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The Converse of Kelly's Lemma and Control-classes in Graph Reconstruction

To Professor Adriano Barlotti on the occasion of his 80th birthday

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Abstract

We prove a converse of the well-known Kelly's Lemma. This motivates the introduction of the general notions of $\mathcal{K}$-table, $\mathcal{K}$-congruence and control-class.

Key words: Graph; Kelly's Lemma; Reconstruction.

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1 Introduction

An *Ulam-subgraph* of a (finite, simple, undirected, labelled) graph G of order n is a subgraph of order $n-1$ obtained from G by deleting a vertex of G and the edges incident to it. Such a subgraph can also be defined as a maximal induced subgraph of G or, simply, as a subgraph induced by $n-1$ vertices of G.

Thus, a graph G of order n gives rise to n distinct Ulam-subgraphs, the set of which is sometimes called the Ulam-deck of G. We shall denote by $G^{(v)}$ the Ulam-subgraph of G obtained by deleting the vertex v of G. Note that distinct Ulam-subgraphs may be isomorphic.

We say that two graphs X, Y *have the same Ulam-deck* if there is a one-to-one correspondence between the Ulam-decks of X and Y, such that corresponding subgraphs are isomorphic.

It is clear that two isomorphic graphs must have the same Ulam-deck. Ulam and Kelly, in 1941, have conjectured that having the same deck is a *sufficient* condition for isomorphism for all graphs of order $n \geq 3$.

Since that time, the conjecture has been verified for X, Y belonging to several classes of graphs and many other related problems have been considered (fairly recent surveys are [2] and [9]).

In Section 2, for the benefit of a reader not too familiar with Reconstruction Theory, we make a few remarks explaining (and improving) current terminology.

In Section 3 we state without proof Kelly's Lemma and one of its well-known generalizations due to Greenwell and Hemminger.

In Section 4 we prove the converse of Kelly's Lemma, a result which—although fairly easy to establish—does not seem to appear in the literature.

In Section 5 we define, for a given class $\mathcal{K}$ of graphs, the notions of *$\mathcal{K}$-table* of a graph G, of *$\mathcal{K}$-homogeneous* graphs, and of *$\mathcal{K}$-congruent* graphs. These notions suggest that we call a class $\mathcal{K}$ an *overall* (resp. *pointwise*) control-class, if two graphs X, Y are isomorphic whenever they are $\mathcal{K}$-homogeneous (resp. $\mathcal{K}$-congruent). We point out that the class of paths is not a pointwise control-class for the trees, and suggest a few classes that might be.

In Section 6 we discuss a possible strengthening of the Kelly–Ulam's conjecture.

2 Remarks on terminology and subproblems

The Kelly–Ulam's Conjecture is stated for two arbitrary given graphs X, Y: if X and Y have the same Ulam-deck, then they should be isomorphic.

In order to obtain partial results, the general problem has been split into subproblems or confined to subclasses of the class of all graphs. The following terminology has been introduced.

First of all, a graph X is called *reconstructible* if any graph Y having the same Ulam-deck as X is isomorphic to X.

Thus, proving the Kelly–Ulam's conjecture is the same as proving that any graph of order ≥ 3 is reconstructible, and partial results regarding the Kelly–Ulam's conjecture may consist in proving that restricted types of graphs (or even interesting individual graphs) are reconstructible. For example, it is easy to prove that a regular graph is reconstructible.

Another useful definition, which generalizes the one given above, is the following.

Definition 1 A graph X is *reconstructible within the class of graphs $\mathcal{A}$* (containing X), if any graph $Y \in \mathcal{A}$, having the same Ulam-deck as X, is isomorphic to X.

Hence, the task of proving that a given graph X is reconstructible may be split into the following two steps, with respect to a suitably chosen class $\mathcal{A}$ (containing X):

- prove that X is reconstructible within $\mathcal{A}$.
- prove that an arbitrary graph Y, having the same Ulam-deck as X, must also belong to $\mathcal{A}$.

The former step is sometimes called *weak-reconstructability* of X, but we prefer to call it reconstructability within $\mathcal{A}$, and the latter *recognizability* of the class $\mathcal{A}$. If $\mathcal{A}$ is characterized by a property P, one also speaks of *recognizability* of P. Thus, the difficulty in establishing the recognizability of a class $\mathcal{A}$ strongly depends on the features of $\mathcal{A}$ and, presumably, it is greater when $\mathcal{A}$ is small. For example, it is not known whether the class of planar graphs is recognizable.

Note that if $\mathcal{A}$ is the class of all graphs isomorphic to a given X, then the recognizability of $\mathcal{A}$ is equivalent to the reconstructability of X.

3 Kelly's Lemma and its generalizations

We first introduce some notation and terminology. If X and Y have the same Ulam-deck, then, by definition, there is a one-to-one correspondence σ between the set of the Ulam-subgraphs of X and the set of the Ulam-subgraphs of Y such that corresponding Ulam-subgraphs are isomorphic. Since an Ulam-subgraph contains all but one vertex, then the one-to-one correspondence σ naturally induces another one-to-one correspondence: the correspondence π between the missing vertices. Thus, we can say that X, Y have the same Ulam-deck if and only if there is a bijection $\pi : V(X) \to V(Y)$ such that $X^{(v)} \simeq Y^{(\pi(v))}$ for all $v \in V(X)$. The bijection π will be referred to as an *Ulam-congruence*, and X will be said *Ulam-congruent* to Y.

