ to the ideal class group of $\mathbf{K}$ is surjective. Now, use Point 6..

8. Let ${\mathbf{H}_{\mathbf{k}}}^+$ denote the narrow Hilbert class field of $\mathbf{k}$ and let $\mathbf{H}_{\mathbf{K}}$ denote the Hilbert class field of $\mathbf{K}$. Since the quadratic extension $\mathbf{K}/\mathbf{k}$ is ramified at least one finite place of $\mathbf{k}$ (see [L01; p. 51]), then $\mathbf{K} \cap {\mathbf{H}_{\mathbf{k}}}^+ = \mathbf{k}$. Therefore, the extension $\mathbf{K}{\mathbf{H}_{\mathbf{k}}}^+/\mathbf{K}$ is a subextension of the abelian extension $\mathbf{H}_{\mathbf{K}}/\mathbf{K}$ whose Galois group $\mathrm{Gal}(\mathbf{K}{\mathbf{H}_{\mathbf{k}}}^+/\mathbf{K})$ is isomorphic to the Galois group $\mathrm{Gal}({\mathbf{H}_{\mathbf{k}}}^+/\mathbf{k})$,

hence isomorphic to $\mathrm{Cl}_{\mathbf{k}}^{+}$. Since $\mathrm{Gal}(\mathbf{H}_{\mathbf{K}}/\mathbf{K})$ is isomorphic to $\mathrm{Cl}_{\mathbf{K}}$, then it has exponent ≤ 2. Hence the exponent of $\mathrm{Gal}(\mathbf{K}\mathbf{H}_{\mathbf{k}}{}^{+}/\mathbf{K}) \simeq \mathrm{Cl}_{\mathbf{k}}^{+}$ is ≤ 2. □

From now on, we let $\mathbf{K}$, $\mathbf{k}$, $h_{\mathbf{K}}^{-}$, $Q_{\mathbf{K}} \in \{1,2\}$ and $\mathbf{N}$ denote a non-normal quartic CM-field, its real quadratic subfield, its relative class number, its Hasse unit index and its normal closure, respectively. In particular, $\mathbf{N}$ is a dihedral octic CM-field. We then let ∞_1, ∞_2, $\mathcal{F}_{\mathbf{K}/\mathbf{k}} = \infty_1\infty_2\mathcal{F}_0$ and $\chi_{\mathbf{K}/\mathbf{k}}$ denote the infinite places of $\mathbf{k}$, the conductor of the quadratic extension $\mathbf{K}/\mathbf{k}$ and the quadratic character associated with this quadratic extension. Hence, $\mathcal{F}_0$ is an integral ideal of $\mathbf{k}$. Finally, we set $d_{\mathbf{K}/\mathbf{k}} = N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0)$ and let $\mathrm{Cl}_{\mathbf{K}}$ and $\mathrm{Cl}_{\mathbf{N}}$ denote the ideal class groups of $\mathbf{K}$ and $\mathbf{N}$, respectively.

2.1. When is $\mathbf{K}_2$ a dual field of $\mathbf{K}_1$?

PROPOSITION 2. *Let $\mathbf{K}_1 = \mathbf{k}_1(\sqrt{-\alpha_1})$ and $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\alpha_2})$ be two non-normal quartic CM-fields, where $\alpha_i = (x_i+y_i\sqrt{d_{\mathbf{k}_i}})/2$ is a totally positive algebraic integer of the real quadratic field $\mathbf{k}_i$ of discriminant $d_i = d_{\mathbf{k}_i}$, $\mathbf{k}_1 \neq \mathbf{k}_2$. Set $N_i = N_{\mathbf{k}_i/\mathbf{Q}}(\alpha_i)$ (hence, $\mathbf{k}_2 = \mathbf{Q}(\sqrt{N_1})$, $\mathbf{k}_1 = \mathbf{Q}(\sqrt{N_2})$) and, in accordance with Point 5. of Proposition 1, assume that $d_{\mathbf{k}_1}d_{\mathbf{K}_1/\mathbf{k}_1} = d_{\mathbf{k}_2}d_{\mathbf{K}_2/\mathbf{k}_2}$ (hence, the positive rational integers N_1d_2 and N_2d_1 are squares of rational integers). Then $\mathbf{K}_1$ and $\mathbf{K}_2$ are dual non-normal quartic fields if and only if at least one of the four rational integers*

$$R(\varepsilon,\varepsilon') = x_1x_2 + 2\varepsilon y_1\sqrt{N_2d_1} + 2\varepsilon' y_2\sqrt{N_1d_2}, \qquad \varepsilon \in \{\pm 1\} \text{ and } \varepsilon' \in \{\pm 1\}$$

is a square of some rational integer.

Proof. $\mathbf{K}_1$ and $\mathbf{K}_2$ are dual quartic fields if and only if $\delta = \alpha_1\alpha_2$ is a square in $\mathbf{N}^+ = \mathbf{k}_1\mathbf{k}_2$. We use [Lou7; Proposition 3.1] with $\mathbf{E} = \mathbf{N}^+$ and $\mathbf{F} = \mathbf{k}_1$ and notice that $N_{\mathbf{N}^+/\mathbf{k}_1}(\alpha_1\alpha_2) = \alpha_1^2N_2$ is always a square in $\mathbf{k}_1$ (use Points 3. and 4. of Proposition 1) and that $\mathrm{Tr}_{\mathbf{N}^+/\mathbf{k}_1}(\alpha_1\alpha_2) = \alpha_1x_2$. Hence, $\delta = \alpha_1\alpha_2$ is a square in $\mathbf{N}^+$ if and only if one of the two $\alpha_\varepsilon(\delta) = \alpha_1(x_2 + 2\varepsilon\sqrt{N_2}) \in \mathbf{k}_1$ is a square in $\mathbf{k}_1$, where $\varepsilon \in \{\pm 1\}$. Now, we use [Lou7; Corollary 3.3] and notice that $N = N_{\mathbf{k}_1/\mathbf{Q}}(\alpha_\varepsilon(\delta)) = N_1(x_2^2 - 4N_2) = N_1d_2y_2^2$ is a square in the field of rational numbers and that $T = \mathrm{Tr}_{\mathbf{k}_1/\mathbf{Q}}(\alpha_\varepsilon(\delta)) = x_1x_2 + 2\varepsilon y_1\sqrt{N_2d_1}$. Hence, $\alpha_\varepsilon(\delta)$ is a square in $\mathbf{k}_1$ if and only if one of the two $T + 2\varepsilon'\sqrt{N} = R(\varepsilon,\varepsilon')$ is the square of some rational number. □

EXAMPLE. $\mathbf{K}_1 = \mathbf{Q}\left(\sqrt{-(104+20\sqrt{24})/2}\right)$ and $\mathbf{K}_2 = \mathbf{Q}\left(\sqrt{-(40+4\sqrt{76})/2}\right)$ are dual CM-quartic fields. Indeed, $N_1 = 304$, $N_2 = 96$, $R(\varepsilon,\varepsilon') = 64(65 + 30\varepsilon + 19\varepsilon')$ and $R(-1,-1) = 64\cdot 16$ is a perfect square. More precisely, $\alpha_1 = 4(13+5\sqrt{6})$, $\alpha_2 = 4(5+\sqrt{19})$, $\alpha_\pm(\delta) = 32(13+5\sqrt{6})(5\pm\sqrt{6})$ and, according to [Lou7; Corollary 3.3], $\alpha_-(\delta) = 32(35 + 12\sqrt{6})$ is a square in $\mathbf{Q}(\sqrt{6})$. More precisely, we have $\alpha_-(\delta) = (16 + 12\sqrt{6})^2$ in $\mathbf{k}_1 = \mathbf{Q}(\sqrt{6})$ and $\delta = \alpha_1\alpha_2 = 16(65 + 25\sqrt{6} + 13\sqrt{19} + 5\sqrt{114}) = 2^2(4 + 3\sqrt{6} + 2\sqrt{19} + \sqrt{114})^2$ in $\mathbf{N}^+ = \mathbf{Q}(\sqrt{6}, \sqrt{19})$.

3. Construction of the quartic CM-fields containing a given real quadratic field

Let $\mathbf{k}$ be a given real quadratic field. The aim of this section is to carefully explain how one can efficiently construct all the non-normal quartic CM-fields $\mathbf{K}$ containing $\mathbf{k}$ and of discriminants $d_{\mathbf{K}}$ less than or equal to any given reasonable large upper bound. This construction is based on the use of class field theory and rests on the construction of all primitive quadratic characters on ray class groups of $\mathbf{k}$.

To begin with, we notice that if $\mathbf{K}$ is a non-normal quartic CM-field with maximal totally real subfield $\mathbf{k}$, a real quadratic field, then there exists some integral ideal $\mathcal{F}_0$ of $\mathbf{k}$ such that $\mathcal{F}_{\mathbf{K}/\mathbf{k}} = \infty_1\infty_2\mathcal{F}_0$ is the conductor of the quadratic extension $\mathbf{K}/\mathbf{k}$, where ∞_1 and ∞_2 denote the two infinite places of $\mathbf{k}$. We let $\mathrm{Cl}_{\mathbf{k}}^+(\mathcal{F}_0)$ denote the unit ray class group of $\mathbf{k}$ modulo $\mathcal{F} = \infty_1\infty_2\mathcal{F}_0$, where ∞_1 and ∞_2 denote the two infinite places of the real quadratic field $\mathbf{k}$. Let χ be a character on the group $\mathrm{Cl}_{\mathbf{k}}^+(\mathcal{F}_0)$. Its associated modular character χ_0 on the group $(\mathbf{A}_{\mathbf{k}}/\mathcal{F}_0)^*$ is defined by $\chi((\alpha)) = \nu(\alpha)\chi_0(\alpha)$ where $\nu(\alpha) = \pm 1$ is the sign of the norm of α. According to class field theory, there is a bijective correspondence between

(1) the non-normal quartic CM-fields $\mathbf{K}$ containing $\mathbf{k}$ such that $\mathcal{F}_{\mathbf{K}/\mathbf{k}} = \mathcal{F} = \infty_1\infty_2\mathcal{F}_0$

and

(2) the quadratic characters on $\mathrm{Cl}_{\mathbf{k}}^+(\mathcal{F}_0)$ whose associated quadratic modular characters χ_0 on $(\mathbf{A}_{\mathbf{k}}/\mathcal{F}_0)^*$ are primitive.

Subsection 3.3 will be devoted to constructing such modular primitive quadratic characters. For the time being, we explain how to recover χ from χ_0.

3.1. Determination of basis of $\mathrm{Cl}_{\mathbf{k}}^+[2]$ and $\mathrm{Cl}_{\mathbf{k}}[2]$.

Recall that $\mathbf{k} = \mathbf{Q}(\sqrt{d})$ denotes a real quadratic number field where $d > 1$ is a positive square-free integer. We then let $\mathbf{A}_{\mathbf{k}}, d_{\mathbf{k}}, \varepsilon_{\mathbf{k}} = (x_{\mathbf{k}} + y_{\mathbf{k}}\sqrt{d_{\mathbf{k}}})/2 > 1$, $\nu(\varepsilon_{\mathbf{k}}) = \pm 1$, $\mathrm{Cl}_{\mathbf{k}}, \mathrm{Cl}_{\mathbf{k}}^+, h_{\mathbf{k}}, h_{\mathbf{k}}^+, t = t_{\mathbf{k}}$, $\mathrm{Cl}_{\mathbf{k}}[2]$ and $\mathrm{Cl}_{\mathbf{k}}^+[2]$ denote its ring of algebraic integers, its discriminant, its fundamental unit, the norm of this fundamental unit, its ordinary and narrow ideal class groups, its ordinary and narrow class numbers, the number of rational primes which are ramified in the quadratic extension $\mathbf{k}/\mathbf{Q}$ and the subgroups of the classes of order ≤ 2 in its ordinary and narrow ideal class groups, respectively. For any square-free rational integer $Q \geq 1$ which divides $d_{\mathbf{k}}$ we let $\mathcal{R}_Q$ denote the only integral ideal of $\mathbf{k}$ of norm $Q \geq 1$ and we let $\tilde{\mathcal{R}}_Q$ denote the primitive ideal of $\mathbf{k}$ such that $(\sqrt{d})\mathcal{R}_Q = (n)\tilde{\mathcal{R}}_Q$ for some rational integer $n \geq 1$. This ideal $\tilde{\mathcal{R}}_Q$ is called the *dual ideal* of the ideal $\mathcal{R}_Q$. Notice that $\mathcal{R}_Q = Q\mathbf{Z} + \big((P + \sqrt{d_{\mathbf{k}}})/2\big)\mathbf{Z}$ where

$$P = \begin{cases} d & \text{if } d \equiv 3 \pmod 4 \text{ and } Q \text{ is even,} \\ d_{\mathbf{k}} & \text{otherwise} \end{cases}$$

and that $\tilde{\mathcal{R}}_Q = \mathcal{R}_{\tilde{Q}}$ where

$$\tilde{Q} = \begin{cases} 4d/Q & \text{if } d \equiv 3 \pmod 4 \text{ and } Q \text{ is even,} \\ d/Q & \text{otherwise.} \end{cases}$$

The 2-rank of the narrow ideal class group $\mathrm{Cl}_{\mathbf{k}}^+$ of $\mathbf{k}$ is equal to $t-1 = t_{\mathbf{k}}-1$. More precisely, $\mathrm{Cl}_{\mathbf{k}}^+[2] = \{\mathcal{C} \in \mathrm{Cl}_{\mathbf{k}}^+ : \mathcal{C}^2 = 1\}$ is generated by the narrow ideal classes of the t prime ramified ideals $\mathcal{P}_p$ of $\mathbf{k}$ where p runs over the prime divisors of $d_{\mathbf{k}}$ and where for such a prime p we let $\mathcal{P}_p$ denote the prime ideal of $\mathbf{k}$ lying above p. Therefore, there is a single relation between these t narrow ideal classes. The following Lemma provides us with this relation and enables us to determine a set $\{\mathcal{C}_j : 1 \leq j \leq t-1\}$ of $t-1$ narrow ideal classes of prime ramified ideals of $\mathbf{k}$ which generate $\mathrm{Cl}_{\mathbf{k}}^+[2]$:

LEMMA 3. (See [Lou1].) *Let $\mathbf{k} = \mathbf{Q}(\sqrt{d})$ be a real quadratic field, $d > 1$ square-free.*

1. *Assume that $\nu(\varepsilon_{\mathbf{k}}) = -1$. Then $\mathcal{R}_d = (\sqrt{d}) \sim 1$ is the single relation in $\mathrm{Cl}_{\mathbf{k}}^+$ between the t prime ramified ideals of $\mathbf{k}$.*
2. *Assume that $\nu(\varepsilon_{\mathbf{k}}) = +1$. Set $(P_0, Q_0) = (d_{\mathbf{k}}, 1)$ and define inductively $a_i =$ the largest integer less than or equal to $\omega_i = \big(P_i + \sqrt{d_{\mathbf{k}}}\big)/2Q_i$, $P_{i+1} = 2a_iQ_i - P_i$ and $Q_{i+1} = \big(d_{\mathbf{k}} - P_{i+1}^2\big)/4Q_i$. Then, the Q_i are positive rational integers and $i_0 = \min\{i \geq 1 : Q_i \mid d\}$ is well defined and Q_{i_0} is a positive square-free divisor of $d_{\mathbf{k}}$. If i_0 is even, then $\mathcal{R}_{Q_{i_0}} \sim 1$ is the single relation in $\mathrm{Cl}_{\mathbf{k}}^+$ between the t prime ramified ideals of $\mathbf{k}$, while if i_0 is odd, then $\tilde{\mathcal{R}}_{Q_{i_0}} \sim 1$ is the single relation in $\mathrm{Cl}_{\mathbf{k}}^+$ between the t prime ramified ideals of $\mathbf{k}$.*

For any prime divisor p of $d_{\mathbf{k}}$, let ψ_p denote the quadratic character on $\mathrm{Cl}_{\mathbf{k}}^+$ defined by

$$\psi_p(\mathcal{I}) = \big(N_{\mathbf{k}/\mathbf{Q}}(\mathcal{I}), d_{\mathbf{k}}\big)_p \qquad \text{(Hilbert's symbol).}$$

These t quadratic characters on $\mathrm{Cl}_{\mathbf{k}}^{+}$ generate the group $\Psi_{\mathbf{k}}$ of order 2^{t-1} of all the quadratic characters on $\mathrm{Cl}_{\mathbf{k}}^{+}$ and satisfy the single relation $\prod_{p|d_{\mathbf{k}}} \psi_p = 1$ (see [Mor; Theorem 2]). Notice that $\mathcal{C} \in \mathrm{Cl}_{\mathbf{k}}^{+}$ is a square in $\mathrm{Cl}_{\mathbf{k}}^{+}$ if and only if $\psi_p(\mathcal{C}) = +1$ for all the t prime divisors p of $d_{\mathbf{k}}$. We then fix a set $\{\psi_i :\ 1 \leq i \leq t-1\}$ of $t-1$ generators of this group $\Psi_{\mathbf{k}}$. We also construe these characters ψ_i as $\mathbf{F}_2$-valued characters. Finally, we fix $\{\mathcal{C}_j :\ 1 \leq j \leq t-1\}$ a set of narrow ideal classes generating $\mathrm{Cl}_{\mathbf{k}}^{+}[2]$ and set

$$M_{\mathbf{k}} = \left[\psi_i(\mathcal{C}_j)\right]_{1 \leq i,\, j \leq t-1} \in \mathrm{GL}_{t-1}(\mathbf{F}_2)\,,$$

where $\mathbf{F}_2$ denotes the finite field with two elements.

3.1.1. Efficient computation of relations in $\mathrm{Cl}_{\mathbf{k}}^{+}[2]$.

Let now $\mathcal{F}_0$ be a given non-zero integral ideal of $\mathbf{k}$ and set $f_0 = N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0)$.

Assumption. From now on, to simplify, we assume that the exponent of $\mathrm{Cl}_{\mathbf{k}}^{+}$ is ≤ 2, i.e., we assume that $\mathrm{Cl}_{\mathbf{k}}^{+} = \mathrm{Cl}_{\mathbf{k}}^{+}[2]$ (see Point 8. of Proposition 1). It may be useful to notice that the exponent of the 2-Sylow subgroup of $\mathrm{Cl}_{\mathbf{k}}^{+}$ is ≤ 2 if and only if for each $j \in \{1, \dots, t-1\}$ there exists $i \in \{1, \dots, t-1\}$ such that $\psi_i(\mathcal{I}_j) = -1$, where $\mathcal{I}_j \in \mathcal{C}_j$.

In particular, an integral ideal $\mathcal{I}$ of $\mathbf{k}$ is principal in $\mathrm{Cl}_{\mathbf{k}}^{+}$ if and only if its ideal class is a square in $\mathrm{Cl}_{\mathbf{k}}^{+}$, i.e.

$$\mathcal{I} \text{ is principal in the narrow sense} \iff \big(\forall \psi \in \Psi_{\mathbf{k}}\big)\big(\psi(\mathcal{I}) = +1\big)\,, \tag{1}$$

$$\mathcal{I} \text{ is principal in the wide sense} \iff \begin{cases} \big(\forall \psi \in \Psi_{\mathbf{k}}\big)\big(\psi(\mathcal{I}) = +1\big) \\ \text{or} \\ \big(\forall \psi \in \Psi_{\mathbf{k}}\big)\big(\psi(\mathcal{I}) = \psi\big((\sqrt{d_{\mathbf{k}}})\big)\big)\,. \end{cases} \tag{2}$$

Therefore, if the $\mathcal{C}_i$'s are the narrow ideal classes of prime ramified ideals $\mathcal{P}_i$ and if $(q_i) = \mathcal{Q}_i \mathcal{Q}_i'$ are rational primes relatively prime to f_0 which split in $\mathbf{k}$ and such that $\psi_j(\mathcal{Q}_i) = \psi_j(\mathcal{P}_i)$ for $1 \leq j \leq t-1$, then $\{\mathcal{Q}_i :\ 1 \leq i \leq t-1\}$ is an easy to construct set of integral ideals of norms relatively prime to f_0 whose narrow ideal classes generate $\mathrm{Cl}_{\mathbf{k}}^{+}[2]$. We fix a set $\{\mathcal{I}_i :\ 1 \leq i \leq t-1\}$ of integral ideals of norms relatively prime to $f_0 = N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0)$ whose narrow ideal classes generate $\mathrm{Cl}_{\mathbf{k}}^{+}[2]$.

