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A Decomposition of Homomorphic Images of Nearlattices*

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Abstract

By a nearlattice is meant a join-semilattice where every principal filter is a lattice with respect to the induced order. The aim of our paper is to show for which nearlattice $\mathcal{S}$ and its element c the mapping $\varphi_c(x) = \langle x \vee c, x \wedge_p c\rangle$ is a (surjective, injective) homomorphism of $\mathcal{S}$ into $[c) \times (c]$.

Key words: Nearlattice; semilattice; distributive element; pseudocomplement; dual pseudocomplement.

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It is well-known (see e.g. [4]) that if L is a bounded distributive lattice and $c \in L$ has a complement in L then L is isomorphic to the direct product $[c) \times (c]$. On the other hand, if c is not complemented then the mapping $\varphi_c(x) = \langle x \vee c, x \wedge c\rangle$ is still an injective homomorphism of L into the mentioned direct product and one can discuss whether the homomorphic image $\varphi_c(L)$ is a subdirect product of $[c) \times (c]$.

In what follows we generalize this setting for the so-called nearlattices (see [1–3, 5–8]) and we investigate which of these results remain true. It turns out that our task is reasonable only for a class of so-called nested nearlattices.

Definition 1 By a *nearlattice* we mean a semilattice $\mathcal{S} = (S; \vee)$ where for each $a \in S$ the principal filter $[a) = \{x \in S; a \leq x\}$ is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$.

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Remark 1 Since the operation meet is defined only in a corresponding principal filter, we will indicate this fact by indices, i.e. $\wedge_x$ denotes the meet in $[x)$. On the other hand, if $a, b \in [x)$ and $y \leq x$ then $a, b \in [y)$ and $a \wedge_x b = a \wedge_y b$ since both are considered with respect to the same (induced) order $\leq$.

Definition 2 Let $\mathcal{S} = (S; \vee)$ be a nearlattice and $\emptyset \neq A \subseteq S$. A is called a *sublattice* of $\mathcal{S}$ if it is a lattice with respect to the induced order $\leq$ of $\mathcal{S}$.

A sublattice M of a nearlattice $\mathcal{S}$ is called *maximal* if M is not a proper sublattice of another sublattice of $\mathcal{S}$.

Let $\mathcal{S} = (S; \vee)$ be a nearlattice. Denote by $\mathcal{M}_{\mathcal{S}} = \{M_\gamma, \gamma \in \Gamma\}$ the set of all maximal sublattices M_γ of $\mathcal{S}$.

Further, if there exists an element $c \in \bigcap \mathcal{M}_{\mathcal{S}}$, $\mathcal{S}$ will be called a *nested nearlattice.*

Remark 2 a) Every finite nearlattice $\mathcal{S}$ is nested, because $\mathcal{S}$ is a join semilattice with 1 and $1 \in \bigcap M_{\mathcal{S}}$.

b) An example of an infinite nearlattice which is not nested is shown in Fig. 1.

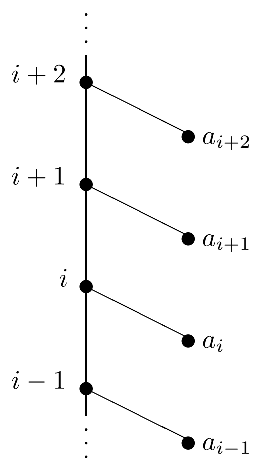

Fig. 1

For any element $c \in S$ we can find a maximal sublattice which does not contain c. In particular, if $c = i$ or $c = a_i$ then c does not belong to the maximal sublattice $[a_{i+1})$.

Let $\mathcal{S}$ be a nested nearlattice and suppose $c \in \bigcap \mathcal{M}_S$. Suppose $x \in \mathcal{S}$. Then there exists $\gamma \in \Gamma$ such that $x \in M_\gamma$. Since M_γ is a lattice and $c \in M_\gamma$, there exists $\inf\{x, c\}$ with respect to the induced order. Suppose $p \in \mathcal{S}$ with $p \leq x, c$. Then clearly $x \wedge_p c = \inf\{x, c\}$. Apparently, this operation does not depend on γ (when x belongs to more than one M_γ). Summarizing, there surely exists $p \in \mathcal{S}$ such that $x \wedge_p c = \inf\{x, c\}$.

