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HOMOGENIZATION OF THE MAXWELL EQUATIONS: CASE II. NONLINEAR CONDUCTIVITY

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Abstract. The Maxwell equations with uniformly monotone nonlinear electric conductivity in a heterogeneous medium, which may be non-periodic, are homogenized by two-scale convergence. We introduce a new set of function spaces appropriate for the nonlinear Maxwell system. New compactness results, of two-scale type, are proved for these function spaces. We prove existence of a unique solution for the heterogeneous system as well as for the homogenized system. We also prove that the solutions of the heterogeneous system converge weakly to the solution of the homogenized system. Furthermore, we prove corrector results, important for numerical implementations.

Keywords: nonlinear PDEs, Maxwell's equations, nonlinear conductivity, homogenization, existence of solution, unique solution, two-scale convergence, corrector results, heterogeneous materials, compactness result, non-periodic medium

MSC 2000: 35B27, 35Q60, 74Q10, 74Q15, 78A25

1. Introduction

In this paper we study the asymptotic behavior of sequences of solutions to the Maxwell equations in the case of nonlinear electric conductivity. The material we have in mind can for example be a ceramic varistor, which is used as a device to protect electrical equipment against surges in power lines. The ceramic varistors consist mainly of sintered ZnO grains whose interior is a good (high) linear electrical conductor. The nonlinear conductivity in the material is caused by the grain boundaries which act as insulators for weak electric fields. Eventually, when the electric field becomes stronger than some threshold the conductivity will suddenly increase very rapidly by several orders of magnitude. For further properties of ceramic metal oxide varistors see [4], [11], [15], [16] and the references given there.

Here we will consider a general nonlinear conducting heterogeneous material, not necessarily a ceramic varistor, contained in the domain Ω in which the electromagnetic field is governed by the Maxwell equations

$$
\begin{aligned}
\partial_t D^\varepsilon(x,t) + J^\varepsilon(x,t) &= \operatorname{rot} H^\varepsilon(x,t) + F^\varepsilon(x,t), \\
\partial_t B^\varepsilon(x,t) &= -\operatorname{rot} E^\varepsilon(x,t), \\
\operatorname{div} B^\varepsilon(x,t) &= 0, \\
\operatorname{div} D^\varepsilon(x,t) &= \varrho^\varepsilon(x,t),
\end{aligned} \tag{1.1}
$$

with boundary condition

$$
n \wedge E^\varepsilon(x,t) = 0 \ \text{ on } \ \partial\Omega \times I, \tag{1.2}
$$

initial conditions

$$
E^\varepsilon(x,0) = E_0^\varepsilon(x), \quad H^\varepsilon(x,0) = H_0^\varepsilon(x),
$$

and constitutive relations

$$
B_i^\varepsilon(x,t) = \mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big) H_j^\varepsilon(x,t), \tag{1.3}
$$

$$
J_i^\varepsilon(x,t) = \sigma_i\Big(x, \frac{x}{\varepsilon}, E^\varepsilon\Big), \tag{1.4}
$$

$$
D_i^\varepsilon(x,t) = \eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big) E_j^\varepsilon(x,t). \tag{1.5}
$$

The unknown quantities are the electric and magnetic fields, E^ε and H^ε, respectively. The electric and magnetic fluxes, denoted by D^ε and B^ε, and the current density, J^ε, are given as mappings of the electric and magnetic fields in the constitutive relations (1.3)–(1.5) in which the magnetic permeability, dielectric permittivity and the nonlinear electric conductivity are denoted by μ, η and σ, respectively. In a heterogeneous material they are nonconstant functions of the spatial variable. In this paper the heterogeneous material is modelled by letting the constitutive mappings depend continuously on the global variable, x, and be periodic in the second local variable, y. It is clearly seen that by scaling $y = x/\varepsilon$ the oscillations will become very rapid when ε, a positive constant, tends to zero, i.e. the parameter ε is a measure of the fine scale variation of the material properties in Ω. Note that (1.4) is the only nonlinear relation where σ is assumed to satisfy certain conditions specified in Section 3. The source term F^ε is a current density source and the charge density, ϱ^ε, is defined by the last equation in (1.1). The boundary condition (1.2) corresponds to the case when the material is in contact with an infinitely good conductor which is an approximation of the contact with a metallic conductor. The system is solved for a finite time interval $(0,T)$ but the fine scale in (1.1)–(1.5) makes it impossible to solve the problem as is, using some standard numerical algorithm. However, it is possible to take care of the fine scales by homogenizing the equations.

Homogenization is a multiscale method in which one studies the convergence of solutions of heterogeneous problems when the fine scale becomes smaller and smaller, i.e. $\varepsilon \to 0$. The limit of these solutions solves the homogenized problem, a system of PDEs with constant coefficients which corresponds to a material with homogeneous material properties.

To the knowledge of the author, the Maxwell equations with nonlinear constitutive relations have never been homogenized before. Existence and uniqueness results for Maxwell's equations with monotone nonlinear conductivity also seem to be unproved until now. Homogenization results for linear Maxwell's equations can be found in [2], [3], [10], [12], [14] and [20]. In [17] the two-scale convergence method was used which turned out to give simple and straightforward proofs in the homogenization procedure.

The nonlinear constitutive relation in (1.4) give difficulties in identifying the corresponding limit of the current density in the homogenization process (and in the limit of the Galerkin approximations in the existence proof). The homogenized current density is identified by the use of perturbed test functions (e.g. see [1]). To identify the limit of the Galerkin approximations we use the weakly sequential lower semicontinuity of the $L^p(\Omega)^3$-norm and the fact that the nonlinear conductivity mapping is uniformly monotone and continuous. The nonlinearity also requires the introduction of a new set of function spaces for the Maxwell system, compared with the linear case.

The paper is organized in the following way: In Section 2 we give some basic definitions. In Section 3 the main homogenization and corrector results as well as the existence and uniqueness results are presented. In Section 4 we analyze the spaces $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$. In Section 5 we give some necessary compactness results of two-scale convergence type and in Section 6 we prove the existence and uniqueness results stated in Section 3. We also prove some important a priori estimates. Finally, in Section 7 we prove the main homogenization and corrector results stated in Section 3.