Let Z be a graph, $v \in V(Z)$. For any graph Q, we set

$\binom{Z}{Q}$ = number of subgraphs of Z isomorphic to Q,

$\binom{Z}{Q}_v$ = number of subgraphs of Z containing vertex v isomorphic to Q.

The so-called Kelly's Lemma is the first result regarding the Kelly-Ulam's conjecture that have been obtained (in [7]). It points out a consequence of the hypothesis that two graphs are Ulam-congruent, quite remarkable in spite of the simplicity of the proof.

Lemma 1 (Kelly's Lemma) *Let X, Y be graphs of order n. Assume that there is a bijection $\pi : V(X) \to V(Y)$ such that*

(i) $X^{(v)} \simeq Y^{(\pi(v))}$ *for all* $v \in V(X)$.

Then

(ii) $\binom{X}{Q} = \binom{Y}{Q}$ *for all graphs Q of order less than n.*

(iii) $\binom{X}{Q}_v = \binom{Y}{Q}_{\pi(v)}$ *for all $v \in V(X)$ and all graphs Q of order less than n.*

We now record a generalization of Kelly's Lemma due to Greenwell and Hemminger ([5]). Let $\mathcal{F}$ be a class of graphs. An *$\mathcal{F}$-subgraph* of a graph G is a subgraph of G isomorphic to some element of $\mathcal{F}$.

Lemma 2 (Greenwell–Hemminger's Lemma) *Let $\mathcal{F}$ be a class of graphs. Let X, Y be Ulam-congruent graphs of order n. Assume that all $\mathcal{F}$-subgraphs of X and Y have order less than n, and that the intersection of two distinct maximal $\mathcal{F}$-subgraphs of X (and Y) is not an $\mathcal{F}$-subgraph. Then, for every $Q \in \mathcal{F}$, the number of maximal $\mathcal{F}$-subgraphs of X isomorphic to Q is equal to the number of maximal $\mathcal{F}$-subgraphs of Y isomorphic to Q.*

Remark 1 When $\mathcal{F}$ consists of a single graph, then the Greenweel–Hemminger's Lemma reduces to Kelly's Lemma. Also, when $\mathcal{F}$ is the set of all subgraphs of X of order exactly $n-1$, the assumption and the conclusion coincide.

Example 1 (Greenwell and Hemminger) Let X, Y and Q be as in Fig. 1.

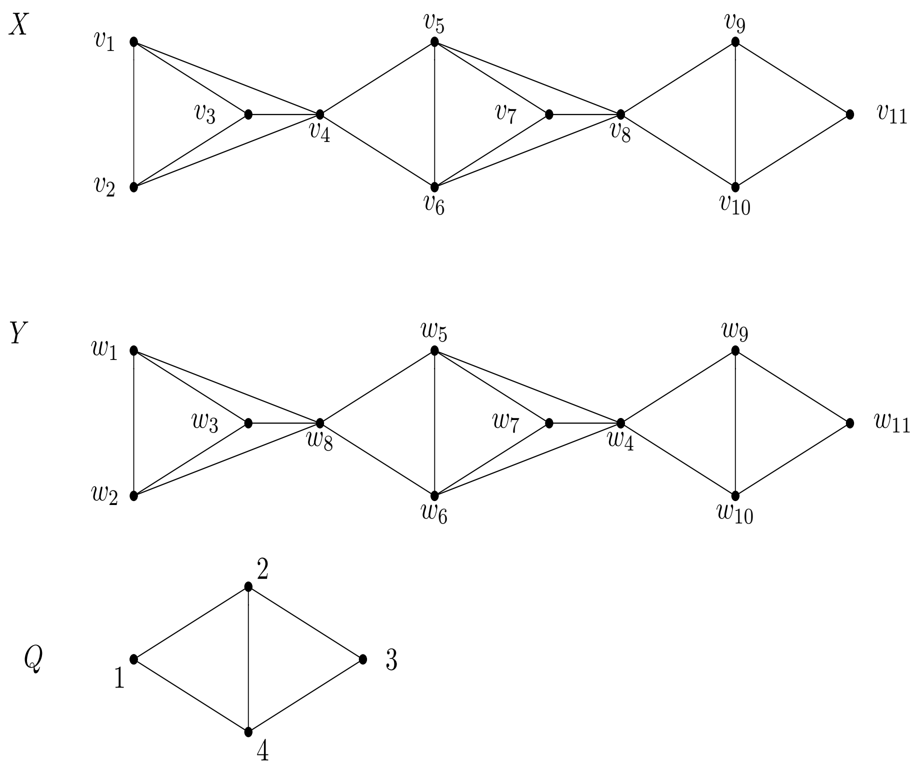

Figure 1: Example of the Greenweel–Hemminger's Lemma.