Now, for any integral ideal $\mathcal{I}$ of $\mathbf{k}$ let

$$\vec{\psi}_{\mathcal{I}} = \left[\psi_i(\mathcal{I})\right]_{1 \leq i \leq t-1}$$

denote the column vector whose ith component is $\psi_i(\mathcal{I}) \in \mathbf{F}_2$. There exists a unique column vector $\vec{e}_{\mathcal{I}} = \left[e_{\mathcal{I}}(i)\right]_{1 \leq i \leq t-1}$ with coefficients in $\mathbf{F}_2$ such that

$$\mathcal{I} \prod_{j=1}^{t-1} \mathcal{I}_j^{e_{\mathcal{I}}(j)} = (\alpha_{\mathcal{I}}) \qquad (\text{with } \alpha_{\mathcal{I}} \in \mathbf{A}_{\mathbf{k}} \text{ totally positive}) \tag{3}$$

is principal in the narrow sense. Since this relation holds if and only if $\psi_i(\mathcal{I}) = \sum_{j=1}^{t-1} \psi_i(\mathcal{I}_j) e_{\mathcal{I}}(j)$ for $1 \leq i \leq t-1$, hence if and only if $\vec{\psi}_{\mathcal{I}} = M_{\mathbf{k}} \vec{e}_{\mathcal{I}}$, we obtain

$$\vec{e}_{\mathcal{I}} = M_{\mathbf{k}}^{-1} \vec{\psi}_{\mathcal{I}},$$

which makes it easy to compute $\vec{e}_{\mathcal{I}}$, the knowledge of which makes it easy to compute $\alpha_{\mathcal{I}}$ (see [L02; p. 539] or Subsection 3.4 below).

3.2. Quadratic characters on ray class groups of $\mathbf{k}$.

Let χ be a quadratic character on the group $\mathrm{Cl}_{\mathbf{k}}^{+}(\mathcal{F}_0)$. Let $\alpha_i \in \mathbf{A}_{\mathbf{k}}$ be totally positive and such that

$$\mathcal{I}_i^2 = (\alpha_i).$$

Since $1 = \chi^2(\mathcal{I}_i) = \chi(\mathcal{I}_i^2) = \chi((\alpha_i)) = \nu(\alpha_i)\chi_0(\alpha_i) = \chi_0(\alpha_i)$, we obtain

$$\textbf{Necessary 1:} \qquad \chi_0(\alpha_i) = +1 \qquad \text{for } 1 \leq i \leq t-1. \tag{4}$$

Moreover, for any unit $\varepsilon \in \mathbf{A}_{\mathbf{k}}^{*}$ we must have $\chi_0(\varepsilon) = \nu(\varepsilon)$ (for $1 = \chi((\varepsilon)) = \nu(\varepsilon)\chi_0(\varepsilon)$), which amounts to having

$$\textbf{Necessary 2:} \qquad \chi_0(\varepsilon_{\mathbf{k}}) = \nu(\varepsilon_{\mathbf{k}}) \quad \text{and} \quad \chi_0(-1) = +1 \tag{5}$$

where $\varepsilon_{\mathbf{k}} > 1$ is the fundamental unit of $\mathbf{k}$.

Finally, if $\nu(\varepsilon_{\mathbf{k}}) = +1$, then the 2-rank of the ordinary ideal class group $\mathrm{Cl}_{\mathbf{k}}$ of $\mathbf{k}$ is equal to $t_{\mathbf{k}}^{w} = t-2$ and there exists a single relation

$$\prod_{i \in I} \mathcal{I}_i = (\alpha) \qquad (\text{with } N_{\mathbf{k}/\mathbf{Q}}(\alpha) < 0) \tag{6}$$

$$\Longleftrightarrow \; (\sqrt{d}) \prod_{i \in I} \mathcal{I}_i = (\beta) \qquad (\text{with } N_{\mathbf{k}/\mathbf{Q}}(\beta) > 0) \tag{7}$$

between the ordinary ideal classes of the $t-1$ ideals whose narrow ideal classes generate $\mathrm{Cl}_{\mathbf{k}}^{+}[2]$. In particular, χ must satisfy $\chi\Big(\prod_{i \in I} \mathcal{I}_i\Big) = \chi((\alpha)) = \nu(\alpha)\chi_0(\alpha) = -\chi_0(\alpha)$ which amounts to asking

$$\textbf{Necessary 3:} \qquad \prod_{i \in I} \varepsilon_i = -\chi_0(\alpha) \qquad (\text{where } \varepsilon_i = \chi(\mathcal{I}_i) \in \{\pm 1\}). \tag{8}$$

Now, for any non-zero integral ideal $\mathcal{I}$ of $\mathbf{k}$ relatively prime to $\mathcal{F}_0$, using 3) we get

$$\chi(\mathcal{I}) = \chi_0(\alpha_{\mathcal{I}}) \prod_{i=1}^{t-1} \varepsilon_i^{e_{\mathcal{I}}(i)} =: \chi_0^{\vec{\varepsilon}}(\mathcal{I}) \qquad (\text{where } \varepsilon_i = \chi(\mathcal{I}_i) \in \{\pm 1\}).$$

Conversely, let χ_0 be a primitive quadratic modular character on $(\mathbf{A_k}/\mathcal{F}_0)^*$ satisfying (4) and (5) and let $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{t-1}) \in \{\pm 1\}^{t-1}$ be given and satisfying the relation (8). Then (9) defines a quadratic character $\chi = \chi_0^{\vec{\varepsilon}}$ on $\mathrm{Cl}_\mathbf{k}^+(\mathcal{F}_0)$ whose associated modular quadratic character is χ_0 and we have proved:

Proposition 4. *Let* $\mathbf{k}$ *and* $\mathcal{F}_0$ *be given. Assume that the exponent of* $\mathrm{Cl}_\mathbf{k}^+$ *is* ≤ 2 *and let* $\mathcal{I}_i$, $1 \leq i \leq t-1$, *be* $t-1$ *integral ideals of* $\mathbf{k}$ *relatively prime to* $\mathcal{F}_0$ *and whose narrow ideal classes generate* $\mathrm{Cl}_\mathbf{k}^+[2] = \mathrm{Cl}_\mathbf{k}^+$.

1. *If there exists a quartic CM-field* $\mathbf{K}$ *containing* $\mathbf{k}$ *of conductor* $\mathcal{F}_{\mathbf{K}\ \mathbf{k}} = \infty_1\infty_2\mathcal{F}_0$, *then* (4) *and* (5) *are satisfied for some primitive quadratic character* χ_0 *on* $(\mathbf{A_k}/\mathcal{F}_0)^*$.
2. *Conversely, let* χ_0 *be a primitive quadratic character* χ_0 *on* $\left(\mathbf{A_k}/\mathcal{F}_0\right)^*$ *such that* (4) *and* (5) *are satisfied. Then there are*

$$h_\mathbf{k} = \begin{cases} 2^{t-1} & \text{if } \nu(\varepsilon_\mathbf{k}) = -1, \\ 2^{t-2} & \text{if } \nu(\varepsilon_\mathbf{k}) = +1 \end{cases}$$

quartic CM-fields $\mathbf{K}$ *containing* $\mathbf{k}$, *of conductor* $\mathcal{F}_{\mathbf{K}/\mathbf{k}} = \infty_1\infty_2\mathcal{F}_0$ *and whose associated modular characters are equal to* χ_0.

3.3. Primitive quadratic modular characters.

Write $\mathcal{F}_{\mathbf{K}/\mathbf{k}} = \infty_1\infty_2\mathcal{F}_2\mathcal{F}_{\mathrm{odd}}$ with $\mathcal{F}_2$ of 2-power norm and $\mathcal{F}_{\mathrm{odd}}$ of odd norm f_{odd}, and let $\chi_0 = \chi_2\chi_{\mathrm{odd}}$ be the associated factorization of χ_0. The aim of this section is to determine the possible choices for these components of χ_0. To begin with, notice that χ_0 is quadratic and primitive if and only if both χ_2 and χ_{odd} are quadratic and primitive.

3.3.1. Primitive quadratic characters modulo $\mathcal{P}_p^e$, $p \geq 3$.

If $p \geq 3$ is an odd prime and n is a rational integer, then $\left(\frac{n}{p}\right)$ denotes Legendre's symbol. If $m \geq 3$ is odd and n is a rational integer, then $\left(\frac{n}{m}\right)$ denotes Jacobi's generalization of Legendre's symbol.

Lemma 5. *Let* $\mathbf{k}$ *be a real quadratic field,* $\mathbf{A_k}$ *be its ring of algebraic integers,* $p \geq 3$ *be an odd prime and let* $\left(\frac{\cdot}{p}\right)$ *denote the Legendre symbol.*

1. *Assume that* $(p) = \mathcal{P}$ *is inert in the quadratic extension* $\mathbf{k}/\mathbf{Q}$. *If* χ *is a primitive quadratic character on the multiplicative group* $\left(\mathbf{A_k}/\mathcal{P}^e\right)^*$, *then* $e = 1$. *Moreover,* $\alpha \mapsto \phi_\mathcal{P}(\alpha) = \left(\frac{N_{\mathbf{k}/\mathbf{Q}}(\alpha)}{p}\right)$ *is the only non trivial quadratic character on* $(\mathbf{A_k}/\mathcal{P})^*$.
2. *Assume that* p *is not inert in the quadratic extension* $\mathbf{k}/\mathbf{Q}$ *and let* $\mathcal{P}$ *be any one of the prime ideals of* $\mathbf{k}$ *lying above* p. *If* χ *is a primitive quadratic character on the multiplicative group* $\left(\mathbf{A_k}/\mathcal{P}^e\right)^*$, *then* $e = 1$.

Moreover, $\alpha \mapsto \phi_{\mathcal{P}}(\alpha) = \left(\frac{a_\alpha}{p}\right)$ *is the only non trivial quadratic character on* $(\mathbf{A_k}/\mathcal{P})^*$*, where* a_α *is any rational integer such that* $\alpha \equiv a_\alpha \pmod{\mathcal{P}}$. *Moreover, if* $(p) = \mathcal{P}\mathcal{P}'$ *splits in* $\mathbf{k}$*, then for any* $\alpha \in \mathbf{A_k}$ *we have* $\phi_{\mathcal{P}}(\alpha)\phi_{\mathcal{P}'}(\alpha) = \left(\frac{N_{\mathbf{k}/\mathbf{Q}}(\alpha)}{p}\right)$.

COROLLARY 6. *Assume that there exists a primitive quadratic character on the multiplicative group* $\left(\mathbf{A_k}/\mathcal{F}_{\text{odd}}\right)^*$ *where* $\mathcal{F}_{\text{odd}} = n_{\text{odd}}Q_{\text{odd}}\mathbf{Z} + n_{\text{odd}}\left((P_{\text{odd}} + \sqrt{d_\mathbf{k}}\right)/2)\mathbf{Z}$ *is an integral ideal of* $\mathbf{k}$ *of odd norm* $f_{\text{odd}} = n_{\text{odd}}^2 Q_{\text{odd}}$. *Then* $\mathcal{F}_{\text{odd}}$ *is square-free, i.e., if a prime ideal* $\mathcal{P}$ *of* $\mathbf{k}$ *divides* $\mathcal{F}_{\text{odd}}$*, then* $\mathcal{P}^2$ *does not divide* $\mathcal{F}_{\text{odd}}$. *Hence,* $Q_{\text{odd}} \geq 1$ *and* $n_{\text{odd}} \geq 1$ *are square-free relatively prime positive rational integers. Conversely, if* $\mathcal{F}_{\text{odd}}$ *of odd norm is square free, then*

$$\alpha = (x + y\sqrt{d_\mathbf{k}})/2 \in \mathbf{A_k} \mapsto \chi_{\text{odd}}(\alpha) = \left(\frac{N_{\mathbf{k}/\mathbf{Q}}(\alpha)}{n_{\text{odd}}}\right)\left(\frac{(x - P_{\text{odd}}y)/2}{Q_{\text{odd}}}\right)$$

is the only primitive quadratic character on $\left(\mathbf{A_k}/\mathcal{F}_{\text{odd}}\right)^*$.

Notice that $\chi_{\text{odd}}(n) = \left(\frac{n}{Q_{\text{odd}}}\right)$ *whenever* $n \in \mathbf{Z}$ *is relatively prime with* f_{odd}.

3.3.2. Primitive quadratic characters modulo $\mathcal{P}_2^e$.

LEMMA 7. *Assume that* $(2) = \mathcal{P}_2\mathcal{P}_2'$ *splits in* $\mathbf{k}$. *Hence* $d_\mathbf{k} \equiv 1 \pmod 8$. *Set* $\mathcal{P}_2^e = 2^e\mathbf{Z} + \omega_e\mathbf{Z}$ *with* $\omega_e = (P_e + \sqrt{d_\mathbf{k}})/2$ *and* $d_\mathbf{k} \equiv P_e^2 \pmod{2^{e+2}}$. *Since* $\alpha = x + y\omega_e \in \mathbf{A_k} \mapsto x \in \mathbf{Z}$ *induces an isomorphism from* $\mathbf{A_k}/\mathcal{P}_2^e$ *to* $\mathbf{Z}/2^e\mathbf{Z}$, *any character on* $\left(\mathbf{A_k}/\mathcal{P}_2^e\right)^*$ *may be construed as a character on* $\left(\mathbf{Z}/2^e\mathbf{Z}\right)^*$. *Therefore, there exists a primitive quadratic character on* $\left(\mathbf{A_k}/\mathcal{P}_2^e\right)^*$ *if and only if* $e \in \{2,3\}$ *and there is only one such character for* $e = 2$ *whereas there are two such characters for* $e = 3$.

LEMMA 8. *Assume that* $(2) = \mathcal{P}_2$ *is inert in* $\mathbf{k}$. *Hence* $d_\mathbf{k} \equiv 5 \pmod 8$. *Let* $\phi_{8,+}$ *denote the primitive quadratic Dirichlet character of conductor* 8 *associated with the real quadratic field* $\mathbf{Q}(\sqrt{2})$. *If there exists a primitive quadratic character* ϕ *on* $\left(\mathbf{A_k}/\mathcal{P}_2^e\right)^*$, *then* $e \in \{2,3\}$. *Conversely, any non trivial quadratic character* ϕ *on* $\left(\mathbf{A_k}/\mathcal{P}_2^2\right)^*$ *is primitive and there are three such primitive quadratic characters. Then, there are four primitive quadratic characters on* $\left(\mathbf{A_k}/\mathcal{P}_2^3\right)^*$*: the characters* $\alpha \mapsto \phi(\alpha)\phi_{8,+}(N_{\mathbf{k}/\mathbf{Q}}(\alpha))$ *where* $\phi \in \{\phi_1, \phi_2, \phi_3, \phi_4\}$ *runs over the following four quadratic characters on* $\left(\mathbf{A_k}/\mathcal{P}_2^3\right)^*$

$d \equiv 5 \pmod 8$ and ϕ quadratic modulo $(4) = \mathcal{P}_2^2$.

ϕ	$\ker\phi$	Remark
ϕ_1		ϕ_1 is trivial
ϕ_2	$\alpha^3 \in \{1, \sqrt{d}\}$	ϕ_2 is primitive
ϕ_3	$\alpha^3 \in \{1, 3\}$	ϕ_3 is primitive
ϕ_4	$\alpha^3 \in \{1, 2+\sqrt{d}\}$	ϕ_4 is primitive

Proof. If $e \geq 4$, then $\alpha = 1 + 2^{e-1}\beta \equiv 1 \pmod{\mathcal{P}_2^{e-1}}$ implies $\alpha \equiv (1 + 2^{e-2}\beta)^2 \pmod{\mathcal{P}_2^e}$ and $\phi(\alpha) = +1$. Hence, ϕ is not primitive. Moreover, since $\ker((\mathbf{A_k}/\mathcal{P}_2^3)^* \to (\mathbf{A_k}/\mathcal{P}_2^2)^*) = \{1, 5, -1+2\sqrt{d_\mathbf{k}}, 5(-1+2\sqrt{d_\mathbf{k}})\}$ and since $5 \equiv (\sqrt{d_\mathbf{k}})^2 \pmod{\mathcal{P}_2^3}$, a quadratic character ϕ on $(\mathbf{A_k}/\mathcal{P}_2^3)^*$ is primitive if and only if $\phi(-1 + 2\sqrt{d_\mathbf{k}}) = -1$, which yields the desired results (since $N_{\mathbf{k}/\mathbf{Q}}(-1 + 2\sqrt{d_\mathbf{k}}) = 1 - 4d_\mathbf{k} \equiv 5 \pmod 8$ and $\phi_{8,+}(5) = -1$). Moreover, if $\alpha = (x_\alpha + y_\alpha\sqrt{d_\mathbf{k}})/2 \in \mathbf{A_k}$, then $\alpha^3 = (x_\alpha(x_\alpha^2 + 3d_\mathbf{k}y_\alpha^2) + y_\alpha(3x_\alpha^2 + d_\mathbf{k}y_\alpha^2)\sqrt{d_\mathbf{k}})/8 = X_\alpha + Y_\alpha\sqrt{d_\mathbf{k}} \in \mathbf{Z}[\sqrt{d_\mathbf{k}}]$, $\phi(\alpha) = \phi(\alpha^3)$ and

$$\alpha^3 = X_\alpha + Y_\alpha\sqrt{d_\mathbf{k}} \equiv \begin{cases} X_\alpha - Y_\alpha \pmod{(4)} & \text{if } Y_\alpha \text{ is even,} \\ X_\alpha - Y_\alpha + 1 + \sqrt{d_\mathbf{k}} \pmod{(4)} & \text{if } Y_\alpha \text{ is odd} \end{cases}$$

is equal to $1, 3, \sqrt{d_\mathbf{k}}$, or $2 + \sqrt{d_\mathbf{k}}$ modulo $(4) = \mathcal{P}_2^2$ and we obtain the desired Table of the four quadratic characters on $(\mathbf{A_k}/(4))^*$. □

Lemma 9. *Assume that $(2) = \mathcal{P}_2^2$ is ramified in $\mathbf{k}$. Hence $d_\mathbf{k} = 4d \equiv 0 \pmod 4$ with $d \equiv 2, 3 \pmod 4$. If there exists a primitive quadratic character ϕ on $(\mathbf{A_k}/\mathcal{P}_2^e)^*$, then $e \in \{2, 4, 5\}$. Conversely,*

1. *The only non trivial quadratic character on $(\mathbf{A_k}/\mathcal{P}_2^2)^*$ is primitive.*
2. *For $d \equiv 2 \pmod 4$, there are two primitive quadratic characters on $(\mathbf{A_k}/\mathcal{P}_2^4)^*$ and four primitive quadratic characters on $(\mathbf{A_k}/\mathcal{P}_2^5)^*$: the characters $\phi_0\phi_i$ where ϕ_i, $1 \leq i \leq 4$, runs over the following four quadratic characters modulo $\mathcal{P}_2^4$*

$d \equiv 2 \pmod 4$ and ϕ quadratic modulo $(4) = \mathcal{P}_2^4$.