2. Preliminaries

In this text we use the Einstein tensor summation convention. Some standard operator symbols will also be used when it simplifies the notation. By C we denote any fixed constant which may take different values in any place it appears in an equation or inequality. Definitions of function spaces used in this paper are found in [17] and [19]. We define the Y-cell as the open unit cube $Y =]0,1[^3$ in $\mathbb{R}^3$. We say that a function $u\colon \mathbb{R}^3 \to \mathbb{R}$ is Y-*periodic* if $u(x+e_i) = u(x)$ for every $x \in \mathbb{R}^3$ and for every $i \in \{1,2,3\}$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$.

Throughout the paper the open interval $(0,T)$ will be denoted by I and by Ω we denote a bounded simply connected domain in $\mathbb{R}^3$. The gradient, divergence and curl operators in $\mathbb{R}^3$ are defined as $Du = (\partial_{x_i} u)_i = \operatorname{grad} u$, $\operatorname{div} u = \partial_{x_i} u_i$ and $\operatorname{rot} u = \operatorname{curl} u = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)$, respectively. The vector product between $n \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$ is denoted by $n \wedge u$.

For any real number $1 < p < \infty$, we define $p' = p/(p-1)$. Further, we define the Banach space $\mathcal{L}^{p,q}(\Omega) = L^p(\Omega)^3 \times L^q(\Omega)^3$, $1 < p, q < \infty$. The duality pairing between $\mathcal{L}^{p,q}(\Omega)$ and $\mathcal{L}^{p',q'}(\Omega)$ is defined by $\langle \Psi, \Psi_* \rangle = \langle \Psi, \Psi_* \rangle_{\mathcal{L}^{p,q}(\Omega)} = (u, u_*) + (w, w_*)$, where $\Psi = \{u, w\} \in \mathcal{L}^{p,q}(\Omega)$, $\Psi_* = \{u_*, w_*\} \in \mathcal{L}^{p',q'}(\Omega)$ and $(\cdot,\cdot)$ is the usual duality pairing between $L^p(\Omega)$ and $L^{p'}(\Omega)$.

The operator $\mathcal{A}$ is defined by $\mathcal{A}\Psi = \{-\operatorname{rot} w, \operatorname{rot} u\}$ on the domain

$$V = \{\Psi = \{u, w\} \in \mathcal{L}^{p,2}(\Omega)\colon \operatorname{rot} u \in L^2(\Omega)^3,\ \operatorname{rot} w \in L^{p'}(\Omega)^3,\ n \wedge u|_{\partial\Omega} = 0\},$$

$2 \leqslant p < \infty$, which is a Banach space.

Consider the evolution triple $V \subseteq \mathcal{L}^{2,2}(\Omega) \subseteq V'$ with continuous and dense embeddings (by Proposition 6.2), where V' is the dual space of V. We define

$$\mathcal{L}^{p,2}(I; \mathcal{L}^{p,2}(\Omega)) = L^p(I; L^p(\Omega)^3) \times L^2(I; L^2(\Omega)^3)$$

and let

$$X = \mathcal{L}^{p,2}(I; V) = L^p(I; \mathcal{L}^{p,2}_{\operatorname{rot}_0}(\Omega)) \times L^2(I; \mathcal{L}^{2,p'}_{\operatorname{rot}}(\Omega)),$$

where $\mathcal{L}^{p,2}_{\operatorname{rot}_0}(\Omega)$ and $\mathcal{L}^{2,p'}_{\operatorname{rot}}(\Omega)$ are defined in Section 4, with the dual $X' = \mathcal{L}^{p',2}(I; V')$. The space X is separable and reflexive, since it is a closed linear subspace of the separable and reflexive space $\mathcal{L}^{p,2}(I; \mathcal{L}^{p,2}(\Omega))$. We also introduce the space $W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$ which is the set of all functions u in X such that $\partial_t u$ belongs to X'.

R e m a r k 2.1. In this nonlinear case we need different function spaces for the different quantities in the Maxwell equations, which we did not need in the linear case in [17]. This is clearly seen in the a priori estimates in Section 6 and in the following observation. If $\{E^\varepsilon, H^\varepsilon\}$ is a solution to (1.1)–(1.5) then $\{E^\varepsilon(\cdot,t), H^\varepsilon(\cdot,t)\} \in V$, i.e. $E^\varepsilon(\cdot,t) \in L^p(\Omega)^3$, $H^\varepsilon(\cdot,t), \operatorname{rot} E^\varepsilon(\cdot,t) \in L^2(\Omega)^3$ and $\operatorname{rot} H^\varepsilon(\cdot,t) \in L^{p'}(\Omega)^3$.

In 1989 Nguetseng [13] presented a new concept for homogenizing scales of partial differential equations (PDEs), the so called *two-scale convergence method*. The two-scale convergence was generalized to the $L^p(\Omega)$-case by Holmbom in [9], where it also was generalized to nonperiodic cases.

Definition 2.2. A sequence $\{u^\varepsilon\}$ in $L^p(\Omega)$, $p \in]1,\infty]$, is said to two-scale converge to $u_0 \in L^p(\Omega \times Y)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x) a\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) a(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{2.1}$$

for every $a \in D(\Omega; C^\infty_\sharp(Y))$.

For some properties of the two-scale convergence method we refer to [1], [9], [13] and [17].

3. The main results

In this section we will state the main results concerning homogenization of the Maxwell equations in the case of uniformly monotone conductivity. We also state existence and uniqueness results for the ε-dependent system as well as for the homogenized system.

3.1. Heterogeneous problem.

The heterogeneous problem is the nonlinear Maxwell system (1.1)–(1.5) with oscillating coefficients. The family of solutions $\{E^\varepsilon, H^\varepsilon\}$ belongs to $W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$ and solve (1.1)–(1.5), one solution for each fixed $\varepsilon > 0$. The solutions are to be understood in the weak sense, i.e. almost everywhere in $\Omega \times I$.