Let $\pi : v_i \to w_i$ for all i. Let $\mathcal{F}$ be the class of all 2-connected graphs. Then π is an Ulam-congruence from X to Y. The assumptions of the Greenwell–

Hemminger's Lemma are verified since the intersection of two maximal 2-connected subgraphs is not 2-connected. In both X and Y the total number of 2-con–nected maximal subgraphs isomorphic to Q equals 1. Also, there are 4 subgraphs of X isomorphic to Q containing v_4: These are

$$\begin{aligned} H_1 &= \{v_1v_4, v_2v_4, v_1v_3, v_2v_3, v_3v_4\}, & H_2 &= \{v_1v_2, v_1v_3, v_2v_3, v_1v_4, v_2v_4\}, \\ H_3 &= \{v_4v_5, v_4v_6, v_5v_7, v_5v_6, v_6v_7\}, & H_4 &= \{v_4v_5, v_4v_6, v_5v_6, v_5v_8, v_6v_8\}. \end{aligned}$$

For $i = 1, 2, 3, 4$, no H_i is a maximal $\mathcal{F}$-subgraph of X, i.e. it is a subgraph of X which can be properly extended to a 2-connected subgraph of X.

There are 4 subgraphs of Y isomorphic to Q containing $w_4 = \pi(v_4)$: These are

$$\begin{aligned} K_1 &= \{w_5w_6, w_5w_4, w_6w_4, w_5w_7, w_6w_7\}, \\ K_2 &= \{w_5w_4, w_5w_7, w_4w_6, w_6w_7, w_4w_7\}, \\ K_3 &= \{w_4w_5, w_4w_6, w_5w_6, w_5w_8, w_6w_8\}, \\ K_4 &= \{w_4w_9, w_4w_{10}, w_9w_{10}, w_9w_{11}, w_{10}w_{11}\}. \end{aligned}$$

Note that K_4 is a maximal $\mathcal{F}$-subgraph of Y. Thus, this example shows that a pointwise version of Greenwell–Hemminger's Lemma does not hold.

Another generalization of Kelly's Lemma is given by Tutte ([12])

4 The Converse of Kelly's Lemma

Recall that a *class* of graphs is a family of graphs closed under isomorphisms. We denote by $\mathcal{G}$ the class of all graphs. In the next theorem we collect the statement of both Kelly's Lemma and its converse and give a complete proof. Regarding the proof of the implication (ii) $\Rightarrow$ (i) (the converse of Kelly's Lemma), the reader may keep in mind the following example, where we have shown how the Ulam-subgraphs are "distributed" among the subgraphs of order $n-1$ of the various *sizes*, i.e. number of edges (see Fig. 2).

Theorem 1 *Let X, Y be graphs of order n and let $\mathcal{K}$ be a class of graphs. Let π be a bijection $V(X) \to V(Y)$. Consider the following conditions:*

(i) $X^{(v)} \simeq Y^{(\pi(v))}$ *for all* $v \in V(X)$.

(ii) $\binom{X}{Q} = \binom{Y}{Q}$ *for all* $Q \in \mathcal{K}$ *of order less than* n.

(iii) $\binom{X}{Q}_v = \binom{Y}{Q}_{\pi(v)}$ *for all* $v \in V(X)$ *and all* $Q \in \mathcal{K}$ *of order less than* n.

Then (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii). If $\mathcal{K} = \mathcal{G}$, the three conditions are equivalent.

Proof Let $\binom{Z}{Q}_{v'}$ = number of subgraphs of Z *not* containing the vertex v and isomorphic to Q, and consider the auxiliary condition

(i)′ There is a bijection $\pi : V(X) \to V(Y)$ such that $\binom{X}{Q}_{v'} = \binom{Y}{Q}_{\pi(v)'}$ for all $v \in V(X)$ and all graphs $Q \in \mathcal{K}$ of order less than n.