ϕ	$\ker\phi$	Remark
ϕ_1		ϕ_1 is trivial
ϕ_2	$1,\ 3+2\sqrt{d},\ 1+3\sqrt{d},\ 3+3\sqrt{d}$	ϕ_2 is primitive
ϕ_3	$1,\ 1+\sqrt{d},\ 3+\sqrt{d},\ 3+2\sqrt{d}$	ϕ_3 is primitive
ϕ_4	$1,\ 3,\ 1+2\sqrt{d},\ 3+2\sqrt{d}$	ϕ_4 is not primitive

and ϕ_0 *is the primitive quadratic character modulo* $\mathcal{P}_2^5$ *satisfying*

$$\ker\phi_0 = \{1, 7, 1+\sqrt{d}, d+5+\sqrt{d}, d+3+2\sqrt{d}, d+1+2\sqrt{d}, d-1+3\sqrt{d}, 7+3\sqrt{d}\}.$$

3. *For* $d \equiv 3 \pmod 4$, *there are two primitive quadratic characters on* $(\mathbf{A_k}/\mathcal{P}_2^4)^*$ *and four primitive quadratic characters on* $(\mathbf{A_k}/\mathcal{P}_2^5)^*$ *: the characters* $\phi_0\phi_i$, $1 \leq i \leq 4$, *where* ϕ_i *runs over the following four quadratic characters modulo* $\mathcal{P}_2^4$

$d \equiv 3 \pmod 4$ and ϕ quadratic modulo $(4) = \mathcal{P}_2^4$.

ϕ	$\ker\phi$	Remark
ϕ_1		ϕ_1 is trivial
ϕ_2	$1, 3, 2+\sqrt{d}, 2+3\sqrt{d}$	ϕ_2 is primitive
ϕ_3	$1, 3, \sqrt{d}, 3\sqrt{d}$	ϕ_3 is primitive
ϕ_4	$1, 3, 1+2\sqrt{d}, 3+2\sqrt{d}$	ϕ_4 is not primitive

and ϕ_0 *is the primitive quadratic character modulo* $\mathcal{P}_2^5$ *satisfying*

$$\ker\phi_0 = \{1, d, \sqrt{d}, 6+\sqrt{d}, 1+2\sqrt{d}, 2-d+2\sqrt{d}, 3-d+3\sqrt{d}, 5-d+3\sqrt{d}\}.$$

Proof. Let ϕ be a quadratic character ϕ on $(\mathbf{A_k}/\mathcal{P}_2^e)^*$. If $e = 2e'+1 \geq 6$ is odd, then $\alpha = 1+2^{e'}\beta \equiv 1 \pmod{\mathcal{P}_2^{e-1}}$ implies $\alpha \equiv (1+2^{e'-1}\beta)^2 \pmod{\mathcal{P}^e}$ and $\phi(\alpha) = +1$. If $e = 2e' \geq 6$ is even, then $\alpha = 1 + 2^{e'-1}\beta \equiv 1 \pmod{\mathcal{P}_2^{e-1}}$ implies $\beta \in \mathcal{P}_2$ and $\alpha \equiv (1+2^{e'-2}\beta)^2 \pmod{\mathcal{P}^e}$ and $\phi(\alpha) = +1$. Hence, if ϕ is primitive, then $e \in \{2,3,4,5\}$. Moreover, since $\ker((\mathbf{A_k}/\mathcal{P}_2^3)^* \to (\mathbf{A_k}/\mathcal{P}_2^2)^*) = \{1, -1 = (d-1+\sqrt{d})^2\}$, if ϕ is primitive, then $e \neq 3$. Finally, the other results follow from the fact that

$$\ker((\mathbf{A_k}/\mathcal{P}_2^4)^* \to (\mathbf{A_k}/\mathcal{P}_2^3)^*) = \begin{cases} \{1, -(1+\sqrt{d})^2\} & \text{if } d \equiv 2 \pmod 4, \\ \{1, (\sqrt{d})^2(1+2\sqrt{d})\} & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and $\ker((\mathbf{A_k}/\mathcal{P}_2^5)^* \to (\mathbf{A_k}/\mathcal{P}_2^4)^*) = \{1, 5\}$ when $d \equiv 2, 3 \pmod 4$. □

3.4. Computation of generators of principal ideals.

We explain how to compute a totally positive generator of a primitive ideal $\mathcal{I} = Q\mathbf{Z} + ((P+\sqrt{d_{\mathbf{k}}})/2)\mathbf{Z}$ which is known to be principal in the narrow sense. Set $\omega_0 = (P+\sqrt{d_{\mathbf{k}}})/2Q$. To compute such a generator, we use the modified continued fraction expansion of ω_0, i.e., when we know ω_i, we let a_i denote the greatest integer less than ω_i and set $\omega_i = a_i + 1 - (1/\omega_{i+1})$. It is easily seen that we can write $\omega_i = (P_i + \sqrt{d_{\mathbf{k}}})/2Q_i$ where the P_i and Q_i are rational integers computed inductively thanks to the following relations:

$$\begin{aligned} P_0 = P \qquad &\text{and} \qquad Q_0 = Q\,, \\ P_{i+1} &= 2(a_i+1)Q_i - P_i\,, \\ Q_{i+1} &= (P_{i+1}^2 - d_{\mathbf{k}})/4Q_i\,. \end{aligned}$$

Set $\mathcal{I}_i = Q_i\mathbf{Z} + ((P_i + \sqrt{d_{\mathbf{k}}})/2)\mathbf{Z}$. Then $\mathcal{I}_i$ is a primitive integral ideal. Since $\mathcal{I}_{i+1} = (Q_{i+1}\omega_{i+1}/Q_i)\mathcal{I}_i$ and $N_{\mathbf{k}/\mathbf{Q}}(Q_{i+1}\omega_{i+1}/Q_i) = Q_{i+1}/Q_i > 0$, all the $\mathcal{I}_i$ are in the narrow ideal class of $\mathcal{I}_0 = \mathcal{I}$. Now, we define inductively matrices $\mathcal{P}_n$ with integral coefficients by

$$\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\mathcal{P}_n = \begin{pmatrix} p_n & -p_{n-1} \\ q_n & -q_{n-1} \end{pmatrix} = \mathcal{P}_{n-1}\begin{pmatrix} a_n+1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\mathcal{P}_n$ is in $\mathrm{SL}_2(\mathbf{Z})$, that $\omega_0 = \mathcal{P}_n\omega_n$ and that $\omega_n = \mathcal{P}_n^{-1}\omega_0$. Since $\mathcal{I}$ is principal, there exists $m \geq 0$ such that $Q_m = 1$. In particular, $\omega_m\mathbf{Z} + \mathbf{Z}$ is equal to the ring of algebraic integers of $\mathbf{k}$. Let β_m and γ_m be defined by

$$\begin{pmatrix} \beta_m \\ \gamma_m \end{pmatrix} = \mathcal{P}_m^{-1}\begin{pmatrix} (P+\sqrt{d_{\mathbf{k}}})/2 \\ Q \end{pmatrix} = \begin{pmatrix} -q_{m-1} & p_{m-1} \\ -q_m & p_m \end{pmatrix}\begin{pmatrix} (P+\sqrt{d_{\mathbf{k}}})/2 \\ Q \end{pmatrix},$$

which yields $\beta_m/\gamma_m = \mathcal{P}_m^{-1}\omega_0 = \omega_m$. Since $\mathcal{P}_m$ is in $\mathrm{SL}_2(\mathbf{Z})$, then

$$(\gamma_m) = \gamma_m\omega_m\mathbf{Z} + \gamma_m\mathbf{Z} = \beta_m\mathbf{Z} + \gamma_m\mathbf{Z} = ((P+\sqrt{d_{\mathbf{k}}})/2)\mathbf{Z} + Q\mathbf{Z} = \mathcal{I}.$$

Therefore,

$$\gamma_m = -q_m((P+\sqrt{d_{\mathbf{k}}})/2) + p_mQ = ((2p_mQ - q_mP) - q_m\sqrt{d_{\mathbf{k}}})/2 \tag{10}$$

is an explicit generator of the principal ideal $\mathcal{I}$ satisfying $N_{\mathbf{k}/\mathbf{Q}}(\gamma_m) > 0$.

3.5. Computation of a generator of a quartic CM-field.

Let $\mathbf{K}$ be a non-normal quartic CM-field which is associated to a primitive quadratic character $\chi_{\mathbf{K}/\mathbf{k}}$ on the unit ray class group $\mathrm{Cl}_{\mathbf{k}}^+(\mathcal{F}_0)$, where $\mathbf{k}$ denotes the real quadratic subfield of $\mathbf{K}$. Hence, $\mathcal{F}_{\mathbf{K}/\mathbf{k}} = \infty_1\infty_2\mathcal{F}_0$. The aim of this section is to explain how one can compute a totally positive $\beta_{\mathbf{K}/\mathbf{k}} \in \mathbf{A}_{\mathbf{k}}$ such that $\mathbf{K} = \mathbf{k}\left(\sqrt{-\beta_{\mathbf{K}/\mathbf{k}}}\right)$. Here again we assume that the exponent of $\mathrm{Cl}_{\mathbf{k}}^+$ is ≤ 2. In that case, the exponent of $\mathrm{Cl}_{\mathbf{k}}$ is ≤ 2 and the 2-rank $t_{\mathbf{k}}^w$ of this class group $\mathrm{Cl}_{\mathbf{k}}$ is given by

$$t_{\mathbf{k}}^w = \begin{cases} t-1 & \text{if } \nu(\varepsilon_{\mathbf{k}}) = -1\,, \\ t-2 & \text{if } \nu(\varepsilon_{\mathbf{k}}) = +1\,. \end{cases} \tag{11}$$

We fix $\Delta_{\mathbf{k}}^{w} = \{\mathcal{P}_i : 1 \leq i \leq t_{\mathbf{k}}^{w}\}$, a set of pairwise distinct prime ramified ideals whose ideal classes generate $\mathrm{Cl}_{\mathbf{k}}[2]$, and set $\delta_{\mathbf{k}}^{w} = \prod_{i=1}^{t_{\mathbf{k}}^{w}} p_i$ (where empty products are equal to 1 and where p_i, which denotes the norm of $\mathcal{P}_i$, is a prime dividing $d_{\mathbf{k}}$). Notice that $\delta_{\mathbf{k}}^{w}$ is square-free and divides $d_{\mathbf{k}}$. Lemma 3 makes it easy to compute these data. Using Point 3. of Proposition 1, we obtain:

PROPOSITION 10. *Let $\varepsilon_{\mathbf{k}} = \left(x_{\mathbf{k}} + y_{\mathbf{k}}\sqrt{d_{\mathbf{k}}}\right)/2 > 1$ be the fundamental unit of a real quadratic field $\mathbf{k}$. Assume that the exponent of the narrow ideal class group of $\mathbf{k}$ is ≤ 2. Let $\mathbf{K}$ range over the non-normal quartic CM-fields containing $\mathbf{k}$. Set*

$$B_{\mathbf{K}/\mathbf{k}} = \begin{cases} y_{\mathbf{k}}\sqrt{d_{\mathbf{K}/\mathbf{k}}} & \text{if } \nu(\varepsilon_{\mathbf{k}}) = +1, \\ x_{\mathbf{k}}\sqrt{d_{\mathbf{K}/\mathbf{k}}/d_{\mathbf{k}}} & \text{if } \nu(\varepsilon_{\mathbf{k}}) = -1. \end{cases}$$

Then, the finite part $\mathcal{F}_0$ of the conductor $\mathcal{F}_{\mathbf{K}/\mathbf{k}}$ of the quadratic extension $\mathbf{K}/\mathbf{k}$ is principal in the narrow sense and there exist a unique totally positive generator $\alpha_{\mathbf{K}/\mathbf{k}} = \left(x_{\mathbf{K}/\mathbf{k}} + y_{\mathbf{K}/\mathbf{k}}\sqrt{d_{\mathbf{k}}}\right)/2 \in \mathcal{F}_0$ such that

$$1/\varepsilon_{\mathbf{k}} < \alpha_{\mathbf{K}/\mathbf{k}}/\sqrt{d_{\mathbf{K}/\mathbf{k}}} \leq \varepsilon_{\mathbf{k}} \iff -B_{\mathbf{K}/\mathbf{k}} < y_{\mathbf{K}/\mathbf{k}} \leq B_{\mathbf{K}/\mathbf{k}} \tag{12}$$

and a unique positive divisor $\delta_{\mathbf{K}/\mathbf{k}}$ of $\delta_{\mathbf{k}}^{w}$ such that $\mathbf{K} = \mathbf{k}\left(\sqrt{-\beta_{\mathbf{K}/\mathbf{k}}}\right)$ where $\beta_{\mathbf{K}/\mathbf{k}} = \delta_{\mathbf{K}/\mathbf{k}}\alpha_{\mathbf{K}/\mathbf{k}}$ is called the canonical generator *of $\mathbf{K}$.*

Note that $B_{\mathbf{K}/\mathbf{k}}$ is never a rational integer, and we let $\tilde{B}_{\mathbf{K}/\mathbf{k}}$ denote the greatest integer less than or equal to $B_{\mathbf{K}/\mathbf{k}}$. Hence, (12) $\iff |y_{\mathbf{K}/\mathbf{k}}| \leq \tilde{B}_{\mathbf{K}/\mathbf{k}}$. The condition (12) insures that $\alpha_{\mathbf{K}/\mathbf{k}}$ is a totally positive generator $\alpha = \left(x_\alpha + y_\alpha\sqrt{d_{\mathbf{k}}}\right)/2$ of $\mathcal{F}_{\mathbf{K}/\mathbf{k}}$ modulo $U_{\mathbf{k}}^2$ with the least absolute value for its second coordinate y_α. Now we explain how one can compute this canonical generator $\beta_{\mathbf{K}/\mathbf{k}}$ when $\mathbf{K}$ is described by using a primitive quadratic character χ on the unit ray class group $\mathrm{Cl}_{\mathbf{k}}(\infty_1\infty_2\mathcal{F}_{\mathbf{K}/\mathbf{k}})$. Let $p \geq 3$ be a prime. Assume that p does not divide $f_0 = N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0)$, that $(p) = \mathcal{P}\mathcal{P}'$ splits in $\mathbf{k}$ and that $\beta \in \mathbf{A}_{\mathbf{k}}$ is totally positive and such that $\mathbf{K} = \mathbf{k}(\sqrt{-\beta})$. We must have

$$\chi(\mathcal{P}) = \left[\frac{-\beta}{\mathcal{P}}\right] = \chi_{\mathcal{P}}(-\beta) \tag{13}$$

and deduce the following algorithm for computing $\beta_{\mathbf{K}/\mathbf{k}}$:

1. Compute a totally positive generator α of $\mathcal{F}_{\mathbf{K}/\mathbf{k}}$ (use Subsection 3.4).
2. Set $\delta_1 = 1$ and $\delta_2 = \delta_{\mathbf{k}}^{w}$.
3. If $\delta_1 = \delta_2$, then go to step 5.
4. Determine the least odd prime $p \geq 3$ relatively prime to f_0 which splits in $\mathbf{k}$ and such that $\left(\frac{\delta_1}{p}\right) \neq \left(\frac{\delta_2}{p}\right)$. Then, (13) cannot be satisfied for both $\beta = \delta_1\alpha$ and $\beta = \delta_2\alpha$. If

(13) is not satisfied for $\delta_1\alpha$, then let δ denote the least divisor of $\delta_{\mathbf{k}}^w$ which is greater than δ_1, set $\delta_1 = \delta$ and go to step 3, else (13) is not satisfied for $\delta_2\alpha$ and let δ denote the greatest divisor of $\delta_{\mathbf{k}}^w$ which is less than δ_2, set $\delta_2 = \delta$ and go to step 3.

5. Set $d = \delta_1 = \delta_2$.
6. If $\nu(\varepsilon_{\mathbf{k}}) = -1$, then set $\delta_{\mathbf{K}/\mathbf{k}} = d$ and go to step 12.
7. Set $\delta_1 = 1$ and $\delta_2 = \delta_{\mathbf{k}}^w$.
8. If $\delta_1 = \delta_2$, then go to step 10.
9. Determine the least odd prime $p \geq 3$ relatively prime to f_0 which splits in $\mathbf{k}$ and such that $(\frac{\delta_1}{p}) \neq (\frac{\delta_2}{p})$, and notice that (13) cannot be satisfied for both $\beta = \delta_1\varepsilon_{\mathbf{k}}\alpha$ and $\beta = \delta_2\varepsilon_{\mathbf{k}}\alpha$. If (13) is not satisfied for $\delta_1\varepsilon_{\mathbf{k}}\alpha$ then let δ denote the least divisor of $\delta_{\mathbf{k}}^w$ which is greater than δ_1, set $\delta_1 = \delta$ and go to step 8, else (13) is not satisfied for $\delta_2\varepsilon_{\mathbf{k}}\alpha$ and let δ denote the greatest divisor of $\delta_{\mathbf{k}}^w$ which is less than δ_2, set $\delta_2 = \delta$ and go to step 8.
10 Set $d' = \delta_1 = \delta_2$.
11. Let p be the least odd prime relatively prime to f_0 which splits in $\mathbf{k}$ and is such that $\chi_{\mathcal{P}}(-d\alpha) \neq \chi_{\mathcal{P}}(-d'\varepsilon_{\mathbf{k}}\alpha)$, and set

$$(\delta_{\mathbf{K}/\mathbf{k}}, \alpha) = \begin{cases} (d, \alpha) & \text{if } \chi(\mathcal{P}) = \chi_{\mathcal{P}}(-d\alpha), \\ (d', \varepsilon_{\mathbf{k}}\alpha) & \text{if } \chi(\mathcal{P}) \neq \chi_{\mathcal{P}}(-d\alpha). \end{cases}$$

12. While $y_\alpha > \tilde{B}_{\mathbf{K}/\mathbf{k}}$ do $\alpha = \alpha/\varepsilon_{\mathbf{k}}^2$.
13. While $y_\alpha < -\tilde{B}_{\mathbf{K}/\mathbf{k}}$ do $\alpha = \alpha\varepsilon_{\mathbf{k}}^2$.
14. Set $\beta_{\mathbf{K}/\mathbf{k}} = \delta_{\mathbf{K}/\mathbf{k}}\alpha_{\mathbf{K}/\mathbf{k}}$.
15. End.

Remarks 11. Since we do not distinguish isomorphic quartic CM-fields, when giving Tables we will assume that $0 \leq y_{\mathbf{K}/\mathbf{k}} < B_{\mathbf{K}/\mathbf{k}}$. Notice also that $\delta_{\mathbf{K}/\mathbf{k}}$ and $\alpha_{\mathbf{K}/\mathbf{k}}$ depend on the choice of $\Delta_{\mathbf{k}}^w$ and $\delta_{\mathbf{k}}^w$. For example, there exists (up to isomorphism) only one non-normal quartic CM-field $\mathbf{K}$ such that $d_{\mathbf{k}} = 60$ and $d_{\mathbf{K}/\mathbf{k}} = 601$ (by Proposition 4). Using Proposition 10, we obtain $\tilde{B}_{\mathbf{K}/\mathbf{k}} = 49$ and $(\delta_{\mathbf{K}/\mathbf{k}}, \alpha_{\mathbf{K}/\mathbf{k}}) = \big(2, 56+13\sqrt{15}\big)$ for $\delta_{\mathbf{k}}^w = 2$, but $(\delta_{\mathbf{K}/\mathbf{k}}, \alpha_{\mathbf{K}/\mathbf{k}}) = \big(3, 29-4\sqrt{15}\big)$ for $\delta_{\mathbf{k}}^w = 3$ (since $2\big(56+13\sqrt{15}\big) = 3\big(29-4\sqrt{15}\big)\big((3+\sqrt{15})/3\big)^2$, we have $\mathbf{k}\Big(\sqrt{-2\big(56+13\sqrt{15}\big)}\,\Big) = \mathbf{k}\Big(\sqrt{-3\big(29-4\sqrt{15}\big)}\,\Big)$).