In Section 3.3 we give the assumptions for the constitutive relations and the driving source F^ε. The existence of unique solution is given in Section 3.4.

3.2. Homogenized problem.

Even if the system has a unique solution it is not possible to obtain it due to the rapidly oscillating coefficients. However, since the fine scale ε is much smaller than all other scales, the solution can be approximated by the solution of the homogenized Maxwell's equations.

The homogenized solution $\{E, H\} \in W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$ solves

$$\begin{aligned} \partial_t D(x,t) + J(x,t) &= \operatorname{rot} H(x,t) + F(x,t), \\ \partial_t B(x,t) &= -\operatorname{rot} E(x,t), \\ \operatorname{div} B(x,t) &= 0, \\ \operatorname{div} D(x,t) &= \varrho(x,t) \end{aligned} \tag{3.1}$$

almost everywhere in $\Omega \times I$, supplied with boundary and initial conditions

$$n \wedge E(x,t) = 0 \quad \text{a.e. on } \partial\Omega \times I,$$
$$E(x,0) = E_0(x) = \int_Y E_0^0(x,y)\,\mathrm{d}y, \quad H(x,0) = H_0(x) = \int_Y H_0^0(x,y)\,\mathrm{d}y,$$

where $E_0^0(x,y)$ and $H_0^0(x,y)$ are the two-scale limits of the initial values $E_0^\varepsilon(x)$ and $H_0^\varepsilon(x)$, respectively. The homogenized constitutive relations are given by

$$\begin{aligned} B_i(x,t) &= \int_Y \mu_{ij}(x,y)(H_j(x,t) + \partial_{y_j}\Phi(x,y,t))\,\mathrm{d}y, \\ D_i(x,t) &= \int_Y \eta_{ij}(x,y)(E_j(x,t) + \partial_{y_j}\varphi(x,y,t))\,\mathrm{d}y, \\ J(x,t) &= \int_Y \sigma(x,y,E(x,t) + D_y\varphi(x,y,t))\,\mathrm{d}y \end{aligned}$$

where $\varphi(x,\cdot,\cdot) \in W^{1,p}(I; W_\sharp^{1,p}(Y)/\mathbb{R}, L_\sharp^2(Y))$ is the solution of

$$\begin{aligned} \int_Y \big(\eta_{ij}(x,y)\partial_t[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)] \\ + \sigma_i(x,y,E(x,t) + D_y\varphi(x,y,t))\big)\partial_{y_i}v_2(y)\,\mathrm{d}y = 0 \end{aligned} \tag{3.2}$$

almost everywhere in $\Omega \times I$ for every $v_2 \in W_\sharp^{1,p}(Y)/\mathbb{R}$. Furthermore, Φ is the unique solution in $L^2(\Omega \times I; W_\sharp^{1,2}(Y)/\mathbb{R})$ of the local problem

$$\int_Y \mu_{ij}(x,y)[H_j(x,t) + \partial_{y_j}\Phi(x,y,t)]\partial_{y_i}v_2(y)\,\mathrm{d}y = 0 \tag{3.3}$$

almost everywhere in $\Omega \times I$ for all $v_2 \in W_\sharp^{1,2}(Y)/\mathbb{R}$.

Note that equation (3.2) is a local conservation law of charges and that equation (3.3) is a local divergence free condition for the magnetic flux, defined on the Y-cell which contain the fine scale information. In the linear case in [17] the local equations could be decoupled from the macroscopic ones. This is not possible in the present case due to the nonlinear conductivity and the explicit dependence of μ, η and σ on the global variable x, i.e. the geometry in the Y-cell depend on where in Ω we solve (3.2) and (3.3). This global dependence allows us to model materials which are nonperiodic. A more precise definition of this is given in the next section.

3.3. Assumptions.

The current density source, F^ε, is assumed to be bounded in $L^{p'}(\Omega \times I)^3$ and converge strongly to $F \in L^{p'}(\Omega \times I)^3$ for $2 \leqslant p < \infty$. Further, we assume that $\partial_t F^\varepsilon$ is bounded in $L^\infty(I; L^{p'}(\Omega)^3)$, $\partial_t^2 F^\varepsilon$ is bounded in $L^{p'}(\Omega \times I)^3$ and that $\operatorname{div} F^\varepsilon$ is bounded in $L^{p'}(\Omega \times I)$. The initial values E_0^ε and H_0^ε are assumed to be admissible test functions and to two-scale converge to $E_0^0(x,y)$ and $H_0^0(x,y)$, respectively. The permeability and permittivity μ and η are bounded, symmetric, Y-periodic and coercive tensors, i.e. $|\mu_{ij}\xi_j| \leqslant c_1|\xi|$, $\mu_{ij} = \mu_{ji}$ and $\mu_{ij}\xi_j\xi_i \geqslant c_2|\xi|^2$ for all vectors ξ. Furthermore, we assume that $\mu_{ij}, \eta_{ij} \in C_0(\Omega; L_\sharp^\infty(Y))$. The assumptions for μ and

η are almost the same as in the linear case in [17] except for the fact that they may depend on the global variable x, which makes it possible to model nonperiodic material.