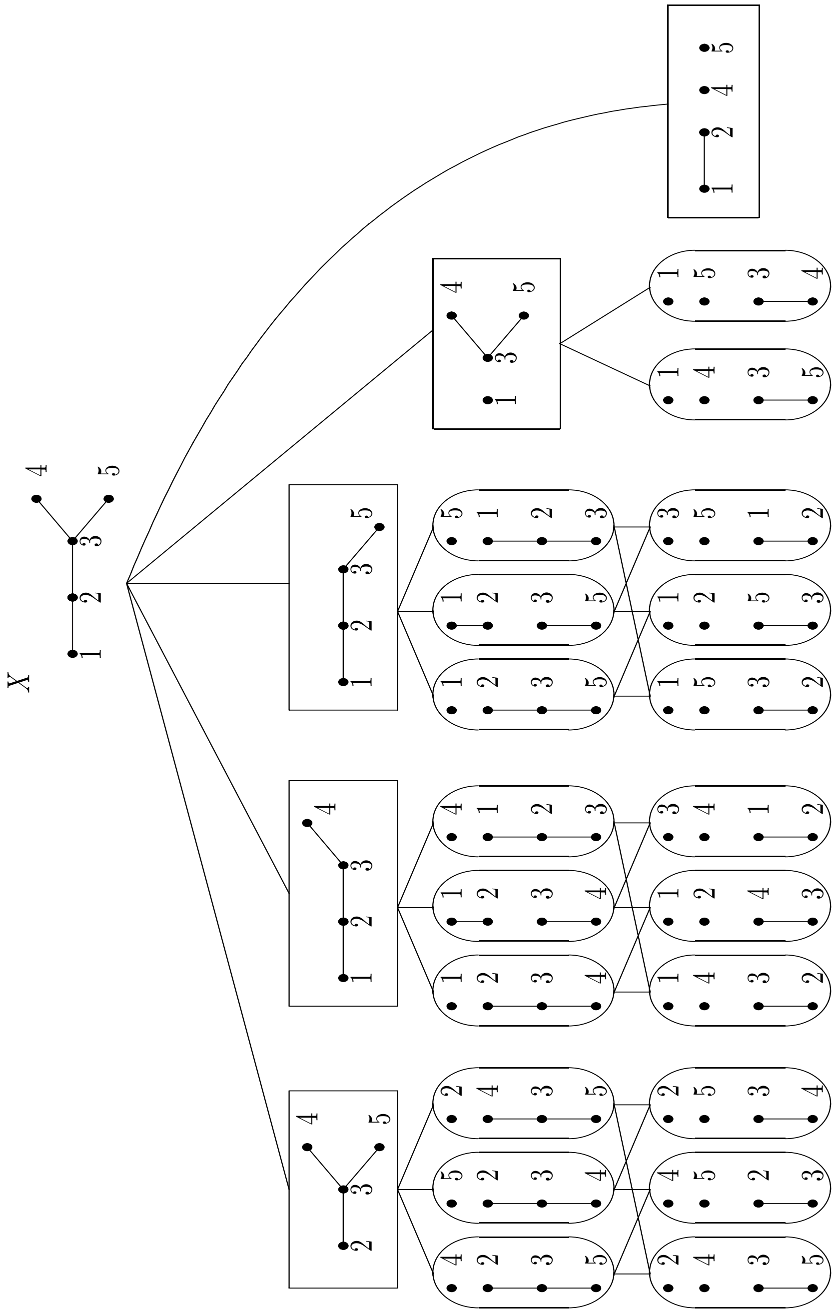

Figure 2: The lattice of the subgraphs of X of order 4. The Ulam-subgraphs are depicted into rectangular frames.

We prove that (i) $\Rightarrow$ (i)$'$ for all $\mathcal{K}$, and (i)$'$ $\Rightarrow$ (i) for $\mathcal{K} = \mathcal{G}$.

Proof of (i)$\Rightarrow$(i)$'$. By (i), $X^{(v)} \simeq Y^{(\pi(v))}$, hence $\binom{X^{(v)}}{Q} = \binom{Y^{(\pi(v))}}{Q}$ for all $Q \in \mathcal{K}$. But since $|Q| < n$, $\binom{X}{Q}_{v'} = \binom{X^{(v)}}{Q}$, and $\binom{Y}{Q}_{\pi(v)'} = \binom{Y^{(\pi(v))}}{Q}$ for all $v \in V(X)$.

Proof of (i)$'$$\Rightarrow$(i). Fix any $v \in V(X)$. Assume (i)$'$ for $\mathcal{K} = \mathcal{G}$. Thus we can replace $X^{(v)}$ for Q, thus obtaining

$$1 = \binom{X^{(v)}}{X^{(v)}} = \binom{X}{X^{(v)}}_{v'} = \binom{Y}{X^{(v)}}_{\pi(v)'} = \binom{Y^{\pi(v)}}{X^{(v)}}.$$

We can also replace $Y^{\pi(v)}$ for Q, obtaining

$$\binom{X^{(v)}}{Y^{(\pi(v))}} = \binom{X}{Y^{(\pi(v))}}_{v'} = \binom{Y}{Y^{(\pi(v))}}_{\pi(v)'} = \binom{Y^{(\pi(v))}}{Y^{(\pi(v))}} = 1.$$

In particular, we get $X^{(v)} \lesssim Y^{\pi(v)}$ from the first equality, and $Y^{(\pi(v))} \lesssim X^{(v)}$ from the second. Hence $Y^{(\pi(v))} \simeq X^{(v)}$.

Note that just one of the inequalities above, together with the finiteness of the graphs involved, would suffice to obtain the same conclusion if one proves that $X^{(v)}$ and $Y^{\pi(v)}$ have the same number of edges.

Proof of (i) $\Rightarrow$ (ii).

$$\binom{X}{Q} = \frac{1}{n - |Q|} \sum_{v \in V(X)} \binom{X^{(v)}}{Q} = \frac{1}{n - |Q|} \sum_{v \in V(X)} \binom{Y^{(\pi(v))}}{Q} = \binom{Y}{Q}.$$

Proof of (i) $\Rightarrow$ (iii). From what above, we can show that (i)$'$$\wedge$ (ii) $\Rightarrow$ (iii). Simply write

$$\binom{X}{Q}_v = \binom{X}{Q} - \binom{X}{Q}_{v'} = \binom{Y}{Q} - \binom{Y}{Q}_{\pi(v)'} = \binom{Y}{Q}_{\pi(v)}.$$

Proof of (iii) $\Rightarrow$ (ii). One can briefly argue as follows.