3.6. Non-isomorphic quartic CM-fields.

Since the conductor of the extension $\mathbf{K}'/\mathbf{k}$ is the conjugate of the conductor of the extension $\mathbf{K}/\mathbf{k}$, we may and we will assume that $\mathcal{F}_0 = (n_0)\mathcal{I}_0$ where $n_0 \geq 1$ is a positive rational integer and where $\mathcal{I}_0 = Q_0\mathbf{Z} + \big((P_0 + \sqrt{d_{\mathbf{k}}})/2\big)\mathbf{Z}$ is a primitive ideal of norm $Q_0 \geq 1$ such that $0 \leq P_0 \leq Q_0$ (and such that $d_{\mathbf{k}} \equiv P_0^2 \pmod{4Q_0}$). Note that $d_{\mathbf{K}/\mathbf{k}} \stackrel{\text{def}}{=} N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0) \stackrel{\text{def}}{=} f_0 = n_0^2 Q_0$, that $\mathbf{K}$ is not normal if and only if $\sqrt{Q_0} \notin \mathbf{k}$ (use Point 4. of Proposition 1) and that $\mathcal{F}_0$ is invariant under the action of the Galois group of the quadratic extension $\mathbf{k}/\mathbf{Q}$ if and only if $P_0 = 0$ or $P_0 = Q_0$.

LEMMA 12. *If the exponent of* $\mathrm{Cl}_{\mathbf{k}}^{+}[2]$ *is* ≤ 2 *and if* $\nu(\varepsilon_{\mathbf{k}}) = -1$, *then* $\mathcal{F}_0$ *cannot be invariant under the action of the Galois group of the quadratic extension* $\mathbf{k}/\mathbf{Q}$.

Proof. Write $\mathcal{F}_0 = (n)\mathcal{R}$ where $n \geq 1$ is a rational integer and $\mathcal{R}$ is a primitive integral ideal of $\mathbf{k}$. If $\mathcal{F}_0$ were invariant under the action of $\mathrm{Gal}(\mathbf{k}/\mathbf{Q})$, then $\mathcal{R}$ would a product of distinct prime ideals of $\mathbf{k}$ ramified in $\mathbf{k}/\mathbf{Q}$ and $\mathcal{R}$ would be principal in the narrow sense, for so is $\mathcal{F}_0$ (see Proposition 10). According to the first point of Lemma 3, we would have $\mathcal{R} = (\sqrt{d})$ and $d_{\mathbf{K}/\mathbf{k}} = N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0) = n^2 d$ would be a square in $\mathbf{k} = \mathbf{Q}(\sqrt{d})$, a contradiction. □

LEMMA 13. *If* $\nu(\varepsilon_{\mathbf{k}}) = +1$ *and* $\mathcal{F}_0$ *is invariant under the action of the Galois group of the quadratic extension* $\mathbf{k}/\mathbf{Q}$, *then* $\mathbf{K}' = \mathbf{k}\left(\sqrt{-\delta_{\mathbf{K}/\mathbf{k}}\varepsilon_{\mathbf{k}}\alpha_{\mathbf{K}/\mathbf{k}}}\right)$ *is the conjugate field of* $\mathbf{K} = \mathbf{k}\left(\sqrt{-\delta_{\mathbf{K}/\mathbf{k}}\alpha_{\mathbf{K}/\mathbf{k}}}\right)$.

14. Computation of h_K^-

We remind the reader with the following technique for computing $h_{\mathbf{K}}^-$:

PROPOSITION 14. (See [Lou3].) *Set* $A_{\mathbf{K}/\mathbf{k}} = \sqrt{d_{\mathbf{K}}/\pi^2 d_{\mathbf{k}}}$ *and*

$$\phi_n = \sum_{N_{\mathbf{K}/\mathbf{k}}(\mathcal{I})=n} \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{I})$$

(*where* $\mathcal{I}$ *ranges over the integral ideals of* $\mathbf{k}$ *of a given norm* $n \geq 1$). *Let* $\gamma = 0.577215664\ldots$ *denote Euler's constant and set*

$$K(A) = 1 + \pi A + 4\sum_{n\geq 0}\left(a_n(\log A + \gamma - s_n) - b_n\right)\frac{A^{2n+2}}{(n!)^2} \qquad (A > 0)$$

where $a_n = (2n+1)^{-1} + (2n+2)^{-1}$, $b_n = (2n+1)^{-2} + (2n+2)^{-2}$ *and* $s_n = \sum_{k=1}^{n} k^{-1}$. *It holds that*

$$h_{\mathbf{K}}^- = \frac{A_{\mathbf{K}/\mathbf{k}}}{2\pi}\sum_{n\geq 1}\frac{\phi_n}{n}K(n/A_{\mathbf{K}/\mathbf{k}}) \tag{14}$$

and $0 \leq K(A) \leq 8e^{-A}$. *Hence, this series* (14) *is absolutely and rapidly convergent and can be used to compute efficiently* $h_{\mathbf{K}}^-$.

Proposition 15. (See [Lou3].) *Let $n = p^k$ be a prime-power.*

1. *Assume that p is inert in $\mathbf{k}/\mathbf{Q}$. Set $\varepsilon_p = \chi_{\mathbf{K}/\mathbf{k}}\big((p)\big) = \chi_0(p)$. Then*

$$\phi_{p^k} = \begin{cases} 0 & \text{if } p \text{ is ramified in } \mathbf{K}/\mathbf{Q}, \\ \varepsilon_p^{k/2}\big(1+(-1)^k\big)/2 & \text{if } p \text{ is not ramified in } \mathbf{K}/\mathbf{Q}. \end{cases}$$

2. *Assume that $(p) = \mathcal{P}^2$ is ramified in $\mathbf{k}/\mathbf{Q}$. Set $\varepsilon_p = \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{P})$. Then*

$$\phi_{p^k} = \varepsilon_p^k .$$

3. *Assume that $(p) = \mathcal{P}\mathcal{P}'$ splits in $\mathbf{k}/\mathbf{Q}$. Set $\varepsilon_{\mathcal{P}} = \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{P})$ and $\varepsilon_{\mathcal{P}'} = \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{P}')$. Then*

$$\phi_{p^k} = \begin{cases} \big(1+(-1)^k\big)/2 & \text{if } \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{P}\mathcal{P}') = \chi_{\mathbf{K}/\mathbf{k}}\big((p)\big) = \chi_0(p) = -1, \\ (k+1)\varepsilon_{\mathcal{P}}^k & \text{if } \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{P}\mathcal{P}') = \chi_{\mathbf{K}/\mathbf{k}}\big((p)\big) = \chi_0(p) = +1, \\ \varepsilon_{\mathcal{P}}^k + \varepsilon_{\mathcal{P}'}^k & \text{if } \chi_{\mathbf{K}/\mathbf{k}}(\mathcal{P}\mathcal{P}') = \chi_{\mathbf{K}/\mathbf{k}}\big((p)\big) = \chi_0(p) = 0. \end{cases}$$

5. A finitness result

The aim of this section is to prove that there are only finitely many non-normal quartic fields $\mathbf{K}$ with ideal class groups of exponent ≤ 2 and to give explicit bounds on their discriminants.

5.1. A necessary condition for the exponent to be ≤ 2.

Proposition 16. (See [Ear; Corollary 3.4].) *Let $\mathbf{K}$ be a CM-field with maximal totally real subfield $\mathbf{k}$. Let $T = t_{\mathbf{K}/\mathbf{k}}$ denote the number of prime ideals of $\mathbf{k}$ which are ramified in the quadratic extension $\mathbf{K}/\mathbf{k}$. Set $\mu = 0$ or 1 according as the canonical map $j = j_{\mathbf{K}/\mathbf{k}}$ from the ideal class group $\mathrm{Cl}_{\mathbf{k}}$ of $\mathbf{k}$ to the ideal class group $\mathrm{Cl}_{\mathbf{K}}$ of $\mathbf{K}$ is injective or not injective.*

If the exponent of the ideal class group of $\mathbf{K}$ is ≤ 2, then

$$h_{\mathbf{K}}^- = 2^{T-1+\rho-\mu} h_{\mathbf{k}} \qquad \text{where} \quad \big[N_{\mathbf{K}/\mathbf{k}}(\mathbf{K}^*) \cap U_{\mathbf{k}} : U_{\mathbf{k}}^2\big] = 2^\rho .$$

Moreover, if $h_{\mathbf{k}} = 1$, then $\mu = 0$ and the exponent of the ideal class group of $\mathbf{K}$ is ≤ 2 if and only if $h_{\mathbf{K}}^- = 2^{T-1+\rho}$.

Proof. The norm map $N = N_{\mathbf{K}/\mathbf{k}}\colon \mathrm{Cl}_{\mathbf{K}} \to \mathrm{Cl}_{\mathbf{k}}$ is onto, its kernel has order $h_{\mathbf{K}}^-$ and $[\ker j \circ N : \ker N] = \#\ker j = 2^\mu$ (remember that $\ker j$ has order 1 or 2 and use the fact that N is onto). Let $\mathrm{Cl}_{\mathbf{K}/\mathbf{k}}^{\mathrm{amb}}$ denote the subgroup of the ambiguous classes. Then, $\#\mathrm{Cl}_{\mathbf{K}/\mathbf{k}}^{\mathrm{amb}} = \big[N_{\mathbf{K}/\mathbf{k}}(\mathbf{K}^*) \cap U_{\mathbf{k}} : U_{\mathbf{k}}^2\big] 2^{T-1} h_{\mathbf{k}}$ (see [Lan1;

Chap. 13, p. 307, Lemma 4.1]). Assume that the exponent of $\mathrm{Cl}_{\mathbf{K}}$ is ≤ 2. Then $\ker j \circ N = \mathrm{Cl}_{\mathbf{K}/\mathbf{k}}^{\mathrm{amb}}$ and we get the desired result from

$$h_{\mathbf{K}} = \#\ker N = \#\ker j \circ N/[\ker j \circ N : \ker N] = \#\mathrm{Cl}_{\mathbf{K}/\mathbf{k}}^{\mathrm{amb}}/[\ker j \circ N : \ker N].$$

Let us prove the second point. If $h_{\mathbf{k}} = 1$, then the subgroup $\mathrm{Cl}_{\mathbf{K}}[2]$ of the classes of order ≤ 2 in $\mathrm{Cl}_{\mathbf{K}}$ is equal to $\mathrm{Cl}_{\mathbf{K}/\mathbf{k}}^{\mathrm{amb}}$, and the exponent of $\mathrm{Cl}_{\mathbf{K}}$ is ≤ 2 if and only if $h_{\mathbf{K}}^- = h_{\mathbf{K}}/h_{\mathbf{k}} = h_{\mathbf{K}} = \#\mathrm{Cl}_{\mathbf{K}/\mathbf{k}}^{\mathrm{amb}} = 2^{T-1+\rho}h_{\mathbf{k}} = 2^{T-1+\rho}$. □

COROLLARY 17. *Let $\mathbf{K}$ be a non-normal quartic CM-field, let $\mathbf{k}$ be its real quadratic subfield and let $\varepsilon_{\mathbf{k}} > 1$ be the fundamental unit of $\mathbf{k}$. Then,*

1. $j_{\mathbf{K}/\mathbf{k}}$ *is injective* (see [L01; p. 51]), $\rho \in \{0,1\}$ *and* $\rho = 1 \iff \varepsilon_{\mathbf{k}} \in N_{\mathbf{K}/\mathbf{k}}(\mathbf{K}^*)$. *In particular,* $\nu(\varepsilon_{\mathbf{k}}) = -1$ *implies* $\rho = 0$.
2. *If the exponent of the ideal class group of* $\mathbf{K}$ *is* ≤ 2, *then the exponent of the narrow ideal class group of* $\mathbf{k}$ *is* ≤ 2 (Point 8. of Proposition 1), *the prime ideals* $\mathcal{P}_{\mathbf{k}}$ *of* $\mathbf{k}$ *which are ramified in* $\mathbf{K}/\mathbf{k}$ *are principal in* $\mathbf{k}$ (use (2) to check it) *and according to* (11) *and Point* 1. *above, we have*
$$h_{\mathbf{K}}^- = 2^{T-1+\rho}h_{\mathbf{k}} \leq 2^{T+t-2}. \tag{15}$$
3. *If* $h_{\mathbf{k}}^+ = 1$, *then* $\nu(\varepsilon_{\mathbf{k}}) = -1$ *and* $\rho = 0$.

5.2. Computation of ρ.

To use (15) we must be able to compute ρ. Let α be any totally positive algebraic integer of $\mathbf{k}$ such that $\mathbf{K} = \mathbf{k}(\sqrt{-\alpha})$. We may assume that $\nu(\varepsilon_{\mathbf{k}}) = +1$ (Point 1. of Corollary 17). Then, $\varepsilon_{\mathbf{k}}$ is totally positive. Hence, $\varepsilon_{\mathbf{k}} \in N_{\mathbf{K}/\mathbf{k}}(\mathbf{K}^*)$ if and only if all the Hilbert norm residue symbols $(\varepsilon_{\mathbf{k}}, -\alpha)_{\mathcal{P}}$ are equal to $+1$ when $\mathcal{P}$ ranges over all but one of the prime ideals of $\mathbf{k}$. Since units are local norms in unramified extensions (see [Lan2; Chap. II, Sec. 4, p. 50, Corollary]), we may restrict $\mathcal{P}$ to range over all but one of the prime ideals of $\mathbf{k}$ which are ramified in $\mathbf{K}/\mathbf{k}$. In particular, we obtain:

LEMMA 18. *Assume that $\nu(\varepsilon_{\mathbf{k}}) = +1$ and that at most one prime ideal of $\mathbf{k}$ is ramified in $\mathbf{K}/\mathbf{k}$. Then $\rho = 1$.*

If $\mathcal{P}$ is ramified in $\mathbf{K}/\mathbf{k}$ and lies above an odd rational prime $p \geq 3$, then $1 = \nu_{\mathcal{P}}(\mathcal{F}_{\mathbf{K}/\mathbf{k}}) \equiv \nu_{\mathcal{P}}(\alpha) \pmod 2$ (see Proposition 10) and we obtain

$$(\varepsilon_{\mathbf{k}}, -\alpha)_{\mathcal{P}} = \left(\frac{\varepsilon_{\mathbf{k}}}{\mathcal{P}}\right)^{\nu_{\mathcal{P}}(\alpha)} = \left(\frac{\varepsilon_{\mathbf{k}}}{\mathcal{P}}\right) = \phi_{\mathcal{P}}(\varepsilon_{\mathbf{k}})$$

(with the notation of Lemma 5 and where $\left(\frac{\cdot}{\mathcal{P}}\right)$ denotes the quadratic residue symbol). In particular, if $\mathcal{P} = (p)$ is inert in $\mathbf{k}/\mathbf{Q}$ and ramified in $\mathbf{K}/\mathbf{k}$, then $(\varepsilon_{\mathbf{k}}, -\alpha)_{\mathcal{P}} = \phi_{\mathcal{P}}(\varepsilon_{\mathbf{k}}) = \left(\frac{\nu(\varepsilon_{\mathbf{k}})}{p}\right) = \left(\frac{+1}{p}\right) = +1$, whereas if $(p) = \mathcal{P}\mathcal{P}'$ splits in $\mathbf{k}$, then $\phi_{\mathcal{P}}(\varepsilon_{\mathbf{k}})\phi_{\mathcal{P}'}(\varepsilon_{\mathbf{k}}) = \left(\frac{\nu(\varepsilon_{\mathbf{k}})}{p}\right) = \left(\frac{+1}{p}\right) = +1$ yields $(\varepsilon_{\mathbf{k}}, -\alpha)_{\mathcal{P}} = \phi_{\mathcal{P}}(\varepsilon_{\mathbf{k}}) = \phi_{\mathcal{P}'}(\varepsilon_{\mathbf{k}}) = (\varepsilon_{\mathbf{k}}, -\alpha)_{\mathcal{P}'}$. Therefore, we obtain:

LEMMA 19. *Assume that $\nu(\varepsilon_{\mathbf{k}}) = +1$ and that at most one of the prime ideals of $\mathbf{k}$ lying above 2 is ramified in $\mathbf{K}/\mathbf{k}$. Then $\rho = 1$ if and only if for each odd rational prime p which divides $d_{\mathbf{K}/\mathbf{k}} = N_{\mathbf{k}/\mathbf{Q}}(\mathcal{F}_0)$ and is non inert in $\mathbf{k}/\mathbf{Q}$ we have $\phi_{\mathcal{P}}(\varepsilon_{\mathbf{k}}) = +1$. Here, $\mathcal{P}$ denotes any one of the prime ideals of $\mathbf{k}$ above p. In particular, ρ depends on $\mathbf{k}$ and $d_{\mathbf{K}/\mathbf{k}}$ only.*

Now, assume that $(2) = \mathcal{P}_2\mathcal{P}_2'$ splits in $\mathbf{k}$ and that both $\mathcal{P}_2$ and $\mathcal{P}_2'$ are ramified in $\mathbf{K}/\mathbf{k}$. It remains to compute the norm residue symbol $(\varepsilon_{\mathbf{k}}, -\beta_{\mathbf{K}/\mathbf{k}})_{\mathcal{P}_2}$. Let $\mathcal{F}_2 = \mathcal{P}_2^a\mathcal{P}_2'^b$ be the 2-part of $\mathcal{F}_0 = (\alpha_{\mathbf{K}/\mathbf{k}})$. Then $\mathcal{F}_2 \in \{(4), (4)\mathcal{P}_2, (4)\mathcal{P}_2', (8)\}$ (use Lemma 7). Since $\delta_{\mathbf{K}/\mathbf{k}}$ is square-free, we can write $\beta_{\mathbf{K}/\mathbf{k}} = \delta_{\mathbf{K}/\mathbf{k}}\alpha_{\mathbf{K}\ \mathbf{k}} = 2^n\gamma_{\mathbf{K}/\mathbf{k}}$ with $n \geq 2$ and $\gamma_{\mathbf{K}/\mathbf{k}}$ some algebraic integer of $\mathbf{k}$ such that $\gamma_{\mathbf{K}/\mathbf{k}} \not\equiv 0 \pmod{\mathcal{P}_2^2}$ and $\gamma_{\mathbf{K}/\mathbf{k}} \not\equiv 0 \pmod{\mathcal{P}_2'^2}$. We are reduced to compute the Hilbert's symbol $(\varepsilon_{\mathbf{k}}, -2^n\gamma_{\mathbf{K}/\mathbf{k}})_{\mathcal{P}_2}$:

LEMMA 20. *Assume that $(2) = \mathcal{P}_2\mathcal{P}_2'$ splits in $\mathbf{k}$, write $\varepsilon_{\mathbf{k}} = \big(x_{\mathbf{k}} + y_{\mathbf{k}}\sqrt{d_{\mathbf{k}}}\,\big)/2$ and $\mathcal{P}_2^3 = 8\mathbf{Z} + \big((P + \sqrt{d_{\mathbf{k}}}\,)/2\big)\mathbf{Z}$ with $d_{\mathbf{k}} \equiv P^2 \pmod{32}$ and let $\alpha = \big(x_\alpha + y_\alpha\sqrt{d_{\mathbf{k}}}\,\big)/2$ be an algebraic integer of $\mathbf{k}$ such that $\alpha \not\equiv 0 \pmod{\mathcal{P}_2^2}$. Then $(\varepsilon_{\mathbf{k}}, 2^n\alpha)_{\mathcal{P}_2} = \big((x_{\mathbf{k}} - Py_{\mathbf{k}})/2, 2^n(x_\alpha - Py_\alpha)/2\big)_2$.*

P r o o f. $\varepsilon_{\mathbf{k}} \equiv \big(x_{\mathbf{k}} - Py_{\mathbf{k}}\big)/2 \pmod{\mathcal{P}_2^3}$ and $\alpha \equiv \big(x_\alpha - Py_\alpha\big)/2 \pmod{\mathcal{P}_2^3}$. □

5.3. Lower bounds on $h_{\mathbf{K}}^-$ and the finitness result.

THEOREM 21. *Let $\mathbf{K}$ be a non-normal quartic CM-field and $\mathbf{k}$ denote its real quadratic subfield. Then $d_{\mathbf{K}} \geq 3 \cdot 10^7$ implies*

$$h_{\mathbf{K}}^- \geq \frac{\sqrt{d_{\mathbf{K}}/d_{\mathbf{k}}}}{12\big(\log(d_{\mathbf{K}}/d_{\mathbf{k}}) + 0.052\big)^2}\,. \tag{16}$$

If the exponent of the ideal class group of $\mathbf{K}$ is ≤ 2, then $h_{\mathbf{K}}^- \leq 2^{15}$ and $d_{\mathbf{K}}/d_{\mathbf{k}} \leq 4 \cdot 10^{16}$. In particular, there are only finitely many non-normal quartic CM-fields $\mathbf{K}$ with ideal class groups of exponents ≤ 2.