The conductivity σ belongs to the class $S_{\sharp,Y}$ defined by

Definition 3.1. Given $2 \leqslant p < \infty$ and two positive real constants c_1 and c_2 we define the class $S_{\sharp,Y} = S_{\sharp,Y}(c_1, c_2)$ of maps consisting of all

$$\sigma\colon\ \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3,$$

such that

(i) $\sigma(x,\cdot,\xi)$ is Y-periodic and Lebesgue measurable for every $x, \xi \in \mathbb{R}^3$;
(ii) $\sigma(\cdot, y, \xi)$ is continuous for almost every y and every ξ in $\mathbb{R}^3$;
(iii) $\sigma(x, y, \cdot)$ is continuous for almost every x in $\mathbb{R}^3$ and y in Y;
(iv) $|\sigma(x, y, 0)| = 0$ a.e. in $\mathbb{R}^3 \times \mathbb{R}^3$;
(v) $|\sigma(x, y, \xi)| \leqslant c_1(1 + |\xi|^{p-1})$, a.e. in $\mathbb{R}^3 \times \mathbb{R}^3$ for any $\xi \in \mathbb{R}^3$;
(vi) $(\sigma(x,y,\xi_1) - \sigma(x,y,\xi_2), \xi_1 - \xi_2) \geqslant c_2|\xi_1 - \xi_2|^p$, a.e. in $\mathbb{R}^3 \times \mathbb{R}^3$ for all $\xi_1, \xi_2 \in \mathbb{R}^3$;
(vii) $(\mathrm{D}_\xi \sigma(\mathrm{x}, \mathrm{y}, \xi_1)\xi_2, \xi_2) \geqslant 0$, a.e. in $\mathbb{R}^3 \times \mathbb{R}^3$ for all $\xi_1, \xi_2 \in \mathbb{R}^3$.

Remark 3.2. Assumptions (i)–(iii) are the well-known Carathéodory conditions. The continuity assumptions on σ, (ii) and (iii), can be weakened. It is enough if $\sigma(x, y, \xi(x, y))$ is an admissible test function for smooth ξ.

3.4. Theorem—existence of unique solution.

Theorem 3.3. *The system* (1.1)–(1.5) *has a unique weak solution* $\{E^\varepsilon, H^\varepsilon\} \in W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$ *for every fixed* $\varepsilon > 0$.

3.5. Theorem—convergence.

Our main theorem for the homogenized Maxwell system reads:

Theorem 3.4. *Any sequence* $\{E^\varepsilon, H^\varepsilon\}$ *of solutions to* (1.1)–(1.5) *converges weakly in* $W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$ *to* $\{E, H\} \in W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$ *which satisfies* (3.1)–(3.3).

3.6. Theorem—correctors.

The weak convergence in Theorem 3.4 can be improved by the following corrector result, which is important for numerical implementations.

Theorem 3.5. *Let the sequence* $\{E^\varepsilon, H^\varepsilon\}$ *of unique solutions to* (1.1)–(1.5) *two-scale converge to* $\{E_j(x,t) + \partial_{y_j}\varphi(x,y,t), H_j(x,t) + \partial_{y_j}\Phi(x,y,t)\}$ *which satisfies* (3.1)–(3.3). *Assume that* $E^\varepsilon(x,0) = E_0^\varepsilon(x)$ *and* $H^\varepsilon(x,0) = H_0^\varepsilon(x)$ *are admissible test functions and two-scale converge to* $E_j(x,0) + \partial_{y_j}\varphi(x,y,0) \in L^p(\Omega; L^p_\sharp(Y))$ *and* $H_j(x,0) + \partial_{y_j}\Phi(x,y,0) \in L^2(\Omega; L^2_\sharp(Y))$*, respectively.*

(a) *If* $E_j(x,t) + \partial_{y_j}\varphi(x,y,t)$ *and* $\partial_{y_j}\Phi(x,y,t)$ *are admissible test functions, then*

(i)

$$\lim_{\varepsilon\to 0}\Big\| E_j^\varepsilon(x,t) - E_j(x,t) - \partial_{y_j}\varphi\Big(x, \frac{x}{\varepsilon}, t\Big)\Big\|_{L^p(\Omega\times I)} = 0$$

and

(ii)

$$\lim_{\varepsilon\to 0}\Big\| H_j^\varepsilon(x,t) - H_j(x,t) - \partial_{y_j}\Phi\Big(x, \frac{x}{\varepsilon}, t\Big)\Big\|_{L^2(\Omega)} = 0$$

for all $t \in I$.

(b) *Let* $\partial_{y_j}\varphi^\delta, \partial_{y_j}\Phi^\delta \in D(\Omega \times I; C^\infty_\sharp(Y))$*,* $E^\delta, H^\delta \in D(\Omega\times I)^3$ *be mollifications of* $\partial_{y_j}\varphi$*,* $\partial_{y_j}\Phi$*,* E *and* H*, respectively.*

Then

(iii)

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\Big\| E_j^\varepsilon(x,t) - E_j^\delta(x,t) - \partial_{y_j}\varphi^\delta\Big(x, \frac{x}{\varepsilon}, t\Big)\Big\|_{L^p(\Omega\times I)} = 0$$

and

(iv)

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\Big\| H_j^\varepsilon(x,t) - H_j^\delta(x,t) - \partial_{y_j}\Phi^\delta\Big(x, \frac{x}{\varepsilon}, t\Big)\Big\|_{L^2(\Omega)} = 0$$

for all $t \in I$.

Theorems 3.4 and 3.5 are proved in Section 7.

3.7. Theorem—existence of unique solution to the homogenized system.

Theorem 3.6. *The homogenized Maxwell equations* (3.1)–(3.3) *have a unique weak solution* $\{E, H\} \in W^1_{p,2}(I; V, \mathcal{L}^{2,2}(\Omega))$.

Theorems 3.3 and 3.6 are proved in Section 6.

4. On the spaces $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$

In this section we define some Banach spaces which are natural generalizations of the Hilbert spaces used in [6] and [17] for the linear Maxwell system.

Definition 4.1. Let Ω be an open and bounded domain in $\mathbb{R}^3$, $1 < p, q < \infty$. We define Banach spaces

$$\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega) = \{u \in L^p(\Omega)^3\colon\ \mathrm{rot}\, u \in L^q(\Omega)^3\},$$
$$\mathcal{L}^{q}_{\mathrm{div}}(\Omega) = \{u \in L^q(\Omega)^3\colon\ \mathrm{div}\, u \in L^q(\Omega)\},$$

with norms

$$\|u\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} = \|u\|_{L^p(\Omega)^3} + \|\mathrm{rot}\, u\|_{L^q(\Omega)^3},$$
$$\|u\|_{\mathcal{L}^{q}_{\mathrm{div}}(\Omega)} = \|u\|_{L^q(\Omega)^3} + \|\mathrm{div}\, u\|_{L^q(\Omega)}.$$

The remaining part of this section is devoted to proving some important embedding and trace properties for the spaces $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$ which will be needed in proof of Theorem 3.4.