If, for each $v \in V(X)$, one counts the number of subgraphs of X containing v and isomorphic to (the given fixed) Q and then sums up the values obtained for various v, one overcounts each such subgraph H (isomorphic to Q) by a factor $|H|$, because for all vertices v of H the same H is counted.

As the subgraphs considered are all isomorphic to Q, they all have the same order $|Q|$. This allows us to obtain $\binom{X}{Q}$ by dividing out by $|Q|$. Then

$$\binom{X}{Q} = \frac{1}{|Q|} \sum_{v \in V(X)} \binom{X}{Q}_v = \frac{1}{|Q|} \sum_{v \in V(Y)} \binom{Y}{Q}_{\pi(v)} = \binom{Y}{Q},$$

which proves (ii).

Proof of (ii) $\Rightarrow$ (i) when $\mathcal{K} = \mathcal{G}$. One has to take into account the fact that the various Ulam-subgraphs may have different sizes (number of edges).

We shall prove (i) by assuming only that $\binom{X}{Q} = \binom{Y}{Q}$ for all graphs Q of order exactly $n-1$. In fact, in this part of the proof, all the subgraphs of X and Y considered will be subgraphs of order $n-1$.

If Q is a graph of order $n-1$, denote by $\mathcal{U}_X(Q)$ (resp. $\mathcal{U}_Y(Q)$) the set of Ulam-subgraphs of X (resp. of Y) isomorphic to Q.

We have to prove that, for any such Q

$$|\mathcal{U}_X(Q)| = |\mathcal{U}_Y(Q)|.$$

(indeed, this amounts to proving that X and Y have the same Ulam-deck, i.e. that (i) holds for some bijection $\pi : V(X) \to V(Y)$).

We shall split the proof into steps, according to the size of Q. We shall procede starting with the maximum size.

So, let l_X (resp. l_Y) be the largest size value of a subgraph of X (resp. of Y) of order $n-1$. By the assumption (ii) it is clear that $l_X = l_Y$: Indeed, if it were, say, $l_X < l_Y$, there would be in Y at least a subgraph U of order $n-1$ and size l_Y, hence $\binom{Y}{U} \geq 1$, whereas in X all subgraphs of order $n-1$ would have size $\leq l_X < l_Y$, hence $\binom{X}{U} = 0$, a contradiction. Thus we set $l := l_X = l_Y$.

Before proceeding, note the important fact that any subgraph of X (resp. of Y) of order $n-1$, and of arbitrary size s, is contained in *exactly one* Ulam-subgraph. In other words, there are no subgraphs of order $n-1$ in the intersection of two distinct Ulam-subgraphs (possibly of different sizes). Let Q be an arbitrary graph of order $n-1$. Denote by $\binom{G}{Q}^{[k]}$ the number of subgraphs of a graph G of order n isomorphic to Q and contained in some Ulam-subgraph of size k.

By the above consideration, it follows that

- if Q_l is a graph of (order $n-1$ and) size equal to l, then

$$\binom{X}{Q_l} = \binom{X}{Q_l}^{[l]} \quad \text{and} \quad \binom{Y}{Q_l} = \binom{Y}{Q_l}^{[l]}. \tag{1}$$

  Indeed, a subgraph of (order $n-1$ and) size l is necessarily contained in (in fact it is equal to) some Ulam-subgraph of size l.

- If Q_{l-1} is a graph of (order $n-1$ and) size equal to $l-1$, then

$$\begin{aligned}\binom{X}{Q_{l-1}} &= \binom{X}{Q_{l-1}}^{[l]} + \binom{X}{Q_{l-1}}^{[l-1]},\\ \binom{Y}{Q_{l-1}} &= \binom{Y}{Q_{l-1}}^{[l]} + \binom{Y}{Q_{l-1}}^{[l-1]}.\end{aligned} \tag{2}$$

  Indeed, a subgraph of (order $n-1$ and) size $l-1$ is either contained in some Ulam-subgraph of size l, or in in some Ulam-subgraph of size $l-1$.

In general, if Q_s is a graph of (order $n-1$ and) size equal to s, we have

$$\binom{X}{Q_s} = \sum_{k=s}^{l} \binom{X}{Q_s}^{[k]} \quad \text{and} \quad \binom{Y}{Q_s} = \sum_{k=s}^{l} \binom{Y}{Q_s}^{[k]}. \tag{3}$$

We shall use the equalities (3) in succession, starting with $s = l$. We begin by considering (one-by-one) representatives of all graphs Q_l of (order $n-1$ and) size l. From (1) (that is (3) with $s = l$) and from assumption (ii), we obtain

$$\binom{X}{Q_l}^{[l]} = \binom{Y}{Q_l}^{[l]},$$

that is $|\mathcal{U}_X(Q_l)| = |\mathcal{U}_Y(Q_l)|$. This equality allows us to set up (at least) a one-to-one iso-correspondence μ_l (i.e. with corresponding objects isomorphic) between the Ulam-subgraphs of X of size l and those of Y. By "restriction", μ_l gives rise to one-to-one iso-correspondences $\mu_{l,r}$ between the set of subgraphs H_r of X of size r *contained in some* Ulam-subgraph of size l, and the analogous set of subgraphs K_r of Y (see Fig. 3, where $r = l-1$, and the action of $\mu_{l,l-1}$ is drawn only partially).