P r o o f. For (16), see [Lou5; Corollary 15] (notice that according to its proof, there is a misprint in the original statement of [Lou5; Corollary 15]). Let $T = t_{\mathbf{K}/\mathbf{k}} \geq 1$ and $t = t_{\mathbf{k}/\mathbf{Q}} \geq 1$ denote the number of prime ideals of $\mathbf{k}$ which are ramified in the quadratic extension $\mathbf{K}/\mathbf{k}$ and the number of rational primes which are ramified in the quadratic extension $\mathbf{k}/\mathbf{Q}$, respectively. Set $p_1 = p_2 = 3$, $p_3 = p_4 = 4$ and let $(p_i)_{i\geq 3}$ denote the increasing sequence of the odd primes greater than or equal to 5, each prime been repeated twice. Set $\delta_r = \prod_{i=1}^{r} p_i$. Then, $d_{\mathbf{K}}/d_{\mathbf{k}} = d_{\mathbf{k}}d_{\mathbf{K}/\mathbf{k}} \geq \delta_{t+T}$. Now, assume that the exponent of the ideal class group of $\mathbf{K}$ is ≤ 2. Then, $h_{\mathbf{K}}^{-} = 2^{T-1}h_{\mathbf{k}} = 2^{T+t-2}$, $d_{\mathbf{K}}/d_{\mathbf{k}} \geq \delta_{t+T}$ and

$$2^{T+t-2} \geq \frac{\sqrt{\delta_{t+T}}}{12(\log(\delta_{t+T}) + 0.052)^2}.$$

The reader can easily check that this implies $t+T \leq 19$, $h_{\mathbf{K}}^{-} = 2^{T+t-2} \leq 2^{17}$ and using (16) we obtain $d_{\mathbf{K}}/d_{\mathbf{k}} \leq 10^{19}$. To obtain the 250-fold improvement $d_{\mathbf{K}}/d_{\mathbf{k}} \leq 4 \cdot 10^{16}$, we use better lower bounds for relative class number of CM-fields (see [Lou8]). □

5.4. Powerful necessary conditions.

PROPOSITION 22. (See [Lou5; Lemma 6].) *Let $\mathbf{k}_1$ be a given real quadratic field and $f_0 > 1$ be a given positive integer. Let $d_1 > 1$ and $d_2 > 1$ be square-free and such that $\mathbf{k}_1 = \mathbf{Q}(\sqrt{d_1})$ and $\mathbf{k}_2 \stackrel{\text{def}}{=} \mathbf{Q}(\sqrt{f_0})$. If there exists a non-normal quartic CM-field $\mathbf{K}_1$ containing $\mathbf{k}_1$ and such that $d_{\mathbf{K}_1/\mathbf{k}_1} = f_0$, then it holds that:*

1. $\left(\frac{d_1}{p_2}\right) = +1$ *for all odd primes p_2 which divide d_2 but do not divide d_1,*
2. $\left(\frac{d_2}{p_1}\right) = +1$ *for all odd primes p_1 which divide d_1 but do not divide d_2,*
3. $\left(\frac{-d_1d_2/p^2}{p}\right) = +1$ *for all odd primes p which divide both d_1 and d_2.*

PROPOSITION 23.

1. *If $\mathbf{K}$ is a CM-field of degree $2n$ with maximal totally real subfield $\mathbf{k}$, if $\mathcal{P}$ is a prime ideal of $\mathbf{K}$ which splits in the quadratic extension $\mathbf{K}/\mathbf{k}$ and if $\mathcal{P}^m$ is principal, then $N_{\mathbf{K}/\mathbf{Q}}(\mathcal{P}^m) \geq \frac{1}{4^n}d_{\mathbf{K}/\mathbf{k}}$.*
2. *Let $p \geq 2$ denote a prime and $\left(\frac{\cdot}{p}\right)$ denote Kronecker's symbol. Let $\mathbf{K}$ be a non-normal quartic CM-field. Let $\mathbf{k}$ denote its real quadratic subfield. Assume that the exponent of the ideal class group of $\mathbf{K}$ is ≤ 2 and that p does not divide $d_{\mathbf{K}} = d_{\mathbf{k}}^2 d_{\mathbf{K}/\mathbf{k}}$. Then,*

$$\left(\frac{d_{\mathbf{k}}}{p}\right) = +1 \quad \textit{and} \quad p^2 < \frac{1}{16}d_{\mathbf{K}/\mathbf{k}} \qquad \textit{imply} \quad \left(\frac{d_{\mathbf{K}/\mathbf{k}}}{p}\right) = +1,$$

$$\left(\frac{d_{\mathbf{k}}}{p}\right) = -1 \quad \textit{and} \quad p^4 < \frac{1}{16}d_{\mathbf{K}/\mathbf{k}} \qquad \textit{imply} \quad \left(\frac{d_{\mathbf{K}/\mathbf{k}}}{p}\right) = -1.$$

Proof.

1. Let α be an algebraic integer of $\mathbf{K}$ such that $\mathcal{P}^m = (\alpha)$ and let $\bar{\alpha}$ denote the conjugate of α in the quadratic extension $\mathbf{K}/\mathbf{k}$. Since $\mathcal{P}$ splits in $\mathbf{K}/\mathbf{k}$, $\alpha \in \mathbf{K} \setminus \mathbf{k}$ and α is a root of $P(X) = X^2 - \mathrm{Tr}_{\mathbf{K}/\mathbf{k}}(\alpha)X + N_{\mathbf{K}/\mathbf{k}}(\alpha)$. Set $\beta = N_{\mathbf{K}/\mathbf{k}}(\alpha+\bar{\alpha}) = (\alpha+\bar{\alpha})^2$ and $\gamma = N_{\mathbf{K}/\mathbf{k}}(\alpha-\bar{\alpha}) = -(\alpha-\bar{\alpha})^2$. Since β and γ are

totally positive elements of $\mathbf{k}$, we have $N_{\mathbf{k}/\mathbf{Q}}(\gamma) \leq N_{\mathbf{k}/\mathbf{Q}}(\beta+\gamma) = N_{\mathbf{k}/\mathbf{Q}}(4\alpha\alpha) = 4^n N_{\mathbf{K}/\mathbf{Q}}(\alpha) = 4^n N_{\mathbf{K}/\mathbf{Q}}(\mathcal{P}^m)$. Now, the different $\mathcal{D}_{\mathbf{K}/\mathbf{k}}$ divides the principal ideal $P'(\alpha) = 2\alpha - \mathrm{Tr}_{\mathbf{K}/\mathbf{k}}(\alpha) = \alpha - \bar{\alpha}$ (see [Lan2; p. 62, Proposition 8]), we obtain $d_{\mathbf{K}/\mathbf{k}} = N_{\mathbf{K}/\mathbf{Q}}(\mathcal{D}_{\mathbf{K}/\mathbf{k}}) \leq N_{\mathbf{K}/\mathbf{Q}}(\alpha - \bar{\alpha}) = N_{\mathbf{k}/\mathbf{Q}}(\gamma) \leq 4^n N_{\mathbf{K}/\mathbf{Q}}(\mathcal{P}^m)$ (see also [Lou2; Proof of Theorem D] and [L01; Proof of Theorem 6] and notice that the result we have just proved is better than the one quoted in [Lou6; Theorem 2.1]).

2. Using ramification groups, the reader will check that if all the prime ideals of $\mathbf{k}$ lying above a rational prime $p \geq 2$ are inert in $\mathbf{K}/\mathbf{k}$, then the inertia field of p is either equal to $\mathbf{N}^+$, in which case p splits completely in $\mathbf{N}^+/\mathbf{Q}$, or is equal to $\mathbf{k}^+$, in which case p is inert in both $\mathbf{k}/\mathbf{Q}$ and $\mathbf{k}'/\mathbf{Q}$. Hence, if $(\frac{d_{\mathbf{k}}}{p}) \neq (\frac{d_{\mathbf{K}/\mathbf{k}}}{p})$, then at least one of the prime ideals $\mathcal{P}$ of $\mathbf{k}$ lying above a rational prime $p \geq 2$ splits in $\mathbf{K}/\mathbf{k}$. Since $\mathcal{P}_{\mathbf{K}}^2$ is principal, the previous point yields $N_{\mathbf{K}/\mathbf{Q}}(\mathcal{P}^2) \geq \frac{1}{16} d_{\mathbf{K}/\mathbf{k}}$, from which we get the desired results. □

Let $\mathbf{k}$ be a given real quadratic field whose narrow ideal class group has exponent ≤ 2. Let $f_0 \geq 1$ be a given rational integer. Assume that there exists at least one non-normal quartic CM-field $\mathbf{K}$ with real quadratic subfield $\mathbf{k}$ and such that $d_{\mathbf{K}/\mathbf{k}} = f_0$. There must exist $n \geq 0$, $a \geq 1$ odd and $b \geq 1$ odd and relatively prime with a such that $f_0 = 2^n a^2 b$. Moreover, b must be such that if a prime p divides b, then p is not inert in $\mathbf{k}/\mathbf{Q}$, i.e. $(\frac{d_{\mathbf{k}}}{p}) \neq -1$. Now, for a given $\mathbf{k}$ and a given such f_0, Propositions 22 and 23 are used to get rid of most of the large values of $f_0 = d_{\mathbf{K}/\mathbf{k}}$, prior to constructing all the $\mathbf{K}$'s of a given $\mathbf{k}$ and a given $f_0 = d_{\mathbf{K}/\mathbf{k}}$, computing their relative class numbers and computing their ideal class group structures whenever the bound $h_{\mathbf{K}}^- \leq 2^{T+t-2}$ is satisfied (see (15)).

6. Tables

We finally list in the following Tables all the non isomorphic non-normal quartic CM-fields with ideal class groups of exponent ≤ 2 (678 such fields) and all the dihedral octic CM-fields with ideal class groups of exponent ≤ 2 (116 such fields).

(1) The first column gives the discriminant $d_{\mathbf{k}}$ of the real quadratic subfield $\mathbf{k}$ of $\mathbf{K}$. This discriminant is in bold type numbers if $h_{\mathbf{k}}^+ = 1$, and is in italic type numbers if $h_{\mathbf{k}} = 1$ but $h_{\mathbf{k}}^+ > 1$ (hence if $h_{\mathbf{k}}^+ = 2$).

(2) The second column gives the norm $d_{\mathbf{K}/\mathbf{k}}$ of the finite part $\mathcal{F}_0$ of the conductor $\mathcal{F}_{\mathbf{K}/\mathbf{k}}$ of the quadratic extension $\mathbf{K}/\mathbf{k}$. It is in bold type numbers if $\mathcal{F}_0$ is invariant under the action of the Galois group $\mathrm{Gal}(\mathbf{k}/\mathbf{Q})$ (10 such cases).

(3) The third column gives the coordinates $n(x,y)$ of the canonical generator $\beta_{\mathbf{K}/\mathbf{k}} = \delta_{\mathbf{K}/\mathbf{k}}\alpha_{\mathbf{K}/\mathbf{k}}$ of $\mathbf{K}$ (see Proposition 10), with $\delta_{\mathbf{K}/\mathbf{k}} = n$ and $\alpha_{\mathbf{K}/\mathbf{k}} = (x+y\sqrt{d_{\mathbf{k}}})/2$. Notice that in all the occurrences in these Tables we have $\delta_{\mathbf{K}/\mathbf{k}} = n = 1$. However, see Remark 11.

(4) The fourth column gives the relative class number $h_{\mathbf{K}}^{-}$ of $\mathbf{K}$.

(5) The fifth column gives the value of $\rho \in \{0,1\}$ defined in Proposition 16. The symbol • in this column means that $\rho = 0$ for we are in the case that $N_{\mathbf{k}/\mathbf{Q}}(\varepsilon_{\mathbf{k}}) = -1$ (see Point 1 of Corollary 17). A value $\rho = \mathbf{1}$ written in bold type letter means that $\rho = 1$ for we are in the case that only one prime ideal of $\mathbf{k}$ is ramified in the quadratic extension $\mathbf{K}/\mathbf{k}$ (see Lemma 18).

(6) The sixth column gives the structure of the ideal class group $\mathrm{Cl}_{\mathbf{K}}$ of $\mathbf{K}$. We did not have to compute it if $h_{\mathbf{k}} = 1$ (see Proposition 16), or if $h_{\mathbf{K}}^{-} = 1$. This class group structure is given in bold type letters for the few fields $\mathbf{K}$ for which we had to compute it (by using Pari). We found 678 non-isomorphic non-normal quartic CM-fields $\mathbf{K}$ with ideal class groups $\mathrm{Cl}_{\mathbf{K}}$ of exponents ≤ 2, with the following related data:

The non-normal quartic CM-fields $\mathbf{K}$ with ideal class groups $\mathrm{Cl}_{\mathbf{K}}$ of exponents ≤ 2.

Structure for $\mathrm{Cl}_{\mathbf{K}}$	[1]	[2]	[2, 2]	[2, 2, 2]	[2, 2, 2, 2]	total
Number of non pairwise isomorphic $\mathbf{K}$'s	37	205	284	140	12	678

(7) Now, we use Point 5. of Proposition 5 to fill in the seventh column of the pairs of invariants $(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$ of the dual fields $(\tilde{\mathbf{k}}, \tilde{\mathbf{K}})$ of the 678 pairs of fields $(\mathbf{k}, \mathbf{K})$, where $\mathbf{K}$ ranges over these 678 non-normal quartic CM-fields $\mathbf{K}$ with ideal class groups of exponents ≤ 2. We also use Points 5. and 6. of Proposition 1 to check whether at least one of the two quadratic extensions $\mathbf{N}^{+}/\tilde{\mathbf{k}}$ or $\mathbf{N}^{+}/\mathbf{k}$ is unramified, in which cases these pairs of invariants $(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$ are asterisked (26 such cases).

(8) Finally, the eighth column gives the structures of the ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ of the normal closures $\mathbf{N}$ of the previous 678 non-normal quartic CM-fields $\mathbf{K}$ with ideal class groups of exponents ≤ 2, which according to Point 7. of Proposition 1 will provide us with the list of all the dihedral octic CM-fields $\mathbf{N}$ with ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ of exponents ≤ 2. However, two points must be emphasized.

First, whereas no two of these 678 quartic fields $\mathbf{K}$ are isomorphic, they can nevertheless be dual non-normal quartic fields, in which case they have the same normal closure $\mathbf{N}$. Therefore, we first searched in our Tables for the fields $\mathbf{K}$ for which there exist non-normal quartic CM-fields $\mathbf{K}_2 \neq \mathbf{K}$ with ideal class groups of exponents ≤ 2 such that $d_{\mathbf{k}} \geq d_{\tilde{\mathbf{k}}} = d_{\mathbf{k}_2}$ and $d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}} = d_{\mathbf{K}_2/\mathbf{k}_2}$. In that case, we used Proposition 2 to check whether $\mathbf{K}$ is a dual non-normal quatic field of at least one of these $\mathbf{K}_2$, in which case we put the symbol *dual* in the eighth columns of our Tables (172 such cases).

Second, we did not have to compute the structure of the ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ of all the remaining $506 = 678 - 172$ possible normal closures $\mathbf{N}$. For the 26 asterisked cases of the seventh column, we did compute them (by using Pari). However, for the $480 = 506 - 26$ remaining fields $\mathbf{K}$, both $\mathbf{N}^+/\tilde{\mathbf{k}}$ and $\mathbf{N}^+/\mathbf{k}$ are ramified at some finite place. Hence, according to Point 7. of Proposition 1, the exponent of the ideal class group $\mathrm{Cl}_{\mathbf{N}}$ of the normal closure $\mathbf{N}$ of a given $\mathbf{K}$ is greater than two if $d_{\tilde{\mathbf{k}}} \neq d_{\mathbf{k}_2}$ or $d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}} \neq d_{\mathbf{K}_2/\mathbf{k}_2}$ for all the 677 non-normal quartic CM-fields $\mathbf{K}_2 \neq \mathbf{K}$ with ideal class groups of exponents ≤ 2. In that case, we put the symbol • in the eighth column (292 such cases). Now, for the $188 = 480 - 292$ remaining non-normal quartic CM-fields $\mathbf{K}$, we used Pari to compute the structure of the ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ of their normal closures $\mathbf{N}$.

We finally give in the eighth columns of our Tables the structure of the $214 = 188 + 26$ ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ we had to compute by using Pari. These class group structures are given in bold type letters for the fields $\mathbf{N}$ for which the exponents of $\mathrm{Cl}_{\mathbf{N}}$ are ≤ 2. We found 116 dihedral octic CM-fields $\mathbf{N}$ with ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ of exponents ≤ 2, with the following related data:

The dihedral octic CM-fields $\mathbf{N}$ with ideal class groups $\mathrm{Cl}_{\mathbf{N}}$ of exponents ≤ 2.