We begin with the definition of the standard mollifier (cf. [7]).

Definition 4.2. Let the function $\varphi\colon\ \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\varphi(x) := \begin{cases} c\exp\Big(\dfrac{1}{|x|^2-1}\Big), & \text{if } |x| < 1,\\ 0, & \text{if } |x| \geqslant 1, \end{cases}$$

where the constant c is chosen such that

$$\int_{R^N} \varphi(x)\,\mathrm{d}x = 1.$$

The standard mollifier is defined by

$$\varphi_\delta(x) := \frac{1}{\delta^3}\varphi\Big(\frac{x}{\delta}\Big), \quad \delta > 0,\ \ x \in \mathbb{R}^3.$$

In the following lemmas we assume that Ω has a bounded Lipschitz boundary $\partial\Omega$.

Lemma 4.3. *Let $\Omega_\delta := \{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) > \delta\}$, φ_δ be the standard mollifier (as in Definition 4.2) and $u^\delta(x) := \int_\Omega \varphi_\delta(x-z)u(z)\,\mathrm{d}z$, (i.e. $u^\delta = \varphi_\delta * u$), $x \in \Omega_\delta$.*

(i) *If $u \in \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ or $u \in \mathcal{L}^{q}_{\mathrm{div}}(\Omega)$, then $u^\delta \in C^\infty(\Omega_\delta)^3$ for each $\delta > 0$.*

(ii) *If* $u \in \mathcal{L}^{p,q}_{\mathrm{rot,loc}}(\Omega)$, *then*

$$\operatorname{rot} u^\delta = \varphi_\delta * \operatorname{rot} u \quad \textit{on } \Omega_\delta,$$
$$u^\delta \to u \quad \textit{in } \mathcal{L}^{p,q}_{\mathrm{rot,loc}}(\Omega).$$

(iii) *If* $u \in \mathcal{L}^{q}_{\mathrm{div,loc}}(\Omega)$, *then*

$$\operatorname{div} u^\delta = \varphi_\delta * \operatorname{div} u \quad \textit{on } \Omega_\delta,$$
$$u^\delta \to u \quad \textit{in } \mathcal{L}^{q}_{\mathrm{div,loc}}(\Omega).$$

Proof. (i) The proof follows from the facts that $\mathcal{L}^{p,q}_{\mathrm{rot,loc}}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)^3$ and that $f^\delta \in C^\infty(\Omega_\delta)$ if $f \in L^1_{\mathrm{loc}}(\Omega)$ (see e.g. [7]).

(ii) We first observe that if $u \in L^p_{\mathrm{loc}}(\Omega)^3$, then

$$u^\delta \to u \quad \text{in } L^p_{\mathrm{loc}}(\Omega)^3$$

(cf. [7]). For $u = (u_1, u_2, u_3) \in \mathcal{L}^{p,q}_{\mathrm{rot},loc}(\Omega)$, we find by integration by parts that

$$\begin{aligned}
\operatorname{rot} u^\delta(x) &= \int_\Omega \begin{pmatrix} \partial_{x_2}\varphi_\delta(x-z)u_3(z) - \partial_{x_3}\varphi_\delta(x-z)u_2(z) \\ \partial_{x_3}\varphi_\delta(x-z)u_1(z) - \partial_{x_1}\varphi_\delta(x-z)u_3(z) \\ \partial_{x_1}\varphi_\delta(x-z)u_2(z) - \partial_{x_2}\varphi_\delta(x-z)u_1(z) \end{pmatrix} \mathrm{d}z \\
&= -\int_\Omega \begin{pmatrix} \partial_{z_2}\varphi_\delta(x-z)u_3(z) - \partial_{z_3}\varphi_\delta(x-z)u_2(z) \\ \partial_{z_3}\varphi_\delta(x-z)u_1(z) - \partial_{z_1}\varphi_\delta(x-z)u_3(z) \\ \partial_{z_1}\varphi_\delta(x-z)u_2(z) - \partial_{z_2}\varphi_\delta(x-z)u_1(z) \end{pmatrix} \mathrm{d}z \\
&= \int_\Omega \varphi_\delta(x-z) \begin{pmatrix} \partial_{z_2}u_3(z) - \partial_{z_3}u_2(z) \\ \partial_{z_3}u_1(z) - \partial_{z_1}u_3(z) \\ \partial_{z_1}u_2(z) - \partial_{z_2}u_1(z) \end{pmatrix} \mathrm{d}z \\
&= \int_\Omega \varphi_\delta(x-z) \operatorname{rot} u(z)\, \mathrm{d}z
\end{aligned}$$

for $x \in \Omega_\delta$. The result now follows from the fact that $\operatorname{rot} u \in L^q_{\mathrm{loc}}(\Omega)^3 \subset L^1_{\mathrm{loc}}(\Omega)^3$.

Assertion (iii) can be proved by similar arguments as in the proof of (ii) and by the Green formula

$$(\operatorname{div} u, \Phi) + (u, \operatorname{grad} \Phi) = \int_{\partial\Omega} (n \cdot u)\varphi \, \mathrm{d}x,$$

and is therefore omitted. □

In the next lemma we prove that we may approximate functions in $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$ by smooth functions in $C^{\infty}(\Omega)^3$.