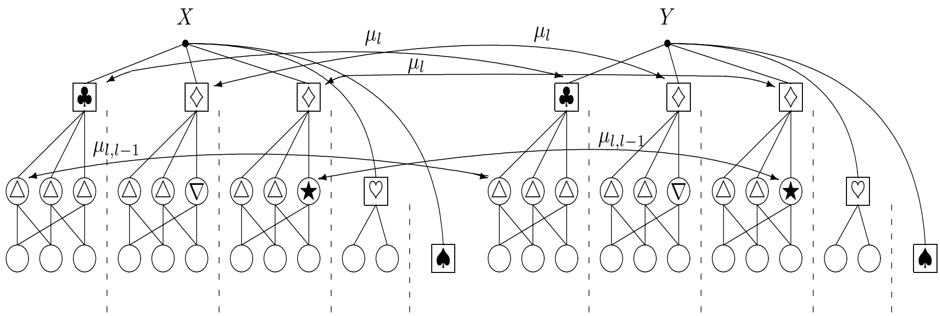

Figure 3: The one-to-one iso-correspondence $\mu_{l,l-1}$ induced by μ_l. The dashed vertical lines stress the fact that any subgraph of order $n-1$ is contained in exactly one Ulam-subgraph (depicted in square frames).

Consequently we have, for any Q_r of (order $n-1$ and) size $r < l$

$$\binom{X}{Q_r}^{[l]} = \binom{Y}{Q_r}^{[l]}. \tag{4}$$

Now, consider equality (2) (that is (3) with $s = l-1$). Applying equality (4) with $r = l-1$ and assumption (ii), we obtain

$$\binom{X}{Q_{l-1}}^{[l-1]} = \binom{Y}{Q_{l-1}}^{[l-1]},$$

that is $|\mathcal{U}_X(Q_{l-1})| = |\mathcal{U}_Y(Q_{l-1})|$. This equality allows us to set up (at least) a one-to-one iso-correspondence μ_{l-1} between the Ulam-subgraphs of X of size $l-1$ and those of Y. By "restriction", μ_{l-1} gives rise to one-to-one iso-correspondences $\mu_{l-1,t}$ between the set of subgraphs H_t of X of size t *contained in some* Ulam-subgraph of size $l-1$ and the analogous set of subgraphs K_t of Y.

Consequently we have, for any Q_t of (order $n-1$ and) size $t < l-1$

$$\binom{X}{Q_t}^{[l-1]} = \binom{Y}{Q_t}^{[l-1]}. \tag{5}$$

Now, consider equality (3) with $s = l-2$. Applying *both* equalities (4) with $r = l-2$ and (5) with $t = l-2$, together with assumption (ii), we obtain

$$\binom{X}{Q_{l-2}}^{[l-2]} = \binom{Y}{Q_{l-2}}^{[l-2]},$$

that is $|\mathcal{U}_X(Q_{l-2})| = |\mathcal{U}_Y(Q_{l-2})|$.

Repeating this argument for $l-3, \ldots, 1, 0$, we obtain the desired conclusion. $\square$

Remark 2 Because of the equivalence (i) $\Leftrightarrow$ (ii) (when $\mathcal{K} = \mathcal{G}$), Kelly–Ulam's conjecture can be rephrased by saying that two graphs of order n are isomorphic if and only if they contain the same number of subgraphs isomorphic to any graph Q of order less than n.

Although Conditions (ii) and (iii) of Theorem 1 are equivalent when considered for *all* graphs Q of order less than n, Conditions (ii) no longer implies Condition (iii) when Q is taken in a class $\mathcal{K}$ smaller than the class $\mathcal{G}$ of all graphs. Thus, for example, when the class $\mathcal{K}$ consists of the single graph K_2 (the connected graph on two vertices) Condition (ii) says that X and Y have the same number of edges, whereas Condition (iii) says that they have the same degree-sequence. As an example, one may take X to be a four cycle and Y a graph of order 4 having exactly one vertex of degree 1. The same X and Y also show that Condition (ii) does not imply Condition (iii) even if the class $\mathcal{K}$ consisted of all Q of order $n-2$.

However, since our proof of (ii) $\Rightarrow$ (i) only uses subgraphs of order $n-1$, we see that (ii) $\Rightarrow$ (iii) when $\mathcal{K}$ consists of all Q of order $n-1$.

5 $\mathcal{K}$-congruent pairs of graphs. Control-classes for a given class of graphs

Because of the equivalence of (i), (ii), and (iii) in Theorem 1 (when $\mathcal{K} = \mathcal{G}$), whenever the Kelly–Ulam's conjecture is proved for two graphs X, Y, such a result can be reformulated in two ways.