Definition 3 Let $\mathcal{S}$ be a nested nearlattice and $c \in \bigcap \mathcal{M}_S$. The mapping $\varphi_c : S \to [c) \times (c]$ defined by

$$\varphi_c(x) = \langle x \vee c, x \wedge_p c \rangle$$

will be called a *decomposition mapping.*

The mapping φ_c is obviously everywhere defined, since $c \in \bigcap \mathcal{M}_\mathcal{S}$.

Definition 4 Let $\mathcal{S}$ be a nearlattice and $\{M_\gamma, \gamma \in \Gamma\}$ be the set of its maximal sublattices.

(i) An element a of $\mathcal{S}$ is called *distributive* if

$$a \vee (x \wedge_p y) = (a \vee x) \wedge_p (a \vee y),$$

for all $x, y, p \in M_\gamma$, $p \leq x, y$ and all $\gamma \in \Gamma$.

(ii) An element a is called *dually distributive* if

$$a \wedge_p (x \vee y) = (a \wedge_p x) \vee (a \wedge_p y),$$

for all $a, x, y, p \in M_\gamma$, $p \leq a, x, y$ and all $\gamma \in \Gamma$.

A nearlattice $\mathcal{S}$ is called *distributive* if

$$a \vee (b \wedge_p c) = (a \vee b) \wedge_p (a \vee c)$$

for all $a, b, c \in S$ with $p \leq b, c$.

Suppose now, that an element c is distributive and also dually distributive. We wonder whether φ_c is a homomorphism.

Definition 5 By a *suitable element* we mean an element c of a nested nearlattice $\mathcal{S} = (S; \vee)$ with $c \in \bigcap \mathcal{M}_\mathcal{S}$, which is distributive and also dually distributive.

Of course, in a nested distributive nearlattice $\mathcal{S}$ every element $c \in \bigcap \mathcal{M}_\mathcal{S}$ is suitable.

Proposition 1 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Then the decomposition mapping φ_c is a homomorphism.*

Proof $\varphi_c(x \vee y) = \langle (x \vee y) \vee c, (x \vee y) \wedge_p c \rangle = \langle (x \vee c) \vee (y \vee c), (x \wedge_p c) \vee (y \wedge_p c) \rangle = \langle x \vee c, x \wedge_p c \rangle \vee \langle y \vee c, y \wedge_p c \rangle = \varphi_c(x) \vee \varphi_c(y)$.

$\varphi_c(x \wedge_p y) = \langle (x \wedge_p y) \vee c, (x \wedge_p y) \wedge_p c \rangle = \langle (x \vee c) \wedge_p (y \vee c), (x \wedge_p c) \wedge_p (y \wedge_p c) \rangle = \langle x \vee c, x \wedge_p c \rangle \wedge_p \langle y \vee c, y \wedge_p c \rangle = \varphi_c(x) \wedge_p \varphi_c(y)$. □

Example 1 Let $\mathcal{S}$ be a nearlattice depicted in Fig 2.

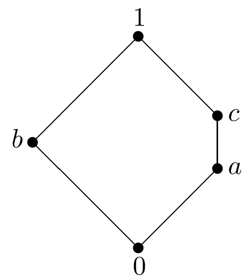

Fig. 2

We can easily check that the elements $0, b, 1$ are distributive and also dually distributive. An element c is distributive, but not dually distributive, an element a is dually distributive, but not distributive.

Consider the decomposition mappings φ_b, φ_a and φ_c. Then for $\varphi_b : \mathcal{S} \mapsto [b) \times (b]$ we have $\varphi_b(1) = \langle 1, b\rangle$, $\varphi_b(0) = \langle b, 0\rangle$, $\varphi_b(b) = \langle b, b\rangle$, $\varphi_b(a) = \langle 1, 0\rangle$ and $\varphi_b(c) = \langle 1, 0\rangle$ (see Fig. 3). Clearly, $[b) = \{b, 1\}$, $(b] = \{0, b\}$.