Lemma 4.4.
(i) *Let $u \in \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$. Then there exists a sequence $\{u_k\} \in C^{\infty}(\Omega)^3$ such that*

$$u_k \to u \ \textit{ in } \ \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega).$$

(ii) *Let $u \in \mathcal{L}^{q}_{\mathrm{div}}(\Omega)$. Then there exists a sequence $\{u_k\} \in C^{\infty}(\Omega)^3$ such that*

$$u_k \to u \ \textit{ in } \ \mathcal{L}^{q}_{\mathrm{div}}(\Omega).$$

Proof. (i) Define

$$\begin{aligned} \Omega_k &:= \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > 1/k\}, \quad && k = 1, 2, \ldots, \\ U_k &:= \Omega_{k+1} - \overline{\Omega}_k, && k = 1, 2, \ldots \end{aligned}$$

and choose $U_0 \subset\subset \Omega$ such that $\Omega = \bigcup\limits_{i=0}^{\infty} U_k$. Let $\{\gamma_k\}_{k=0}^{\infty}$ be a partition of unity, i.e., $\gamma_k \in C_0^1(U_k)$, $0 \leqslant \gamma_k \leqslant 1$ for every $k \in \{0, 1, 2, \ldots\}$ and $\sum\limits_{k=0}^{\infty} \gamma_k = 1$ everywhere on Ω. For $u \in \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ we clearly have $u\gamma_k \in \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ with compact support in U_k. By using Lemma 4.3 we conclude that, for each $k = 0, 1, 2 \ldots$, there exists $\delta_k > 0$ such that

$$\begin{cases} \operatorname{supp}(\varphi_{\delta_k} * (u\gamma_k)) \subset U_k \\ \|u\gamma_k - \varphi_{\delta_k} * (u\gamma_k)\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} < \frac{\delta}{2^{k+1}}. \end{cases}$$

Put $u^{\delta} := \sum\limits_{k=0}^{\infty} \varphi_{\delta_k} * (u\gamma_k)$, and note that $u^{\delta} \in C^{\infty}(\Omega)$. This follows from the fact that for each $x \in \Omega$ there are neighbourhoods such that there is only a finite number of nonzero terms in the sum. From the definitions of γ_k it follows that $u = \sum\limits_{k=0}^{\infty} (u\gamma_k)$. Therefore

$$\|u - u^{\delta}\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} \leqslant \sum_{k=0}^{\infty} \|(u\gamma_k) - \varphi_{\delta_k} * (u\gamma_k)\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} < \delta \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \delta,$$

and the statement follows. The statement in (ii) can be proved by adopting similar arguments and using the corresponding result in Lemma 4.3. The proof is complete. □

A direct consequence of Lemma 4.4 is the following result.

Lema 4.5. *The space $C^1(\overline{\Omega})^3$ is dense in $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$.*

For the trace results we need the dual spaces $W^{\frac{1}{r},r'}(\partial\Omega)$ and $W^{-\frac{1}{r},r}(\partial\Omega)$, $r = \min\{p,q\}$, $1 < p,q < \infty$.

Lemma 4.6. *Let n be the outer normal to $\partial\Omega$. The linear mappings*
(i) $T_0\colon C^1(\overline{\Omega})^3 \to C^1(\partial\Omega)^3$, $u \mapsto n \wedge u|_{\partial\Omega} = n \wedge u$,
(ii) $T_1\colon C^1(\overline{\Omega})^3 \to C^1(\partial\Omega)$, $u \mapsto n \cdot u|_{\partial\Omega} = n \cdot u$,
can be extended by continuity to continuous linear mappings
(i) $T_0'\colon \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega) \to W^{-\frac{1}{r},r}(\partial\Omega)^3$,
(ii) $T_1'\colon \mathcal{L}^{q}_{\mathrm{div}}(\Omega) \to W^{-\frac{1}{q},q}(\partial\Omega)$.

Proof. The proof is based on the usual estimate of the norm of traces by the norm in the corresponding function spaces, using Green's theorem, and is therefore omitted (see [18], or [6] for the L^2-case). □

Lemma 4.7. *The linear mapping*

$$T\colon\ C^1(\overline{\Omega})^3 \to C^1(\partial\Omega)^3, \quad u \mapsto u|_{\partial\Omega},$$

can be extended by continuity to the continuous linear mapping

$$T'\colon\ \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega) \to W^{\frac{1}{s'},s}(\partial\Omega)^3,$$

where $s = \max\{p,q\}$.

Proof. The proof is similar to the previous one and is found in [18]. □

We continue with the following obvious lemma:

Lemma 4.8. *The space $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ is dense in $L^p(\Omega)^3$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$ is dense in $L^q(\Omega)^3$.*

We note that $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$ are closed in their norms and that one consequence of Lemma 4.8 is that $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$ and $\mathcal{L}^{q}_{\mathrm{div}}(\Omega)$ are separable, reflexive Banach spaces. Next, we consider the space

$$\mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega) := \{u \in \mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega) \mid n \wedge u = 0 \text{ on } \partial\Omega\}.$$

Lemma 4.9. *$\mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega)$ is a closed subspace of $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$.*

Proof. Let $u_i \in \mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega)$ be a sequence such that $u_i \to u$ in $L^p(\Omega)^3 \subset L^1(\Omega)^3$ and $\mathrm{rot}\, u_i \to w$ in $L^q(\Omega)^3 \subset L^1(\Omega)^3$. It follows that $u_i \to u$ and $\mathrm{rot}\, u_i \to \mathrm{rot}\, u$ in $D'(\Omega)^3$. This gives $w = \mathrm{rot}\, u$. Furthermore, $0 = n \wedge u_i \to n \wedge u = 0$ in $W^{-\frac{1}{r},r}(\partial\Omega)^3$, $r = \min\{p,q\}$. The lemma is proved. □

Lemma 4.10. *The space*

$$\{v \in C^1(\overline{\Omega})^3 \mid n \wedge v = 0 \ \text{ on } \ \partial\Omega\}$$

is dense in $\mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega)$.