For example, Kelly's Theorem on trees ([7]) can be reformulated in the following ways (omitting the recognition part of his statement)

(O) *Let T_1, T_2 be two trees of order n. If for all graphs $Q \in \mathcal{G}$ of order less than n it holds $\binom{T_1}{Q} = \binom{T_2}{Q}$, then $T_1 \simeq T_2$.*

(P) *Let T_1, T_2 be two trees of order n. If there is a bijection $\pi : V(T_1) \to V(T_2)$ such that $\binom{T_1}{Q}_v = \binom{T_2}{Q}_{\pi(v)}$ for all $v \in V(T_1)$ and for all $Q \in \mathcal{G}$ of order less than n, then $T_1 \simeq T_2$.*

This leads to the following definitions.

Definition 2 Let $\mathcal{K}$ be a class of graphs, and X, Y graphs of order n. We say that X, Y are *$\mathcal{K}$-homogeneous* if $\binom{X}{Q} = \binom{Y}{Q}$ for all Q in $\mathcal{K}$ of order less than n. We say that X, Y are *$\mathcal{K}$-congruent* if there is a bijection $\pi : V(X) \to V(Y)$, called *$\mathcal{K}$-congruence*, such that $\binom{X}{Q}_v = \binom{Y}{Q}_{\pi(v)}$ for all $v \in V(X)$ and all Q in $\mathcal{K}$ of order less than n.

Definition 3 Let $\mathcal{K}$ be a class of graphs. The *$\mathcal{K}$-table* of a graph G is the array whose rows are labelled by the vertices of G, whose columns are labelled by representatives of the isomorphism classes of the graphs of $\mathcal{K}$ such that, for $v \in V(G)$, $Q \in \mathcal{K}$, the entry at position (v, Q) is the number of subgraphs of G containing v isomorphic to Q.

From this definition it follows that two graphs X and Y are $\mathcal{K}$-congruent if and only if their $\mathcal{K}$-tables are equal, up to reordering of the rows.

Definition 4 Let $\mathcal{A}$ be a class of graphs of order n. A class of graphs $\mathcal{K}$ is called an *overall control-class* for $\mathcal{A}$ if two graphs $G_1, G_2 \in \mathcal{A}$ are isomorphic whenever $\binom{G_1}{Q} = \binom{G_2}{Q}$, for all $Q \in \mathcal{K}$ of order less than n, i.e. whenever they are $\mathcal{K}$-homogeneous.

Similarly, $\mathcal{K}$ is called a *pointwise control-class* for $\mathcal{A}$ if two graphs $G_1, G_2 \in \mathcal{A}$ are isomorphic whenever there is a bijection $\pi : V(G_1) \to V(G_2)$ for which $\binom{G_1}{Q}_v = \binom{G_2}{Q}_{\pi(v)}$, for all $v \in V(G_1)$ and all $Q \in \mathcal{K}$ of order less than n, i.e. whenever they are $\mathcal{K}$-congruent.

Remark 3 Since

$$\binom{G}{Q} = \frac{1}{|Q|} \sum_{v \in V(G)} \binom{G}{Q}_v$$

then, clearly, if $\mathcal{K}$ is an overall control-class for $\mathcal{A}$, then $\mathcal{K}$ is also a control-class for $\mathcal{A}$. Moreover, if Kelly–Ulam's Conjecture is true, then the class $\mathcal{G}$ of all graphs is a control-class for any $\mathcal{A}$.

A generalization of the Reconstruction Problem which seems interesting to us is the following.

Problem 1 Find *minimal* control-classes for $\mathcal{A}$, when $\mathcal{A}$ is a class of reconstructible graphs, for instance the class of trees ([7]), cacti ([4], [10]), maximal planar graphs ([8]) and so on (minimal control-classes may not be unique).

In the special case when $\mathcal{A}$ is the class of all trees of a fixed order n, one may consider several interesting candidates for control-classes (either *overall* or *pointwise*)

- the class $\mathcal{P}$ of paths,
- the class $\mathcal{P}_\sigma$ of σ-paths (i.e. disjoint unions of paths),

- the class $\mathcal{C}$ of caterpillars,
- the class $\mathcal{C}_\sigma$ of σ-caterpillars,
- the class $\mathcal{O}$ of octopi (i.e. trees with at most one vertex of degree greater than 2).

Each class listed above has the feature that it contains the connected subgraphs of its elements. The classes $\mathcal{P}_\sigma$ and $\mathcal{C}_\sigma$ in fact contain all subgraphs of their elements.

It is not known if the classes $\mathcal{P}_\sigma$, $\mathcal{C}$, $\mathcal{C}_\sigma$, $\mathcal{O}$ listed above are pointwise control-classes for the trees. In [3] example-pairs are given that show that $\mathcal{P}$ is not a pointwise control-class for the trees of order n for many values of n. The minimal pair ($n = 20$) is shown in Figure 4. This pair also shows that $\mathcal{O}$ is not an overall control-class for the trees.