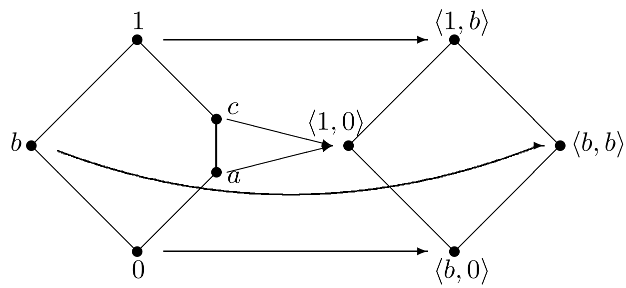

Fig. 3

One can see that the mapping φ_b is a surjective homomorphism which is not injective.

For the decomposition mapping $\varphi_a : \mathcal{S} \mapsto [a) \times (a]$ we have $\varphi_a(1) = \langle 1, a\rangle$, $\varphi_a(0) = \langle a, 0\rangle$, $\varphi_a(a) = \langle a, a\rangle$, $\varphi_a(b) = \langle 1, 0\rangle$ and $\varphi_a(c) = \langle c, a\rangle$ (see Fig. 4). Obviously, $[a) = \{a, c, 1\}$ and $(a] = \{0, a\}$.

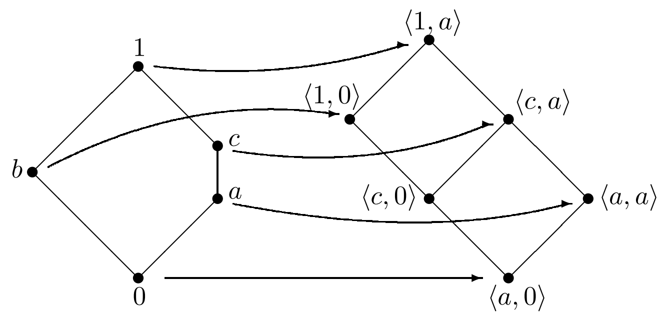

Fig. 4

The mapping φ_a is not a homomorphism, because

$$\varphi_a(c \wedge b) = \langle a, 0 \rangle \neq \langle c, 0 \rangle = \varphi_a(c) \wedge \varphi_a(b).$$

Similarly, the decomposition mapping $\varphi_c : \mathcal{S} \mapsto [c) \times (c]$ is not a homomorphism.

Now, we will check, whether φ_c is an injection. Let $\varphi_c(x) = \varphi_c(y)$. Then $x \vee c = y \vee c$ and $x \wedge_p c = y \wedge_p c$. If the mapping φ_c is injective, then $x = y$. Thus the mapping φ_c is injective only if for each $x, y \in M_\gamma$ ($x \vee c = y \vee c$ and $x \wedge_p c = y \wedge_p c$) implies $x = y$.

Remark 3 Distributivity and dual distributivity of the element c is not enough to ensure injectivity of the mapping φ_c (see Fig. 3). If we swap b and c, in Fig. 2, we obtain $b \vee c = a \vee c$ and also $b \wedge_0 c = a \wedge_0 c$, but $a \neq b$.

Let us note that for injectivity of φ_c it is not necessary that each maximal sublattice is distributive.

Proposition 2 *If $\mathcal{S} = (S; \vee)$ is a nested distributive nearlattice and $c \in \bigcap \mathcal{M}_{\mathcal{S}}$, then the decomposition mapping φ_c is injective.*

Proof If $\mathcal{S}$ is distributive then each maximal sublattice is a distributive lattice, in which ($x \vee c = y \vee c$ and $x \wedge_p c = y \wedge_p c$) implies $x = y$. $\square$

If φ_c is an injective homomorphism, then φ_c is an embedding of S into $[c) \times (c]$, i.e. $\mathcal{S}$ is isomorphic to a subnearlattice of this direct product.

Example 2 Denote by $M_1 = \{a, c, 1\}, M_2 = \{b, c, 1\}$ the maximal sublattices of the finite distributive nearlattice $\mathcal{S}$ visualized in Fig. 5.