Structure for $\mathrm{Cl}_{\mathbf{N}}$	[1]	[2]	[2, 2]	[2, 2, 2]	[2, 2, 2, 2]	[2, 2, 2, 2, 2]	[2, 2, 2, 2, 2, 2]	total
Number of $\mathbf{N}$'s	17	7	50	31	3	3	5	116

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^{-}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
5	41	(13, 1)	1	•	[1]	(41, 5)	**[1]**
5	61	(17, 3)	1	•	[1]	(61, 5)	**[1]**
5	109	(21, 1)	1	•	[1]	(109, 5)	**[1]**
5	145	(25, 3)	2	•	[2]	(145, 5)*	[4]
5	149	(26, 4)	1	•	[1]	(149, 5)	**[1]**
5	176	(28, 4)	2	•	[2]	(44, 20)	**[2, 2]**
5	209	(29, 1)	2	•	[2]	(209, 5)	**[2, 2]**
5	261	(33, 3)	2	•	[2]	(29, 45)	[4]
5	269	(34, 4)	1	•	[1]	(269, 5)	[3]
5	304	(36, 4)	2	•	[2]	(76, 20)	**[2, 2]**
5	341	(37, 1)	2	•	[2]	(341, 5)	[4, 2]
5	389	(41, 5)	1	•	[1]	(389, 5)	**[1]**
5	445	(45, 7)	2	•	[2]	(445, 5)*	[4]
5	464	(44, 4)	2	•	[2]	(29, 80)	**[2, 2]**
5	589	(49, 3)	2	•	[2]	(589, 5)	**[2, 2]**
5	704	(56, 8)	2	•	[2]	(44, 80)	**[2, 2]**
5	869	(61, 7)	2	•	[2]	(869, 5)	[6, 2]
5	880	(60, 4)	4	•	[2, 2]	(220, 20)*	**[2, 2, 2]**
5	909	(66, 12)	2	•	[2]	(101, 45)	[4]
5	944	(64, 8)	2	•	[2]	(236, 20)	**[2, 2]**
5	1045	(65, 3)	4	•	[2, 2]	(1045, 5)*	[4, 2, 2]
5	1189	(69, 1)	2	•	[2]	(1189, 5)	**[2, 2, 2]**
5	1349	(74, 4)	2	•	[2]	(1349, 5)	**[2, 2]**
5	1520	(80, 8)	4	•	[2, 2]	(380, 20)*	**[2, 2, 2]**
5	1584	(84, 12)	4	•	[2, 2]	(44, 180)	**[2, 2, 2]**
5	1845	(90, 12)	4	•	[2, 2]	(205, 45)*	**[2, 2, 2]**
5	2480	(100, 4)	4	•	[2, 2]	(620, 20)*	**[2, 2, 2]**
5	3069	(114, 12)	4	•	[2, 2]	(341, 45)	[8, 2, 2]
5	3245	(125, 23)	4	•	[2, 2]	(3245, 5)*	[4, 2, 2]
5	3344	(124, 20)	4	•	[2, 2]	(209, 80)	[4, 2, 2]
5	3509	(121, 11)	4	•	[2, 2]	(29, 605)	•
5	7920	(180, 12)	8	•	[2, 2, 2]	(220, 180)*	[4, 2, 2, 2]
5	9405	(210, 36)	8	•	[2, 2, 2]	(1045, 45)*	[4, 4, 2, 2]
8	17	(10, 2)	1	•	[1]	(17, 8)	**[1]**
8	73	(18, 2)	1	•	[1]	(73, 8)	**[1]**
8	89	(22, 4)	1	•	[1]	(89, 8)	**[1]**
8	112	(24, 4)	2	•	[2]	(28, 32)	**[2, 2]**
8	164	(28, 4)	2	•	[2]	(41, 32)	**[2, 2]**
8	217	(34, 6)	2	•	[2]	(217, 8)	**[2, 2]**
8	224	(32, 4)	2	•	[2]	(56, 32)	**[2, 2]**
8	233	(38, 8)	1	•	[1]	(233, 8)	**[1]**
8	281	(34, 2)	1	•	[1]	(281, 8)	**[1]**
8	329	(38, 4)	2	•	[2]	(329, 8)	**[2, 2]**
8	368	(40, 4)	2	•	[2]	(92, 32)	**[2, 2]**
8	425	(50, 10)	2	•	[2]	(17, 200)	**[2]**
8	548	(52, 8)	2	•	[2]	(137, 32)	**[2, 2]**
8	553	(50, 6)	2	•	[2]	(553, 8)	**[2, 2]**
8	612	(60, 12)	4	•	[2, 2]	(17, 288)	**[2, 2, 2]**
8	644	(68, 16)	4	•	[2, 2]	(161, 32)	[4, 4, 2]
8	697	(54, 4)	2	•	[2]	(697, 8)	[6, 2, 2]
8	713	(70, 16)	2	•	[2]	(713, 8)	**[2, 2]**
8	868	(60, 4)	4	•	[2, 2]	(217, 32)	[4, 2, 2]
8	1008	(72, 12)	4	•	[2, 2]	(28, 288)	**[2, 2, 2]**

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^-$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
8	1017	(66, 6)	2	•	[2]	(113, 72)	[8]
8	1316	(92, 20)	4	•	[2, 2]	(329, 32)	•
8	1337	(86, 16)	2	•	[2]	(1337, 8)	**[2, 2]**
8	1449	(102, 24)	4	•	[2, 2]	(161, 72)	[4, 2, 2]
8	1904	(88, 4)	4	•	[2, 2]	(476, 32)	[4, 2, 2, 2, 2]
8	1988	(100, 16)	4	•	[2, 2]	(497, 32)	•
8	2009	(98, 14)	4	•	[2, 2]	(41, 392)	•
8	3332	(140, 28)	8	•	[2, 2, 2]	(17, 1568)	•
8	3689	(134, 20)	4	•	[2, 2]	(3689, 8)	•
8	4025	(130, 10)	4	•	[2, 2]	(161, 200)	[4, 2, 2]
8	5796	(156, 12)	8	•	[2, 2, 2]	(161, 288)	•
8	7497	(210, 42)	8	•	[2, 2, 2]	(17, 3528)	•
8	7812	(204, 36)	8	•	[2, 2, 2]	(217, 288)	[4, 2, 2, 2, 2]
12	33	(18, 4)	2	0	[2]	(33, 12)	**[2]**
12	52	(16, 2)	2	0	[2]	(13, 48)	**[2, 2]**
12	**96**	(24, 4)	2	0	[2]	(24, 48)	**[2]**
12	97	(34, 8)	2	**1**	[2]	(97, 12)	**[2, 2]**
12	148	(32, 6)	2	0	[2]	(37, 48)	**[2, 2]**
12	177	(30, 4)	2	0	[2]	(177, 12)	**[2]**
12	208	(32, 4)	2	0	[2]	(13, 192)	**[2, 2]**
12	244	(32, 2)	2	0	[2]	(61, 48)	**[2, 2]**
12	276	(48, 10)	4	0	[2, 2]	(69, 48)	[4, 2]
12	321	(66, 16)	2	0	[2]	(321, 12)	[6]
12	352	(56, 12)	2	0	[2]	(88, 48)	**[2, 2]**
12	393	(42, 4)	2	0	[2]	(393, 12)	**[2]**
12	433	(50, 8)	2	**1**	[2]	(433, 12)	[8, 2]
12	528	(48, 4)	4	0	[2, 2]	(33, 192)	**[2, 2, 2]**
12	537	(54, 8)	2	0	[2]	(537, 12)	**[2]**
12	564	(48, 2)	4	0	[2, 2]	(141, 48)	[4, 2]
12	628	(80, 18)	2	0	[2]	(157, 48)	**[2, 2]**
12	673	(98, 24)	2	**1**	[2]	(673, 12)	[4, 2]
12	736	(56, 4)	4	1	[2, 2]	(184, 48)	[4, 2, 2]
12	825	(90, 20)	4	0	[2, 2]	(33, 300)	**[2, 2, 2]**
12	852	(96, 22)	4	0	[2, 2]	(213, 48)	[4, 2]
12	897	(66, 8)	4	0	[2, 2]	(897, 12)	[8, 2, 2]
12	913	(62, 4)	2	0	[2]	(913, 12)	[4, 2]
12	937	(74, 12)	2	**1**	[2]	(937, 12)	[4, 2]
12	1012	(80, 14)	4	0	[2, 2]	(253, 48)	[4, 2, 2]
12	1081	(86, 16)	4	1	[2, 2]	(1081, 12)	[4, 4, 2]
12	1104	(96, 20)	4	0	[2, 2]	(69, 192)	[4, 2]
12	1168	(80, 12)	4	1	[2, 2]	(73, 192)	**[2, 2, 2, 2]**
12	1248	(72, 4)	4	0	[2, 2]	(312, 48)	[4, 2, 2]
12	1300	(80, 10)	4	0	[2, 2]	(13, 1200)	**[2, 2, 2]**
12	1504	(104, 20)	4	1	[2, 2]	(376, 48)	[4, 2, 2]
12	1552	(80, 4)	4	1	[2, 2]	(97, 192)	**[2, 2, 2, 2]**
12	1617	(126, 28)	4	0	[2, 2]	(33, 588)	**[2, 2, 2]**
12	1633	(98, 16)	4	1	[2, 2]	(1633, 12)	•
12	1716	(96, 14)	8	0	[2, 2, 2]	(429, 48)	•
12	1716	(144, 34)	8	0	[2, 2, 2]	(429, 48)	•
12	1825	(110, 20)	4	1	[2, 2]	(73, 300)	[4, 2, 2]
12	1833	(102, 16)	4	0	[2, 2]	(1833, 12)	•
12	2409	(102, 8)	4	0	[2, 2]	(2409, 12)	[4, 2, 2, 2]
12	2553	(150, 32)	4	0	[2, 2]	(2553, 12)	•

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^{-}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
12	3289	(122, 12)	4	0	[2, 2]	(3289, 12)	•
12	3337	(170, 36)	4	1	[2, 2]	(3337, 12)	•
12	4884	(144, 10)	8	0	[2, 2, 2]	(1221, 48)	•
12	5577	(234, 52)	8	0	[2, 2, 2]	(33, 2028)	•
12	6292	(176, 22)	8	0	[2, 2, 2]	(13, 5808)	•
13	17	(9, 1)	1	•	[1]	(17, 13)	**[1]**
13	29	(18, 4)	1	•	[1]	(29, 13)	**[1]**
13	48	(20, 4)	2	•	[2]	(12, 52)	*dual*
13	69	(17, 1)	2	•	[2]	(69, 13)	**[2, 2]**
13	157	(41, 9)	1	•	[1]	(157, 13)	**[1]**
13	181	(29, 3)	1	•	[1]	(181, 13)	**[1]**
13	192	(40, 8)	2	•	[2]	(12, 208)	*dual*
13	237	(34, 4)	2	•	[2]	(237, 13)	[4, 2]
13	381	(61, 13)	2	•	[2]	(381, 13)	**[2, 2]**
13	477	(45, 3)	4	•	[2, 2]	(53, 117)	•
13	549	(57, 9)	4	•	[2, 2]	(61, 117)	[4, 2, 2]
13	597	(49, 1)	2	•	[2]	(597, 13)	**[2, 2]**
13	624	(52, 4)	4	•	[2, 2]	(156, 52)*	**[2, 2, 2]**
13	688	(68, 12)	2	•	[2]	(172, 52)	**[2, 2]**
13	816	(64, 8)	4	•	[2, 2]	(204, 52)	**[2, 2, 2, 2, 2]**
13	901	(74, 12)	2	•	[2]	(901, 13)	•
13	909	(81, 15)	4	•	[2, 2]	(101, 117)	•
13	1104	(88, 16)	4	•	[2, 2]	(69, 208)	[4, 2, 2]
13	1173	(73, 7)	4	•	[2, 2]	(1173, 13)	•
13	1200	(100, 20)	4	•	[2, 2]	(12, 1300)	*dual*
13	1392	(76, 4)	4	•	[2, 2]	(348, 52)	•
13	1557	(90, 12)	4	•	[2, 2]	(173, 117)	•
13	1677	(130, 28)	4	•	[2, 2]	(1677, 13)*	[4, 2, 2]
13	1725	(85, 5)	4	•	[2, 2]	(69, 325)	[4, 2, 2]
13	1989	(117, 21)	8	•	[2, 2, 2]	(221, 117)*	[4, 4, 2]
13	2448	(108, 12)	8	•	[2, 2, 2]	(17, 1872)	[4, 2, 2, 2]
13	3312	(144, 24)	8	•	[2, 2, 2]	(92, 468)	•
13	4437	(186, 36)	8	•	[2, 2, 2]	(493, 117)	•
13	4437	(165, 27)	8	•	[2, 2, 2]	(493, 117)	•
17	8	(7, 1)	1	•	[1]	(8, 17)	*dual*
17	13	(18, 4)	1	•	[1]	(13, 17)	*dual*
17	136	(51, 11)	2	•	[2]	(136, 17)*	**[2]**
17	137	(70, 16)	1	•	[1]	(137, 17)	**[1]**
17	152	(59, 13)	2	•	[2]	(152, 17)	[8, 2]
17	172	(39, 7)	2	•	[2]	(172, 17)	**[2, 2]**
17	200	(35, 5)	2	•	[2]	(8, 425)	*dual*
17	236	(31, 1)	2	•	[2]	(236, 17)	**[2, 2]**
17	257	(46, 8)	1	•	[1]	(257, 17)	[3]
17	288	(60, 12)	4	•	[2, 2]	(8, 612)	*dual*
17	332	(79, 17)	2	•	[2]	(332, 17)	**[2, 2]**
17	416	(44, 4)	4	•	[2, 2]	(104, 68)	•
17	416	(92, 20)	4	•	[2, 2]	(104, 68)	•
17	536	(91, 19)	2	•	[2]	(536, 17)	**[2, 2]**
17	608	(188, 44)	4	•	[2, 2]	(152, 68)	[8, 2, 2, 2]
17	689	(62, 8)	2	•	[2]	(689, 17)	•
17	848	(88, 16)	4	•	[2, 2]	(53, 272)	•
17	936	(219, 51)	4	•	[2, 2]	(104, 153)	[4, 2, 2, 2]
17	988	(63, 1)	4	•	[2, 2]	(988, 17)	**[2, 2, 2, 2, 2, 2]**

$d_{\mathbf{k}}$	$d_{\mathbf{K/k}}$	$\beta_{\mathbf{K/k}}$	$h^-_{\mathbf{K}}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
17	1292	(119, 23)	4	•	[2, 2]	(1292, 17)*	[4, 2, 2]
17	988	(63, 1)	4	•	[2, 2]	(988, 17)	**[2, 2, 2, 2, 2, 2]**
17	1292	(119, 23)	4	•	[2, 2]	(1292, 17)*	[4, 2, 2]
17	1352	(91, 13)	4	•	[2, 2]	(8, 2873)	•
17	1368	(75, 3)	4	•	[2, 2]	(152, 153)	[8, 2, 2]
17	1548	(87, 9)	4	•	[2, 2]	(172, 153)	[4, 2, 2]
17	1692	(159, 33)	4	•	[2, 2]	(188, 153)	[8, 2, 2]
17	1872	(216, 48)	8	•	[2, 2, 2]	(13, 2448)	*dual*
21	37	(13, 1)	2	**1**	[2]	(37, 21)	**[2, 2]**
21	85	(37, 7)	2	0	[2]	(85, 21)	•
21	85	(26, 4)	2	0	[2]	(85, 21)	•
21	105	(21, 1)	4	0	[2, 2]	(105, 21)*	[4, 2]
21	141	(33, 5)	4	1	[2, 2]	(141, 21)	•
21	177	(57, 11)	4	1	[2, 2]	(177, 21)	•
21	205	(29, 1)	2	0	[2]	(205, 21)	•
21	240	(48, 8)	4	0	[2, 2]	(60, 84)	•
21	240	(36, 4)	4	0	[2, 2]	(60, 84)	•
21	277	(53, 9)	2	**1**	[2]	(277, 21)	**[2, 2]**
21	357	(42, 4)	4	0	[2, 2]	(357, 21)*	[4, 2]
21	421	(65, 11)	2	**1**	[2]	(421, 21)	**[2, 2]**
21	445	(53, 7)	2	0	[2]	(445, 21)	•
21	501	(45, 1)	4	1	[2, 2]	(501, 21)	•
21	541	(50, 4)	2	**1**	[2]	(541, 21)	**[2, 2]**
21	645	(117, 23)	4	0	[2, 2]	(645, 21)	•
21	645	(93, 17)	4	0	[2, 2]	(645, 21)	•
21	816	(60, 4)	4	0	[2, 2]	(204, 81)	•
21	861	(105, 19)	4	0	[2, 2]	(861, 21)*	[4, 2]
21	925	(65, 5)	4	0	[2, 2]	(37, 525)	[4, 2, 2]
21	960	(72, 8)	4	0	[2, 2]	(60, 336)	•
21	1149	(114, 20)	4	1	[2, 2]	(1149, 21)	•
21	1360	(92, 12)	4	0	[2, 2]	(85, 336)	•
21	1509	(81, 5)	4	1	[2, 2]	(1509, 21)	•
21	1645	(98, 12)	4	0	[2, 2]	(1645, 21)	•
21	1680	(168, 32)	8	0	[2, 2, 2]	(105, 336)*	[4, 2, 2, 2]
21	2256	(120, 16)	8	1	[2, 2, 2]	(141, 336)	•
21	**2800**	(140, 20)	8	0	[2, 2, 2]	(28, 2100)	[4, 2, 2, 2]
21	3525	(165, 25)	8	0	[2, 2, 2]	(141, 525)	[4, 2, 2, 2]
21	3885	(273, 53)	8	0	[2, 2, 2]	(3885, 21)*	[4, 2, 2, 2, 2, 2]
21	3984	(156, 20)	8	1	[2, 2, 2]	(249, 336)	[4, 2, 2, 2]
21	6549	(186, 20)	8	1	[2, 2, 2]	(6549, 21)	•
21	8925	(210, 20)	16	0	[2, 2, 2, 2]	(357, 525)*	[4, 4, 4, 2]
24	**48**	(24, 4)	2	0	[2]	(12, 96)	*dual*
24	145	(26, 2)	2	0	[2]	(145, 24)	•
24	160	(32, 4)	2	0	[2]	(40, 96)	•
24	193	(74, 14)	2	**1**	[2]	(193, 24)	**[2, 2]**
24	228	(36, 4)	4	0	[2, 2]	(57, 96)	**[2, 2, 2]**
24	265	(38, 4)	2	0	[2]	(265, 24)	•
24	292	(68, 12)	4	1	[2, 2]	(73, 96)	**[2, 2, 2, 2]**
24	304	(104, 20)	2	0	[2]	(76, 96)	**[2, 2]**
24	313	(86, 16)	2	**1**	[2]	(313, 24)	**[2, 2]**
24	345	(54, 8)	4	0	[2, 2]	(345, 24)	•
24	480	(48, 4)	4	0	[2, 2]	(120, 96)*	[4, 2]
24	505	(182, 36)	2	0	[2]	(505, 24)	•