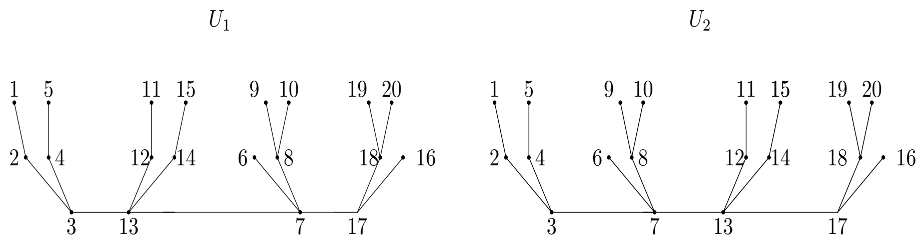

Figure 4: Minimal pair of non-isomorphic $\mathcal{P}$-congruent trees.

Remark 4 In view of the remark at the end of Section 4, if Kelly-Ulam's Conjecture is true, not only the class $\mathcal{G}$ of all graphs, but also the class $\mathcal{G}_{n-1}$ of all graphs of order exactly $n-1$ is an overall control-class for the class of all graphs of order n. However, in general, if $\mathcal{K}$ is a control-class for a class $\mathcal{A}$ of graphs of order n, it may not be true that also the class $\mathcal{K} \cap \mathcal{G}_{n-1}$ is a control-class for $\mathcal{A}$. In fact, $\mathcal{K} \cap \mathcal{G}_{n-1}$ may well be empty. For example, several trees of order n will contain no octopus or caterpillar of order $n-1$: Thus these trees could never be distinguished by the octopi or caterpillars of order $n-1$.

6 The Ulam-ladder

There are several ways of strengthening Kelly–Ulam's conjecture. The first and most natural is to ask whether fewer than n Ulam-subgraphs suffice to determine a graph (up to isomorphism). It has been proved that three suitably selected Ulam-subgraphs suffice "almost always" ([1]). For an arbitrary graph G of order n, Harary and Plantholt ([6]) have conjectured that $[\frac{n}{2}] + 2$ well-selected Ulam-subgraphs should suffice to determine G, and in fact 3 should suffice if n is prime.

To discuss another strengthening of Kelly–Ulam's conjecture we premise a definition.

Definition 5 The *Ulam-ladder* is the function $L : \mathbb{N} \to \mathbb{N}$ defined by setting $L(n)$ to be the smallest positive integers m such that all graphs of order n are determined by their induced subgraphs of order m.

There is some evidence to contend that

$$\lim_{n \to \infty} n - L(n) = \infty.$$

However, Nýdl has proved that for any fixed rational number $q < 1$, there is a positive integer n and a graph G of order n such that the knowledge of all induced subgraphs of G of order less than or equal to qn does not allow to determine G ([11]). In other words, if the Kelly–Ulam's conjecture is true, the graph of $L(n)$ lies below the straight line $y = x - 1$, but, by Nýdl's result, it does *not* lie below any straight line passing through the origin of slope $q < 1$. However, a shape for the graph of $L(n)$ like the one hinted at in Figure 5 would be compatible with Nýdl's result (the first eight values of $L(n)$ that we have drawn have been verified by computer).

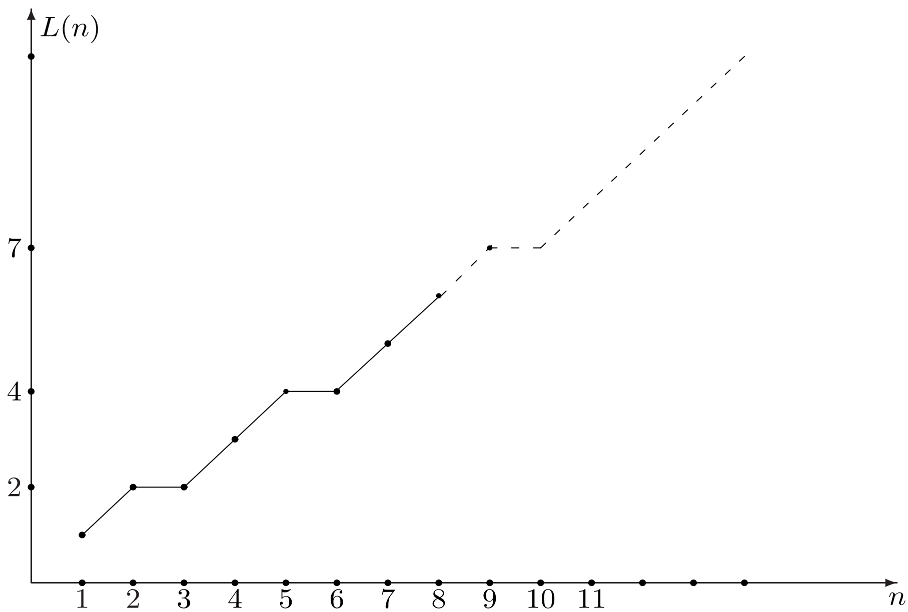

Figure 5: The Ulam-ladder.

We believe that the determination of the Ulam-ladder is one of the most charming problems in graph reconstruction.

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