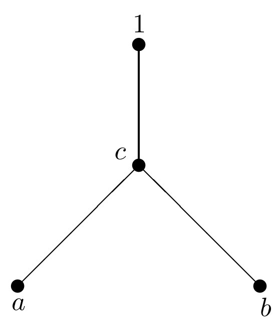

Fig. 5

Evidently $c \in M_1 \cap M_2$. Further, $[c) = \{c, 1\}$ and $(c] = \{c, a, b\}$.
The direct product $[c) \times (c]$ is depicted in Fig. 6.

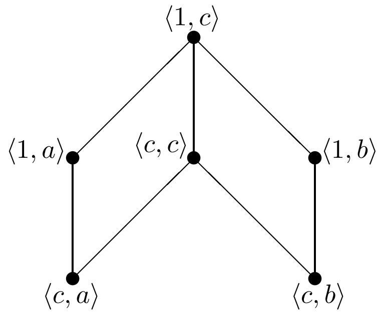

Fig. 6

We have $\varphi_c(1) = \langle 1, c\rangle$, $\varphi_c(c) = \langle c, c\rangle$, $\varphi_c(a) = \langle c, a\rangle$ and $\varphi_c(b) = \langle c, b\rangle$.
One can see that φ_c is an injective homomorphism, which is not surjective.

Remark 4 For $c = 1$ (where 1 is the greatest element of S), we obtain: $[c) = \{c\}$, $(c] = S$, and thus $[c) \times (c] \cong \mathcal{S}$.

Now we are interested in assumptions under which the mapping φ_c is surjective. Suppose $\mathcal{S}$ is a nested nearlattice with the set $\{M_\gamma; \gamma \in \Gamma\}$ of its maximal sublattices. An element $a \in S$ has a *complement* b_γ in M_γ if M_γ is a bounded lattice (with 0_γ or 1 as the least or greatest element, respectively) and $a \vee b_\gamma = 1$, $a \wedge_{0_\gamma} b_\gamma = 0_\gamma$.

Proposition 3 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Suppose that c has a complement p_γ in each maximal sublattice M_γ of the nearlattice S. Then the decomposition mapping φ_c is a surjective homomorphism.*

Proof We need only to prove, that for each $\langle x, y\rangle \in [c) \times (c]$, there exists an element $z \in S$, such that $\varphi_c(z) = \langle x, y\rangle$.

Since $\langle x, y\rangle \in [c) \times (c]$, then clearly $y \leq c \leq x$ and there exists $\gamma \in \Gamma$ such that $[y) \subseteq M_\gamma$. Denote by $\wedge_\gamma$ the operation symbol $\wedge_y$ (because it in fact does not depend on y in the following computation).

Take $z = (y \vee p_\gamma) \wedge_\gamma x$. Then

$$\begin{aligned}\varphi_c(z) &= \langle z \vee c, z \wedge_\gamma c\rangle = \langle ((y \vee p_\gamma) \wedge_\gamma x) \vee c, (y \vee p_\gamma) \wedge_\gamma x \wedge_\gamma c\rangle \\ &= \langle (y \vee p_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (p_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle \\ &= \langle (y \vee 1) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (p_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle \\ &= \langle x \vee c, y \wedge_\gamma c \wedge_\gamma x\rangle = \langle x, y\rangle,\end{aligned}$$

proving that φ_c is surjective. □

Corollary 1 *Let $\mathcal{S} = (S; \vee)$ be a nested distributive nearlattice and $\mathcal{M}_\mathcal{S} = \{M_\gamma; \gamma \in \Gamma\}$ the set of its maximal sublattices. If there exists an element $c \in \bigcap \mathcal{M}_\mathcal{S}$ such that c has a complement in each M_γ then the decomposition mapping φ_c is the isomorphism of $\mathcal{S}$ onto $[c) \times (c]$.*

Example 3 The nearlattice S in Fig. 7 is a nested distributive nearlattice which has exactly two distinct maximal sublattices $M_1 = \{a, c, p_1, 1\}$ and $M_2 = \{b, c, p_2, 1\}$. Of course, $c \in M_1 \cap M_2$.