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^{-}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
24	516	(60, 8)	4	0	[2, 2]	(129, 96)	**[2, 2, 2]**
24	580	(92, 16)	4	0	[2, 2]	(145, 96)	•
24	705	(138, 26)	4	0	[2, 2]	(705, 24)	•
24	736	(80, 12)	4	1	[2, 2]	(184, 96)	[4, 2]
24	769	(74, 10)	2	**1**	[2]	(769, 24)	**[2, 2]**
24	804	(204, 40)	4	0	[2, 2]	(201, 96)	**[2, 2, 2]**
24	964	(116, 20)	4	1	[2, 2]	(241, 96)	[4, 2, 2, 2]
24	1060	(68, 4)	4	0	[2, 2]	(265, 96)	•
24	1065	(102, 16)	4	0	[2, 2]	(1065, 24)	•
24	**1200**	(120, 20)	8	0	[2, 2, 2]	(12, 2400)	•
24	1380	(228, 44)	8	0	[2, 2, 2]	(345, 96)	•
24	1380	(108, 16)	8	0	[2, 2, 2]	(345, 96)	•
24	1425	(90, 10)	8	0	[2, 2, 2]	(57, 600)	[4, 2, 2, 2]
24	1825	(170, 30)	4	0	[2, 2]	(73, 600)	[4, 2, 2]
24	2001	(90, 2)	4	0	[2, 2]	(2001, 24)	•
24	2185	(422, 84)	4	0	[2, 2]	(2185, 24)	•
24	3225	(150, 20)	8	0	[2, 2, 2]	(129, 600)	[4, 2, 2, 2]
24	4324	(236, 40)	8	1	[2, 2, 2]	(1081, 96)	•
24	4324	(164, 20)	8	1	[2, 2, 2]	(1081, 96)	•
24	5700	(180, 20)	16	0	[2, 2, 2, 2]	(57, 2400)	•
28	**32**	(24, 4)	2	**1**	[2]	(8, 112)	*dual*
28	57	(86, 16)	2	0	[2]	(57, 28)	[4, 2]
28	84	(56, 10)	4	0	[2, 2]	(21, 112)	•
28	113	(30, 4)	2	**1**	[2]	(113, 28)	**[2, 2, 2]**
28	177	(34, 4)	2	0	[2]	(177, 28)	[4, 2]
28	249	(214, 40)	2	0	[2]	(249, 28)	[4, 2]
28	**288**	(72, 12)	4	0	[2, 2]	(8, 1008)	*dual*
28	336	(112, 20)	4	0	[2, 2]	(21, 448)	•
28	337	(194, 36)	2	**1**	[2]	(337, 28)	**[2, 2]**
28	372	(40, 2)	4	0	[2, 2]	(93, 112)	•
28	372	(184, 34)	4	0	[2, 2]	(93, 112)	•
28	393	(58, 8)	2	0	[2]	(393, 28)	[4, 2]
28	417	(46, 4)	2	0	[2]	(417, 28)	[4, 2]
28	457	(134, 24)	2	**1**	[2]	(457, 28)	**[2, 2, 2]**
28	532	(56, 6)	4	0	[2, 2]	(133, 112)	•
28	564	(88, 14)	4	0	[2, 2]	(141, 112)	•
28	609	(238, 44)	4	0	[2, 2]	(609, 28)	•
28	672	(280, 52)	4	0	[2, 2]	(168, 112)	•
28	777	(70, 8)	4	0	[2, 2]	(777, 28)	•
28	912	(64, 4)	4	0	[2, 2]	(57, 448)	[4, 2, 2, 2]
28	1044	(72, 6)	8	0	[2, 2, 2]	(29, 1008)	•
28	1233	(450, 84)	4	0	[2, 2]	(137, 252)	[4, 2, 2]
28	1332	(120, 18)	8	0	[2, 2, 2]	(37, 1008)	**[2, 2, 2, 2, 2]**
28	1393	(98, 12)	4	1	[2, 2]	(1393, 28)	•
28	1425	(130, 20)	4	0	[2, 2]	(57, 700)	[4, 4, 2]
28	1561	(266, 48)	4	1	[2, 2]	(1561, 28)	•
28	1953	(210, 36)	8	0	[2, 2, 2]	(217, 252)	[4, 2, 2, 2]
28	2100	(280, 50)	8	0	[2, 2, 2]	(21, 2800)	*dual*
28	2569	(518, 96)	4	1	[2, 2]	(2569, 28)	•
28	2961	(126, 12)	8	0	[2, 2, 2]	(329, 252)	•
28	3193	(170, 24)	4	1	[2, 2]	(3193, 28)	•
28	3472	(224, 36)	8	1	[2, 2, 2]	(217, 448)	•
28	4788	(168, 18)	16	0	[2, 2, 2, 2]	(133, 1008)	•

$d_{\mathbf{k}}$	$d_{\mathbf{K/k}}$	$\beta_{\mathbf{K/k}}$	$h^-_{\mathbf{K}}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
28	5425	(350, 60)	8	1	[2, 2, 2]	(217, 700)	•
29	13	(9, 1)	1	•	[1]	(13, 29)	*dual*
29	45	(21, 3)	2	•	[2]	(5, 261)	*dual*
29	53	(26, 4)	1	•	[1]	(53, 29)	**[1]**
29	65	(17, 1)	2	•	[2]	(65, 29)	**[2, 2, 2]**
29	80	(28, 4)	2	•	[2]	(5, 464)	*dual*
29	245	(49, 7)	4	•	[2, 2]	(5, 1421)	•
29	325	(45, 5)	4	•	[2, 2]	(13, 725)	•
29	560	(52, 4)	4	•	[2, 2]	(140, 116)	•
29	805	(57, 1)	4	•	[2, 2]	(805, 29)	[4, 2, 2, 2, 2]
33	12	(75, 13)	2	0	[2]	(12, 33)	*dual*
33	37	(26, 4)	2	**1**	[2]	(37, 33)	**[2, 2, 2]**
33	124	(23, 1)	2	0	[2]	(124, 33)	[4, 2]
33	**192**	(96, 16)	4	0	[2, 2]	(12, 528)	*dual*
33	232	(35, 3)	2	0	[2]	(232, 33)	•
33	264	(627, 109)	4	0	[2, 2]	(264, 33)*	[4, 2]
33	268	(455, 79)	2	0	[2]	(268, 33)	[4, 2]
33	300	(375, 65)	4	0	[2, 2]	(12, 825)	*dual*
33	313	(278, 48)	2	**1**	[2]	(313, 33)	**[2, 2, 2]**
33	328	(83, 13)	2	0	[2]	(328, 33)	•
33	352	(44, 4)	4	0	[2, 2]	(88, 132)	**[2, 2, 2]**
33	408	(75, 11)	4	0	[2, 2]	(408, 33)	•
33	408	(387, 67)	4	0	[2, 2]	(408, 33)	•
33	412	(239, 41)	2	0	[2]	(412, 33)	[4, 2]
33	433	(62, 8)	2	**1**	[2]	(433, 33)	**[2, 2]**
33	444	(183, 31)	4	0	[2, 2]	(444, 33)	•
33	544	(668, 116)	4	0	[2, 2]	(136, 132)	•
33	588	(63, 7)	4	0	[2, 2]	(12, 1617)	*dual*
33	664	(491, 85)	4	1	[2, 2]	(664, 33)	•
33	664	(59, 5)	4	1	[2, 2]	(664, 33)	•
33	696	(219, 37)	4	0	[2, 2]	(696, 33)	•
33	748	(143, 23)	4	0	[2, 2]	(748, 33)	•
33	1056	(396, 68)	8	0	[2, 2, 2]	(264, 132)*	[4, 2, 2, 2]
33	1177	(374, 64)	4	1	[2, 2]	(1177, 33)	•
33	1353	(198, 32)	4	0	[2, 2]	(1353, 33)*	[4, 2]
33	1488	(120, 16)	8	0	[2, 2, 2]	(93, 528)	[4, 2, 2, 2]
33	1632	(1404, 244)	8	0	[2, 2, 2]	(408, 132)	•
33	2200	(275, 45)	8	1	[2, 2, 2]	(88, 825)	•
33	2368	(560, 96)	8	1	[2, 2, 2]	(37, 2112)	•
33	2425	(470, 80)	4	1	[2, 2]	(97, 825)	[8, 2, 2]
33	2497	(110, 8)	4	1	[2, 2]	(2497, 33)	•
33	3256	(1859, 323)	8	1	[2, 2, 2]	(3256, 33)	•
33	4312	(539, 91)	8	1	[2, 2, 2]	(88, 1617)	[4, 2, 2, 2]
37	21	(26, 4)	2	•	[2]	(21, 37)	*dual*
37	33	(13, 1)	2	•	[2]	(33, 37)	*dual*
37	48	(28, 4)	2	•	[2]	(12, 148)	*dual*
37	77	(81, 13)	2	•	[2]	(77, 37)	**[2, 2]**
37	141	(34, 4)	2	•	[2]	(141, 37)	**[2, 2]**
37	213	(73, 11)	2	•	[2]	(213, 37)	[6, 2]
37	525	(130, 20)	4	•	[2, 2]	(21, 925)	*dual*
37	693	(90, 12)	8	•	[2, 2, 2]	(77, 333)	•
37	861	(274, 44)	4	•	[2, 2]	(861, 37)	•
37	1008	(228, 36)	8	•	[2, 2, 2]	(28, 1332)	*dual*

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^{-}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
37	1584	(108, 12)	8	•	[2, 2, 2]	(44, 1332)	•
37	2352	(196, 28)	8	•	[2, 2, 2]	(12, 7252)	•
37	2541	(121, 11)	8	•	[2, 2, 2]	(21, 4477)	•
40	41	(18, 2)	2	•	**[2, 2]**	(41, 40)	**[2, 2, 2]**
40	89	(54, 8)	2	•	**[2, 2]**	(89, 40)	**[2, 2, 2]**
40	409	(86, 12)	2	•	**[2, 2]**	(409, 40)	**[2, 2, 2]**
41	5	(26, 4)	1	•	[1]	(5, 41)	*dual*
41	32	(28, 4)	2	•	[2]	(8, 164)	*dual*
41	40	(59, 9)	2	•	[2]	(40, 41)	*dual*
41	92	(583, 91)	2	•	[2]	(92, 41)	**[2, 2, 2]**
41	124	(39, 5)	2	•	[2]	(124, 41)	**[2, 2, 2]**
41	296	(35, 1)	2	•	[2]	(296, 41)	**[2, 2, 2]**
41	305	(62, 8)	2	•	[2]	(305, 41)	•
41	320	(208, 32)	4	•	[2, 2]	(5, 2624)	•
41	460	(47, 3)	4	•	[2, 2]	(460, 41)	•
41	620	(343, 53)	4	•	[2, 2]	(620, 41)	•
41	720	(312, 48)	8	•	[2, 2, 2]	(5, 5904)	•
41	800	(140, 20)	8	•	[2, 2, 2]	(8, 4100)	•
41	860	(1007, 157)	4	•	[2, 2]	(860, 41)	•
41	920	(307, 47)	4	•	[2, 2]	(920, 41)	•
41	1440	(108, 12)	8	•	[2, 2, 2]	(40, 1476)	•
41	1800	(1635, 255)	8	•	[2, 2, 2]	(8, 9225)	•
44	20	(16, 2)	2	0	[2]	(5, 176)	*dual*
44	80	(32, 4)	2	0	[2]	(5, 704)	*dual*
44	180	(48, 6)	4	0	[2, 2]	(5, 1584)	*dual*
44	224	(136, 20)	4	1	[2, 2]	(56, 176)	•
44	308	(176, 26)	4	0	[2, 2]	(77, 176)	•
44	401	(190, 28)	2	**1**	[2]	(401, 44)	•
44	665	(118, 16)	4	0	[2, 2]	(665, 44)	•
44	665	(58, 4)	4	0	[2, 2]	(665, 44)	•
44	889	(534, 80)	4	1	[2, 2]	(889, 44)	•
44	980	(112, 14)	8	0	[2, 2, 2]	(5, 8624)	•
44	1169	(274, 40)	4	1	[2, 2]	(1169, 44)	•
44	2016	(120, 12)	8	1	[2, 2, 2]	(56, 1584)	•
44	3465	(198, 24)	8	0	[2, 2, 2]	(385, 396)	•
53	29	(13, 1)	1	•	[1]	(29, 53)	*dual*
53	77	(34, 4)	2	•	[2]	(77, 53)	**[2, 2, 2]**
56	**32**	(32, 4)	2	**1**	[2]	(8, 224)	*dual*
56	65	(106, 14)	2	0	[2]	(65, 56)	•
56	113	(26, 2)	2	**1**	[2]	(113, 56)	**[2, 2, 2]**
56	217	(182, 24)	4	1	[2, 2]	(217, 56)	•
56	260	(212, 28)	4	0	[2, 2]	(65, 224)	•
56	385	(42, 2)	4	0	[2, 2]	(385, 56)	•
56	585	(102, 12)	4	0	[2, 2]	(65, 504)	•
56	1001	(70, 4)	4	0	[2, 2]	(1001, 56)	•
56	1316	(140, 16)	8	1	[2, 2, 2]	(329, 224)	•
57	24	(39, 5)	4	1	[2, 2]	(24, 57)	•
57	28	(823, 109)	2	0	[2]	(28, 57)	*dual*
57	96	(876, 116)	4	0	[2, 2]	(24, 228)	*dual*
57	168	(27, 1)	4	0	[2, 2]	(168, 57)	•
57	232	(251, 33)	2	0	[2]	(232, 57)	•
57	348	(447, 59)	4	0	[2, 2]	(348, 57)	•
57	448	(128, 16)	4	0	[2, 2]	(28, 912)	*dual*

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h^-_{\mathbf{K}}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
57	456	(2907, 385)	4	0	[2, 2]	(456, 57)*	[4, 2]
57	457	(2054, 272)	2	1	[2]	(457, 57)	**[2, 2]**
57	472	(179, 23)	4	1	[2, 2]	(472, 57)	●
57	492	(2439, 323)	4	0	[2, 2]	(492, 57)	●
57	537	(246, 32)	4	1	[2, 2]	(537, 57)	●
57	568	(83, 9)	4	1	[2, 2]	(568, 57)	●
57	600	(195, 25)	8	1	[2, 2, 2]	(24, 1425)	*dual*
57	609	(78, 8)	4	0	[2, 2]	(609, 57)	●
57	672	(60, 4)	8	0	[2, 2, 2]	(168, 228)	●
57	672	(396, 52)	8	0	[2, 2, 2]	(168, 228)	●
57	681	(486, 64)	4	1	[2, 2]	(681, 57)	●
57	700	(1775, 235)	4	0	[2, 2]	(28, 1425)	*dual*
57	856	(59, 1)	4	1	[2, 2]	(856, 57)	●
57	1113	(4470, 592)	4	0	[2, 2]	(1113, 57)	●
57	1176	(1323, 175)	8	0	[2, 2, 2]	(24, 2793)	●
57	1464	(363, 47)	8	1	[2, 2, 2]	(1464, 57)	●
57	1752	(99, 7)	8	1	[2, 2, 2]	(1752, 57)	●
57	4425	(4230, 560)	8	1	[2, 2, 2]	(177, 1425)	●
60	241	(98, 12)	4	**1**	**[2, 2, 2]**	(241, 60)	[4, 2, 2, 2, 2]
60	649	(134, 16)	4	0	**[2, 2, 2]**	(649, 60)	[4, 4, 2, 2, 2]
61	5	(9, 1)	1	●	[1]	(5, 61)	*dual*
61	48	(64, 8)	2	●	[2]	(12, 244)	*dual*
61	117	(282, 36)	4	●	[2, 2]	(13, 549)	*dual*
61	141	(25, 1)	2	●	[2]	(141, 61)	**[2, 2]**
61	205	(98, 12)	2	●	[2]	(205, 61)	●
61	240	(532, 68)	4	●	[2, 2]	(60, 244)	●
61	285	(229, 29)	4	●	[2, 2]	(285, 61)	[4, 2, 2, 2, 2]
61	720	(108, 12)	8	●	[2, 2, 2]	(5, 8784)	●
61	741	(970, 124)	4	●	[2, 2]	(741, 61)	●
61	2925	(1410, 180)	16	●	[2, 2, 2, 2]	(13, 13725)	●
65	29	(34, 4)	2	●	**[2, 2]**	(29, 65)	*dual*
69	13	(34, 4)	2	**1**	[2]	(13, 69)	*dual*
69	85	(77, 9)	2	0	[2]	(85, 69)	●
69	165	(234, 28)	4	0	[2, 2]	(165, 69)	●
69	165	(42, 4)	4	0	[2, 2]	(165, 69)	●
69	208	(44, 4)	4	1	[2, 2]	(13, 1104)	*dual*
69	253	(161, 19)	4	1	[2, 2]	(253, 69)	●
69	325	(170, 20)	4	0	[2, 2]	(13, 1725)	*dual*
69	949	(461, 55)	4	1	[2, 2]	(949, 69)	**[2, 2, 2, 2, 2, 2]**
73	8	(675, 79)	1	●	[1]	(8, 73)	*dual*
73	96	(172, 20)	4	●	[2, 2]	(24, 292)	*dual*
73	97	(206, 24)	1	●	[1]	(97, 73)	**[1]**
73	184	(131, 15)	2	●	[2]	(184, 73)	**[2, 2]**
73	192	(1504, 176)	4	●	[2, 2]	(12, 1168)	*dual*
73	288	(108, 12)	8	●	[2, 2, 2]	(8, 2628)	●
73	300	(55, 5)	4	●	[2, 2]	(12, 1825)	*dual*
73	444	(103, 11)	4	●	[2, 2]	(444, 73)	**[2, 2, 2, 2, 2, 2]**
73	492	(1735, 203)	4	●	[2, 2]	(492, 73)	**[2, 2, 2, 2, 2, 2]**
73	552	(1291, 151)	4	●	[2, 2]	(552, 73)	[8, 2, 2, 2, 2, 2]
73	600	(4315, 505)	4	●	[2, 2]	(24, 1825)	*dual*
73	684	(11919, 1395)	8	●	[2, 2, 2]	(76, 657)	●
73	828	(63, 3)	8	●	[2, 2, 2]	(92, 657)	●
73	912	(38824, 4544)	8	●	[2, 2, 2]	(57, 1168)	●

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h^-_{\mathbf{K}}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
73	1368	(747, 87)	8	•	[2, 2, 2]	(152, 657)	•
76	20	(192, 22)	2	0	[2]	(5, 304)	*dual*
76	96	(40, 4)	2	0	[2]	(24, 304)	*dual*
76	180	(576, 66)	8	0	[2, 2, 2]	(5, 2736)	•
76	340	(368, 42)	4	0	[2, 2]	(85, 304)	**[2, 2, 2, 2, 2]**
76	465	(82, 8)	4	0	[2, 2]	(465, 76)	•
76	720	(1152, 132)	8	0	[2, 2, 2]	(5, 10944)	•
76	969	(874, 100)	4	0	[2, 2]	(969, 76)	[4, 2, 2, 2]
76	1425	(190, 20)	8	0	[2, 2, 2]	(57, 1900)	•
76	3060	(192, 18)	16	0	[2, 2, 2, 2]	(85, 2736)	•
76	3825	(8370, 960)	16	0	[2, 2, 2, 2]	(17, 17100)	•
77	37	(29, 3)	2	**1**	[2]	(37, 77)	*dual*
77	53	(17, 1)	2	**1**	[2]	(53, 77)	*dual*
77	133	(42, 4)	4	1	[2, 2]	(133, 77)	•
85	229	(41, 3)	2	•	**[2, 2]**	(229, 85)	•
85	304	(116, 12)	4	•	**[2, 2, 2]**	(76, 340)	*dual*
88	48	(40, 4)	2	0	[2]	(12, 352)	*dual*
88	97	(170, 18)	2	**1**	[2]	(97, 88)	[4, 2]
88	132	(1276, 136)	4	0	[2, 2]	(33, 352)	*dual*
88	273	(2458, 262)	4	0	[2, 2]	(273, 88)	•
88	553	(230, 24)	4	1	[2, 2]	(553, 88)	•
88	609	(106, 10)	4	0	[2, 2]	(609, 88)	•
88	1057	(3434, 366)	4	1	[2, 2]	(1057, 88)	•
88	1092	(604, 64)	8	0	[2, 2, 2]	(273, 352)	•
88	1617	(154, 14)	8	0	[2, 2, 2]	(33, 4312)	*dual*
89	8	(11, 1)	1	•	[1]	(8, 89)	*dual*
89	40	(387, 41)	2	•	[2]	(40, 89)	*dual*
89	80	(152, 16)	4	•	[2, 2]	(5, 1424)	•
89	425	(1510, 160)	4	•	[2, 2]	(17, 2225)	•
89	440	(43, 1)	4	•	[2, 2]	(440, 89)	•
89	1100	(8255, 875)	8	•	[2, 2, 2]	(44, 2225)	•
92	**32**	(40, 4)	2	**1**	[2]	(8, 368)	*dual*
92	41	(154, 16)	2	**1**	[2]	(41, 92)	*dual*
92	161	(46, 4)	4	1	[2, 2]	(161, 92)	•
93	33	(78, 8)	4	1	[2, 2]	(33, 93)	•
93	69	(42, 4)	4	1	[2, 2]	(69, 93)	•
93	109	(194, 20)	2	**1**	[2]	(109, 93)	**[2, 2]**
93	528	(60, 4)	8	1	[2, 2, 2]	(33, 1488)	*dual*
97	12	(719, 73)	2	•	[2]	(12, 97)	*dual*
97	73	(3782, 384)	1	•	[1]	(73, 97)	*dual*
97	88	(35, 3)	2	•	[2]	(88, 97)	*dual*
97	192	(160, 16)	4	•	[2, 2]	(12, 1552)	*dual*
97	264	(523, 53)	4	•	[2, 2]	(264, 97)	•
97	288	(828, 84)	8	•	[2, 2, 2]	(8, 3492)	•
97	528	(82888, 8416)	8	•	[2, 2, 2]	(33, 1552)	•
97	792	(10371, 1053)	8	•	[2, 2, 2]	(88, 873)	•
97	825	(790, 80)	4	•	[2, 2]	(33, 2425)	*dual*
97	1116	(111, 9)	8	•	[2, 2, 2]	(124, 873)	•
97	1452	(175615, 17831)	8	•	[2, 2, 2]	(12, 11737)	•
101	45	(33, 3)	2	•	[2]	(5, 909)	*dual*
104	153	(66, 6)	4	•	**[2, 2, 2]**	(17, 936)	*dual*
105	1009	(254, 24)	4	**1**	**[2, 2, 2]**	(1009, 105)	•