P r o o f. By using the regularization results in Lemma 4.5, we find that $C^\infty(\overline{\Omega})^3$ is dense in $\mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega)$. But the elements in $C^\infty(\overline{\Omega})^3$ do not necessarily satisfy the boundary condition. Let $\{\Omega_k\}$ be an invading sequence in Ω, i.e., $\Omega_k \subset \Omega$ is such that $\operatorname{dist}(\partial\Omega_k, \partial\Omega) < 1/k$, and let $v^k \in D(\Omega)$ be such that $v^k|_{\overline{\Omega}_k} = u^\delta|_{\overline{\Omega}_k}$, where $u^\delta \in C^\infty(\overline{\Omega})^3$ is the regularization of $u \in \mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega)$. Clearly, $v^k \in \{v \in C^1(\overline{\Omega})^3 \mid n \wedge v = 0 \ \text{on} \ \partial\Omega\}$ and we have, for any $\varepsilon > 0$,

$$\|v^k - u\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} \leqslant \|v^k - u^\delta\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} + \|u^\delta - u\|_{\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)} < 2\varepsilon$$

for k large enough and δ sufficiently small. We have proved that $D(\Omega)$ is dense in $\mathcal{L}^{p,q}_{\mathrm{rot}_0}(\Omega)$. The statement now follows from the fact that $D(\Omega) \subset \{v \in C^1(\overline{\Omega})^3 \mid n \wedge v = 0 \ \text{on} \ \partial\Omega\}$. □

5. Compactness results

Up to now, we have defined and characterized some suitable function spaces in order to be able to prove the fundamental compactness results needed in the proof of Theorem 3.4.

The following lemma is a natural generalization of a corresponding result proved in [8] for the $W^{1,2}(I; L^2(\Omega)^3)$-case.

Lemma 5.1. *Let u^ε and $\partial_t u^\varepsilon$ be bounded in $L^p(\Omega \times I)^3$ and $L^q(\Omega \times I)^3$, $1 < p, q < \infty$, respectively. Further, let $u_0(x, y, t)$ be the two-scale limit of u^ε in $L^p(\Omega \times Y \times I)^3$. Then $\partial_t u_0 \in L^q(\Omega \times Y \times I)^3$ is the two-scale limit of $\partial_t u^\varepsilon$.*

P r o o f. By assumption $\|u^\varepsilon\|_{L^p(\Omega\times I)^3} < C$ and $\|\partial_t u^\varepsilon\|_{L^q(\Omega\times I)^3} < C$ for all $\varepsilon > 0$. It follows that

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \Big(u^\varepsilon(x,t), a\Big(x, \frac{x}{\varepsilon}, t\Big)\Big) \,\mathrm{d}x \,\mathrm{d}t = \int_0^T\!\!\int_\Omega\!\int_Y \big(u_0(x,y,t), a(x,y,t)\big) \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t$$

and

$$\lim_{\varepsilon\to 0} \int_0^T\!\!\int_\Omega \Big(\partial_t u^\varepsilon(x,t), a\Big(x, \frac{x}{\varepsilon}, t\Big)\Big) \,\mathrm{d}x \,\mathrm{d}t = \int_0^T\!\!\int_\Omega\!\int_Y \big(\chi_0(x,y,t), a(x,y,t)\big) \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t$$

for all $a \in D(\Omega \times Y \times I)^3$. By integrating by parts and using the fact that a has compact support we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_0^T\!\!\int_\Omega \Big(\partial_t u^\varepsilon(x,t), a\Big(x,\frac{x}{\varepsilon},t\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t &= -\lim_{\varepsilon\to 0}\int_0^T\!\!\int_\Omega \Big(u^\varepsilon(x,t), \partial_t a\Big(x,\frac{x}{\varepsilon},t\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T\!\!\int_\Omega\int_Y \big(u_0(x,y,t), \partial_t a(x,y,t)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\!\!\int_\Omega\int_Y \big(\partial_t u_0(x,y,t), a(x,y,t)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Therefore $\partial_t u_0 = \chi_0 \in L^q(\Omega \times Y \times I)^3$ and the proof follows. □

Proposition 5.2. *Let $\{u^\varepsilon\}$ be a bounded sequence in $\mathcal{L}^{p,q}_{\mathrm{rot}}(\Omega)$, $1 < p, q < \infty$. Then $\{u^\varepsilon\}$, up to a subsequence, two-scale converges to $u_0(x,y) = u(x) + D_y\varphi(x,y) \in L^p(\Omega; L^p_\sharp(Y)^3)$, where $\varphi \in L^p(\Omega; W^{1,p}_\sharp(Y))$ is a scalar valued function satisfying $\int_Y D_y\varphi(x,y)\,\mathrm{d}y = 0$. Moreover, $\operatorname{rot} u^\varepsilon \rightharpoonup \operatorname{rot} u(x)$ weakly in $L^q(\Omega)^3$.*

P r o o f. The proof follows by using Lemmas 4.6 and 4.7 and the same kind of arguments as in the proof of Proposition 4.3 in [17]. □

R e m a r k 5.3. If $D_y\varphi(x,y) \neq 0$ then, by Proposition 2.6 in [17], the sequence $\{u^\varepsilon\}$ will never contain a strongly convergent subsequence in $L^p(\Omega)^3$. Moreover, it clearly follows that $\varphi(x,\cdot)$ is Y-periodic.

Proposition 5.4. *Let $\{u^\varepsilon\}$ be a bounded sequence in $\mathcal{L}^q_{\mathrm{div}}(\Omega)$. Then, up to a subsequence, u^ε two-scale converges to $u_0(x,y) \in L^q(\Omega; L^q_\sharp(Y))^3$,*

$$\operatorname{div}_y u_0(x,y) = 0,$$

and

$$\operatorname{div} u^\varepsilon \rightharpoonup \operatorname{div} u(x) = \operatorname{div}\int_Y u_0(x,y)\,\mathrm{d}y \ \text{ weakly in } L^q(\Omega).$$

P r o o f. The proof is similar to that of Proposition 4.5 in [17] and is therefore omitted. □

6. A priori estimates and proofs of existence and uniqueness of solutions

The operator $\mathcal{A}$, defined in Section 2, is characterized in the following remark and proposition.