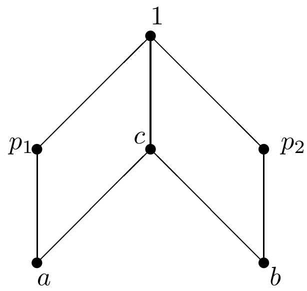

Fig. 7

The complement of c in M_1, is p_1. The complement of c in M_2, is p_2. Clearly, $[c) = \{c, 1\}$, $(c] = \{a, b, c\}$. The direct product $[c) \times (c]$ is depicted in Fig. 6. Obviously, the decomposition mapping $\varphi_c : S \mapsto [c) \times (c]$ is an isomorphism.

Remark 5 If the element c has not a complement in any M_γ, then the mapping φ_c need not be surjective (see Example 2).

Definition 6 Let $(L; \vee, 0, 1)$ be a lattice with the greatest element 1 and the least element 0. An element $c^* \in L$ is called a *pseudocomplement* of $c \in L$, if it is the greatest element such that $c \wedge c^* = 0$. An element $c^+ \in L$ will be called a *dual pseudocomplement* of $c \in L$, if it is the least element for which $c \vee c^+ = 1$.

Proposition 4 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Suppose that an element c has a pseudocomplement c^*_γ and a dual pseudocomplement c^+_γ in each maximal sublattice M_γ. Then the homomorphic image $\varphi_c(S)$ is a subdirect product of $[c), (c]$.*

Proof By Proposition 1, φ_c is a homomorphism of $\mathcal{S}$ into $[c) \times (c]$, thus $\varphi_c(S)$ is a subnearlattice of the nearlattice $[c) \times (c]$. We need only to prove that φ_c is surjective in the both components. Let $\langle x, y\rangle \in [c) \times (c]$, i.e. $y \leq c \leq x$. By the assumption, there exist $c^*_\gamma, c^+_\gamma \in M_\gamma$.

Put $z_1 = (y \vee c^+_\gamma) \wedge_\gamma x$. Then

$$\varphi_c(z_1) = \langle z_1 \vee c, z_1 \wedge_\gamma c\rangle = \langle((y \vee c^+_\gamma) \wedge_\gamma x) \vee c, ((y \vee c^+_\gamma) \wedge_\gamma x) \wedge_\gamma c\rangle$$
$$= \langle(y \vee c^+_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (c^+_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle = \langle x, y \vee (c^+_\gamma \wedge_\gamma c)\rangle,$$

thus $\varphi_c(z_1)$ is surjective in the first component.

Consider $z_2 = (y \vee c^*_\gamma) \wedge_\gamma x$. Then

$$\varphi_c(z_2) = \langle z_2 \vee c, z_2 \wedge_\gamma c\rangle = \langle((y \vee c^*_\gamma) \wedge_\gamma x) \vee c, ((y \vee c^*_\gamma) \wedge_\gamma x) \wedge_\gamma c\rangle$$
$$= \langle(y \vee c^*_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (c^*_\gamma \wedge_\gamma c)) \wedge_\gamma x\rangle = \langle(c \vee c^*_\gamma) \wedge_\gamma x, y\rangle,$$

i.e. $\varphi_c(z_2)$ is surjective in the second component. □

On the other hand, we are able to get a surjective mapping of $S \times S$ onto $[c) \times (c]$ for a nested nearlattice S and its suitable element c which need not be a homomorphism.

Proposition 5 *Let $\mathcal{S} = (S; \vee)$ be a nested nearlattice and c its suitable element. Suppose that an element c has a pseudocomplement c^*_γ and a dual pseudocomplement c^+_γ in each maximal sublattice M_γ. Denote by ψ_c a mapping from $S \times S$ into $[c) \times (c]$, defined by*

$$\psi_c(z_1, z_2) = \langle z_1 \vee c, z_2 \wedge_\gamma c \rangle,$$

where $\gamma \in \Gamma$, such that $z_2 \in M_\gamma$. Then ψ_c is a surjective mapping of $S \times S$ onto $[c) \times (c]$.

Proof Let $\langle x, y \rangle \in [c) \times (c]$, then $y \leq c \leq x$. Hence there exists $\gamma \in \Gamma$ such that $[y) \subseteq M_\gamma$.