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^-$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
109	5	(42, 4)	1	•	[1]	(5, 109)	*dual*
109	5	(42, 4)	1	•	[1]	(5, 109)	*dual*
109	93	(1681, 161)	2	•	[2]	(93, 109)	*dual*
109	240	(52, 4)	4	•	[2, 2]	(60, 436)	•
109	261	(45, 3)	4	•	[2, 2]	(29, 981)	•
109	525	(265, 25)	8	•	[2, 2, 2]	(21, 2725)	•
109	2205	(882, 84)	16	•	[2, 2, 2, 2]	(5, 48069)	•
113	28	(15, 1)	2	•	[2]	(28, 113)	*dual*
113	56	(139, 13)	2	•	[2]	(56, 113)	*dual*
113	72	(3795, 357)	2	•	[2]	(8, 1017)	*dual*
124	33	(46, 4)	2	0	[2]	(33, 124)	*dual*
124	41	(90, 8)	2	**1**	[2]	(41, 124)	*dual*
124	180	(72, 6)	8	0	[2, 2, 2]	(5, 4464)	•
124	276	(40, 2)	4	0	[2, 2]	(69, 496)	•
124	660	(1048, 94)	8	0	[2, 2, 2]	(165, 496)	•
124	825	(36970, 3320)	8	0	[2, 2, 2]	(33, 3100)	•
129	40	(95099, 8373)	2	0	[2]	(40, 129)	•
129	60	(2919, 257)	4	0	[2, 2]	(60, 129)	•
129	96	(14220, 1252)	4	0	[2, 2]	(24, 516)	*dual*
129	160	(1772, 156)	4	0	[2, 2]	(40, 516)	•
129	312	(603, 53)	8	1	[2, 2, 2]	(312, 129)	•
129	465	(1182, 104)	4	0	[2, 2]	(465, 129)	•
129	600	(75, 5)	8	0	[2, 2, 2]	(24, 3225)	*dual*
129	780	(7167, 631)	8	0	[2, 2, 2]	(780, 129)	•
129	2769	(476574, 41960)	8	1	[2, 2, 2]	(2769, 129)	•
129	4056	(195, 13)	16	1	[2, 2, 2, 2]	(24, 21801)	•
133	57	(646, 56)	4	1	[2, 2]	(57, 133)	•
133	93	(50, 4)	4	1	[2, 2]	(93, 133)	•
133	309	(58, 4)	4	1	[2, 2]	(309, 133)	•
137	17	(94, 8)	1	•	[1]	(17, 137)	*dual*
137	32	(796, 68)	2	•	[2]	(8, 548)	*dual*
137	252	(35079, 2997)	4	•	[2, 2]	(28, 1233)	*dual*
141	37	(17, 1)	2	**1**	[2]	(37, 141)	*dual*
141	61	(50, 4)	2	**1**	[2]	(61, 141)	*dual*
141	253	(146, 12)	4	1	[2, 2]	(253, 141)	•
141	525	(1485, 125)	8	0	[2, 2, 2]	(21, 3525)	*dual*
149	5	(13, 1)	1	•	[1]	(5, 149)	*dual*
152	17	(26, 2)	2	**1**	[2]	(17, 152)	*dual*
152	68	(52, 4)	4	1	[2, 2]	(17, 608)	*dual*
152	153	(150, 12)	4	1	[2, 2]	(17, 1368)	*dual*
156	337	(62, 4)	4	**1**	**[2, 2, 2]**	(337, 156)	[4, 2, 2, 2, 2]
157	13	(113, 9)	1	•	[1]	(13, 157)	*dual*
157	48	(52, 4)	2	•	[2]	(12, 628)	*dual*
161	72	(267, 21)	4	1	[2, 2]	(8, 1449)	*dual*
161	140	(15239, 1201)	4	0	[2, 2]	(140, 161)	•
161	200	(10595, 835)	4	0	[2, 2]	(8, 4025)	*dual*
161	1064	(91, 5)	8	1	[2, 2, 2]	(1064, 161)	•
172	17	(210, 16)	2	**1**	[2]	(17, 172)	*dual*
172	52	(80, 6)	2	0	[2]	(13, 688)	*dual*
172	153	(630, 48)	4	0	[2, 2]	(17, 1548)	*dual*
177	12	(15, 1)	2	0	[2]	(12, 177)	*dual*
177	28	(13823, 1039)	2	0	[2]	(28, 177)	*dual*

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h^-_{\mathbf{K}}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
177	184	(241643, 18163)	4	1	[2, 2]	(184, 177)	•
177	193	(110, 8)	2	**1**	[2]	(193, 177)	**[2, 2, 2]**
181	13	(41, 3)	1	•	[1]	(13, 181)	*dual*
181	45	(162, 12)	4	•	[2, 2]	(5, 1629)	•
181	165	(12337, 917)	4	•	[2, 2]	(165, 181)	•
184	73	(326, 24)	2	**1**	[2]	(73, 184)	*dual*
184	105	(23494, 1732)	4	0	[2, 2]	(105, 184)	•
184	345	(598, 44)	4	0	[2, 2]	(345, 184)	•
184	420	(7108, 524)	8	0	[2, 2, 2]	(105, 736)	•
184	420	(182092, 13424)	8	0	[2, 2, 2]	(105, 736)	•
184	721	(28730, 2118)	4	1	[2, 2]	(721, 184)	•
188	153	(330, 24)	4	1	[2, 2]	(17, 1692)	*dual*
193	24	(403, 29)	2	•	[2]	(24, 193)	*dual*
193	177	(134590, 9688)	2	•	[2]	(177, 193)	*dual*
193	217	(1334, 96)	2	•	[2]	(217, 193)	**[2, 2]**
193	252	(25892751, 1863801)	8	•	[2, 2, 2]	(28, 1737)	•
193	288	(677340, 48756)	8	•	[2, 2, 2]	(8, 6948)	•
193	504	(111987, 8061)	8	•	[2, 2, 2]	(56, 1737)	•
193	588	(11767, 847)	8	•	[2, 2, 2]	(12, 9457)	•
201	24	(1035, 73)	4	1	[2, 2]	(24, 201)	•
201	60	(3514263, 247877)	4	0	[2, 2]	(60, 201)	•
201	96	(60, 4)	4	0	[2, 2]	(24, 804)	*dual*
201	220	(105863, 7467)	4	0	[2, 2]	(220, 201)	•
201	825	(1494870, 105440)	8	0	[2, 2, 2]	(33, 5025)	•
201	2904	(262779, 18535)	16	1	[2, 2, 2, 2]	(24, 24321)	•
204	52	(32, 2)	4	0	**[2, 2, 2]**	(13, 816)	*dual*
209	5	(58, 4)	2	**1**	[2]	(5, 209)	*dual*
209	80	(32152, 2224)	4	0	[2, 2]	(5, 3344)	*dual*
213	37	(161, 11)	2	**1**	[2]	(37, 213)	*dual*
217	8	(18723, 1271)	2	**1**	[2]	(8, 217)	*dual*
217	193	(3182, 216)	2	**1**	[2]	(193, 217)	*dual*
217	204	(79, 5)	4	0	[2, 2]	(204, 217)	•
217	249	(406102, 27568)	4	1	[2, 2]	(249, 217)	•
217	252	(399, 27)	8	0	[2, 2, 2]	(28, 1953)	*dual*
217	273	(9310, 632)	4	0	[2, 2]	(273, 217)	•
217	288	(12412332, 842604)	8	0	[2, 2, 2]	(8, 7812)	*dual*
217	744	(1132027, 76847)	8	1	[2, 2, 2]	(744, 217)	•
217	1017	(4950, 336)	8	1	[2, 2, 2]	(113, 1953)	•
217	1800	(49275, 3345)	16	1	[2, 2, 2, 2]	(8, 48825)	•
233	8	(107, 7)	1	•	[1]	(8, 233)	*dual*
236	17	(62, 4)	2	**1**	[2]	(17, 236)	*dual*
236	20	(32, 2)	2	0	[2]	(5, 944)	*dual*
237	13	(17, 1)	2	**1**	[2]	(13, 237)	*dual*
241	60	(803671, 51769)	4	•	[2, 2]	(60, 241)	*dual*
241	96	(1180, 76)	4	•	[2, 2]	(24, 964)	*dual*
241	145	(1118, 72)	2	•	[2]	(145, 241)	•
241	160	(188, 12)	4	•	[2, 2]	(40, 964)	•
241	300	(4735, 305)	8	•	[2, 2, 2]	(12, 6025)	•
241	360	(16347, 1053)	8	•	[2, 2, 2]	(40, 2169)	•
241	720	(13069368, 841872)	16	•	[2, 2, 2, 2]	(5, 34704)	•
249	28	(190919, 12099)	2	0	[2]	(28, 249)	*dual*
249	40	(14206163, 900279)	2	0	[2]	(40, 249)	•

$d_{\mathbf{k}}$	$d_{\mathbf{K/k}}$	$\beta_{\mathbf{K/k}}$	$h_{\mathbf{K}}^{-}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
249	120	(27, 1)	4	0	[2, 2]	(120, 249)	•
249	300	(1815, 115)	8	0	[2, 2, 2]	(12, 6225)	•
249	336	(910680, 57712)	8	0	[2, 2, 2]	(21, 3984)	*dual*
249	700	(95, 5)	8	0	[2, 2, 2]	(28, 6225)	•
253	133	(53, 3)	4	1	[2, 2]	(133, 253)	•
265	649	(7814, 480)	4	•	**[2, 2, 2]**	(649, 265)	**[2, 2, 2, 2, 2, 2]**
268	33	(177262, 10828)	2	0	[2]	(33, 268)	*dual*
268	84	(1408, 86)	4	0	[2, 2]	(21, 1072)	•
269	5	(17, 1)	1	•	[1]	(5, 269)	*dual*
273	337	(926, 56)	4	**1**	**[2, 2, 2]**	(337, 273)	[8, 2, 2, 2, 2, 2]
277	21	(1681, 101)	2	•	[2]	(21, 277)	*dual*
281	8	(4811, 287)	1	•	[1]	(8, 281)	*dual*
284	161	(13010, 772)	4	1	[2, 2]	(161, 284)	•
285	61	(53, 3)	4	**1**	**[2, 2, 2]**	(61, 285)	*dual*
296	41	(70, 4)	2	•	**[2, 2]**	(41, 296)	*dual*
301	21	(6853, 395)	4	1	[2, 2]	(21, 301)	•
301	165	(2377, 137)	4	0	[2, 2]	(165, 301)	•
301	336	(280, 16)	8	1	[2, 2, 2]	(21, 4816)	•
309	141	(405, 23)	4	1	[2, 2]	(141, 309)	•
313	24	(6258211, 353735)	2	•	[2]	(24, 313)	*dual*
313	33	(142, 8)	2	•	[2]	(33, 313)	*dual*
329	8	(19, 1)	2	**1**	[2]	(8, 329)	*dual*
329	65	(2122478, 117016)	2	0	[2]	(65, 329)	•
332	17	(146, 8)	2	**1**	[2]	(17, 332)	*dual*
337	28	(1487, 81)	2	•	[2]	(28, 337)	*dual*
337	72	(292491, 15933)	4	•	[2, 2]	(8, 3033)	•
337	156	(31, 1)	4	•	[2, 2]	(156, 337)	*dual*
337	273	(238942, 13016)	4	•	[2, 2]	(273, 337)	*dual*
341	5	(74, 4)	2	**1**	[2]	(5, 341)	*dual*
341	45	(57, 3)	4	1	[2, 2]	(5, 3069)	*dual*
381	13	(449, 23)	2	**1**	[2]	(13, 381)	*dual*
389	5	(217, 11)	1	•	[1]	(13, 389)	*dual*
393	12	(54599511, 2754181)	2	0	[2]	(12, 393)	*dual*
393	28	(575, 29)	2	0	[2]	(28, 393)	*dual*
409	40	(63374476667, 3133666191)	2	•	[2]	(40, 409)	*dual*
409	60	(223, 11)	4	•	[2, 2]	(60, 409)	•
412	33	(82, 4)	2	0	[2]	(33, 412)	*dual*
417	24	(1383147, 67733)	4	1	[2, 2]	(24, 417)	•
417	28	(23, 1)	2	0	[2]	(28, 417)	*dual*
417	312	(242331, 11867)	8	1	[2, 2, 2]	(312, 417)	•
417	897	(174, 8)	8	1	[2, 2, 2]	(897, 417)	•
421	21	(230605, 11239)	2	•	[2]	(21, 421)	*dual*

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433	12	(10930015, 525263)	2	•	[2]	(12, 433)	*dual*
433	33	(2830, 136)	2	•	[2]	(33, 433)	*dual*
444	73	(506, 24)	4	**1**	**[2, 2, 2]**	(73, 444)	*dual*
457	28	(7247, 339)	2	•	[2]	(28, 457)	*dual*
457	57	(835606, 39088)	2	•	[2]	(57, 457)	*dual*
476	**32**	(88, 4)	4	**1**	**[2, 2, 2]**	(8, 1904)	*dual*
489	33	(1415948046, 64031384)	4	1	[2, 2]	(33, 489)	•
492	73	(710, 32)	4	**1**	**[2, 2, 2]**	(73, 492)	*dual*
536	17	(602, 26)	2	**1**	[2]	(17, 536)	*dual*

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h^-_{\mathbf{K}}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
537	12	(255, 11)	2	0	[2]	(12, 537)	*dual*
541	21	(25, 1)	2	•	[2]	(21, 541)	*dual*
541	45	(16677, 717)	4	•	[2, 2]	(5, 4869)	•
552	73	(50, 2)	4	**1**	**[2, 2, 2]**	(73, 552)	*dual*
553	8	(387416115, 16474609)	2	**1**	[2]	(8, 553)	*dual*
553	177	(190, 8)	4	1	[2, 2]	(177, 553)	•
589	5	(81642, 3364)	2	**1**	[2]	(5, 589)	*dual*
589	45	(801, 33)	8	1	[2, 2, 2]	(5, 5301)	•
597	13	(98, 4)	2	**1**	[2]	(13, 597)	*dual*
601	60	(7183, 293)	4	0	[2, 2]	(60, 601)	•
604	105	(152230618, 6194176)	4	0	[2, 2]	(105, 604)	•
633	24	(27, 1)	4	•	[2, 2]	(24, 633)	•
649	60	(371407, 14579)	4	0	[2, 2]	(60, 649)	*dual*
649	88	(4818411299339, 189139187991)	4	1	[2, 2]	(88, 649)	•
649	265	(4228623302, 165987984)	4	1	[2, 2]	(265, 649)	*dual*
649	300	(517244815, 20303635)	8	0	[2, 2, 2]	(12, 16225)	•
673	12	(51142572151, 1971401281)	2	•	[2]	(12, 673)	*dual*
713	8	(187, 7)	2	**1**	[2]	(8, 713)	*dual*
721	60	(8202918288583,	4	0	[2, 2]	(60, 721)	•
		305492641063)	4	0			
721	72	(58563, 2181)	8	1	[2, 2, 2]	(8, 6489)	•
721	105	(569212373116198, 21198576539648)	4	0	[2, 2]	(105, 721)	•
753	88	(63086147, 2298987)	4	1	[2, 2]	(88, 753)	•
769	24	(139, 5)	2	•	[2]	(24, 769)	*dual*
805	29	(114, 4)	4	**1**	**[2, 2, 2]**	(29, 805)	*dual*
849	60	(344727, 11831)	4	0	[2, 2]	(60, 849)	•
849	105	(1350233033401638, 46339877741824)	4	0	[2, 2]	(105, 849)	•

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869	5	(2093, 71)	2	**1**	[2]	(5, 869)	*dual*
889	60	(10407336813043783, 349050873682621)	4	0	[2, 2]	(60, 889)	•
889	168	(50726686186939, 1701318449729)	8	1	[2, 2, 2]	(168, 889)	•
913	12	(31, 1)	2	0	[2]	(12, 913)	*dual*
921	24	(92766648500739, 3056764996055)	4	1	[2, 2]	(24, 921)	•
937	12	(155949895, 5094661)	2	•	[2]	(12, 937)	*dual*
949	69	(10474, 340)	4	•	**[2, 2, 2]**	(69, 949)	*dual*
969	76	(95, 3)	4	0	**[2, 2, 2]**	(76, 969)	*dual*
988	17	(126, 4)	4	**1**	**[2, 2, 2]**	(17, 988)	*dual*
1057	72	(99, 3)	8	1	[2, 2, 2]	(8, 9513)	•
1137	33	(270, 8)	4	1	[2, 2]	(33, 1137)	•
1189	5	(138, 4)	2	•	**[2, 2]**	(5, 1189)	*dual*
1201	60	(3604859809658740063, 104020006431274577)	4	•	[2, 2]	(60, 1201)	•
1201	105	(300098932051942, 8659502585488)	4	•	[2, 2]	(105, 1201)	•
1265	209	(286, 8)	8	1	**[2, 2, 2, 2]**	(209, 1265)	•
1273	153	(14404662, 403728)	8	1	[2, 2, 2]	(17, 11457)	•

$d_{\mathbf{k}}$	$d_{\mathbf{K}/\mathbf{k}}$	$\beta_{\mathbf{K}/\mathbf{k}}$	$h_{\mathbf{K}}^{-}$	ρ	$\mathrm{Cl}_{\mathbf{K}}$	$(d_{\tilde{\mathbf{k}}}, d_{\tilde{\mathbf{K}}/\tilde{\mathbf{k}}})$	$\mathrm{Cl}_{\mathbf{N}}$
1337	8	(594818275, 16267417)	2	**1**	[2]	(8, 1337)	*dual*
1349	5	(37, 1)	2	**1**	[2]	(5, 1349)	*dual*
1477	21	(154, 4)	4	1	[2, 2]	(21, 1477)	•
1497	33	(5262, 136)	4	1	[2, 2]	(33, 1497)	•
1501	45	(117, 3)	8	1	[2, 2, 2]	(5, 13509)	•
1969	385	(3192837128276626252 4462, 7195382854445651357 04)	8	1	[2, 2, 2]	(385, 1969)	•
2409	12	(30087, 613)	4	0	**[2, 2, 2]**	(12, 2409)	*dual*
2641	120	(20080027, 390733)	8	1	**[2, 2, 2]**	(120, 2641)	•

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