Take $z_1 = (y \vee c^+_\gamma) \wedge_\gamma x$, $z_2 = (y \vee c^*_\gamma) \wedge_\gamma x$. Then

$$\begin{aligned} \psi_c(z_1, z_2) &= \langle z_1 \vee c, z_2 \wedge_\gamma c \rangle \\ &= \langle ((y \vee c^+_\gamma) \wedge_\gamma x) \vee c, ((y \vee c^*_\gamma) \wedge_\gamma x) \wedge_\gamma c \rangle \\ &= \langle (y \vee c^+_\gamma \vee c) \wedge_\gamma (x \vee c), ((y \wedge_\gamma c) \vee (c^*_\gamma \wedge_\gamma c)) \wedge_\gamma x \rangle \\ &= \langle x \vee c, y \wedge_\gamma c \rangle = \langle x, y \rangle, \end{aligned}$$

thus ψ_c is a surjective mapping of $S \times S$ onto $[c) \times (c]$. □

We finish with a note concerning lattices.

Remark 6 Let $\mathcal{L} = (L; \vee, \wedge)$ be a bounded lattice and suppose that an element $c \in \mathcal{L}$ has a pseudocomplement c^* and a dual pseudocomplement c^+. Let the elements c^+ and c^* are distributive and dually distributive. Introduce a mapping:

$$\psi_{c^+,c^*} : L \mapsto [c^+) \times (c^*], \psi_{c^+,c^*}(z) = \langle z \vee c^+, z \wedge c^* \rangle.$$

Since the decomposition mappings φ^*_c and φ^+_c are homomorphisms by Proposition 1, also ψ_{c^+,c^*} is a homomorphism.

Further, analogously as in the Proposition 4 and the Proposition 5, it is easy to show that the mapping φ_{c^+} is surjective in the first component, the mapping φ_{c^*} is surjective in the second component and the mapping ψ_{c^+,c^*} is a surjective homomorphism of the lattice $\mathcal{L}$ onto $[c^+) \times (c^*]$.

Example 4 Let $\mathcal{L}$ be the eight element lattice depicted on the left hand side in Fig. 8.

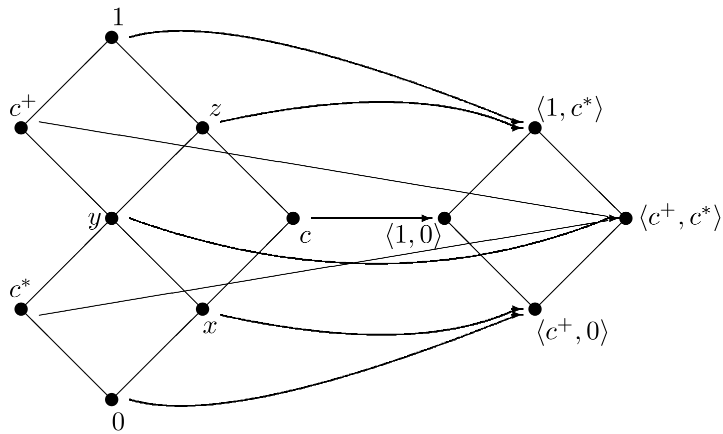

Fig. 8

$\mathcal{L}$ is obviously distributive. Clearly $[c^+) = \{c^+, 1\}$ and $(c^*] = \{0, c^*\}$ (see the lattice $(c^+] \times (c^*]$ on the right hand side of Fig. 8). The mapping ψ_{c^+,c^*} is a surjective homomorphism of the lattice $\mathcal{L}$ onto $[c^+) \times (c^*]$, given by

$$\psi_{c^+,c^*}(1) = \psi_{c^+,c^*}(z) = \langle 1, c^* \rangle,$$

$$\psi_{c^+,c^*}(c^+) = \psi_{c^+,c^*}(y) = \psi_{c^+,c^*}(c^*) = \langle c^+, c^* \rangle,$$

$$\psi_{c^+,c^*}(c) = \langle 1, 0 \rangle,$$

$$\psi_{c^+,c^*}(x) = \psi_{c^+,c^*}(0) = \langle c^+, 0 \rangle.$$

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