Remark 6.1. Since $\mathcal{L}^{p,2}_{\mathrm{rot}_0}(\Omega)$ is a linear closed subspace of $\mathcal{L}^{p,2}_{\mathrm{rot}}(\Omega)$ it follows that $V = \mathcal{L}^{p,2}_{\mathrm{rot}_0}(\Omega) \times \mathcal{L}^{2,p'}_{\mathrm{rot}}(\Omega)$ is a reflexive and separable Banach space.

Proposition 6.2. *The space V is dense in $\mathcal{L}^{p,2}(\Omega)$ and the graph of the operator $\mathcal{A}$ is closed. Moreover,*

$$\mathcal{A}' = -\mathcal{A}, \quad D(\mathcal{A}') = V,$$

where $\mathcal{A}'$ is the dual operator to $\mathcal{A}$ and $D(\mathcal{A}')$ is the domain of $\mathcal{A}'$.

Proof. The complete proof can be found in [18] and is similar to that of the $L^2(\Omega)$-case in [6]. □

As for the linear case in [17], we have the following obvious consequence of Proposition 6.2 which will be useful in the sequel:

Corollary 6.3. $\langle \mathcal{A}\Psi, \Psi\rangle = 0$ *for all* $\Psi \in V$.

Definition 6.4. Let $\Psi = \{u, w\} \in \mathcal{L}^{p,2}(\Omega)$ and define the operator $\mathcal{M}^{\varepsilon}$: $\mathcal{L}^{p,2}(\Omega) \to \mathcal{L}^{p',2}(\Omega)$ by $\mathcal{M}^{\varepsilon}(\Psi) = \{\sigma^{\varepsilon}(u), 0\}$.

By the assumptions made in Definition 3.1, the restriction $\mathcal{M}^{\varepsilon}_1 = \mathcal{M}^{\varepsilon}|_{L^p(\Omega)^3}$ such that $\mathcal{M}^{\varepsilon}_1(u) = \sigma^{\varepsilon}(u) = \sigma(x, x/\varepsilon, u)$ is bounded and uniformly monotone.

We are now prepared to prove the existence and uniqueness results in Theorem 3.3.

Proof of Theorem 3.3. The proof follows as a consequence of Lemmas 6.5–6.8 below. □

By definition $U^{\varepsilon} = \{E^{\varepsilon}, H^{\varepsilon}\}$ and $\mathcal{N}^{\varepsilon}\partial_t U^{\varepsilon} = \{\eta^{\varepsilon}\partial_t E^{\varepsilon}, \mu^{\varepsilon}\partial_t H^{\varepsilon}\}$, where $\eta^{\varepsilon} = \eta(x, x/\varepsilon)$ and $\mu^{\varepsilon} = \mu(x, x/\varepsilon)$. Summation of the first and second equations in (1.1) yields

$$\mathcal{N}^{\varepsilon}\partial_t U^{\varepsilon} + \mathcal{A}U^{\varepsilon} + \mathcal{M}^{\varepsilon}(U^{\varepsilon}) = G^{\varepsilon}, \tag{6.1}$$

where $G^{\varepsilon} = \{F^{\varepsilon}, 0\}$. By multiplying (6.1) with $v \in V$ and integrating over Ω we obtain a weak formulation of (6.1)

$$\langle \mathcal{N}^{\varepsilon}\partial_t U^{\varepsilon}(t), v\rangle + \langle \mathcal{A}U^{\varepsilon}(t), v\rangle + \langle \mathcal{M}^{\varepsilon}(U^{\varepsilon}(t)), v\rangle = \langle G^{\varepsilon}(t), v\rangle.$$

Let $\{w_1, w_2, \dots w_n\}$ be a base in V and set

$$U_n^\varepsilon(t) := \sum_{k=1}^{n} c_{kn}(t) w_k.$$

The Galerkin equation reads, for almost every $t \in I$,

$$\langle \mathcal{N}^\varepsilon \partial_t U_n^\varepsilon(t), w_j \rangle + \langle \mathcal{A} U_n^\varepsilon(t), w_j \rangle + \langle \mathcal{M}^\varepsilon(U_n^\varepsilon(t)), w_j \rangle = \langle G^\varepsilon(t), w_j \rangle, \tag{6.2}$$

$j = 1, \dots, n$, $U_n^\varepsilon(0) = U_{n0}^\varepsilon \in \mathcal{L}_n$, $U_n^\varepsilon \in \mathcal{L}^{p,2}(I; \mathcal{L}_n)$, $\partial_t U_n^\varepsilon \in \mathcal{L}^{p',2}(I; \mathcal{L}_n)$, $w_j \in \mathcal{L}_n = \operatorname{span}\{w_1, \dots, w_n\}$.

We note that we have continuous and dense embeddings, $\mathcal{L}_n \subseteq V \subseteq \mathcal{L}^{p,2}(\Omega) \subseteq \mathcal{L}^{2,2}(\Omega) \subseteq \mathcal{L}^{p',2}(\Omega) \subseteq V'$ and that (6.2) is an ordinary nonlinear differential equation which has a unique solution, which is proved in Lemma 6.6 below where we use Carathéodory's theorem (cf. [5] and [19]), with some minor changes in the proof.

The following a priori estimates of the Galerkin solutions are needed in the proof:

Lemma 6.5. *There exists a constant $C > 0$ (independent of n) such that every solution of* (6.2) *satisfies*

$$\begin{aligned} \|U_n^\varepsilon\|_{(L^p(I;L^p(\Omega)^3)\cap L^\infty(I;L^2(\Omega)^3))\times L^\infty(I;L^2(\Omega)^3)} &\leqslant C, \\ \|\mathcal{A}U_n^\varepsilon\|_{L^{p'}(I;L^{p'}(\Omega)^3)\times L^\infty(I;L^2(\Omega)^3)} &\leqslant C, \\ \|\mathcal{M}^\varepsilon(U_n^\varepsilon)\|_{L^{p'}(\Omega\times I)^6} &\leqslant C \end{aligned}$$

and

$$\max_{t\in[0,T]} \|U_n^\varepsilon(t)\|_{L^2(\Omega)^6} \leqslant C.$$

Proof. By choosing $w_j = U_n^\varepsilon(t)$ in (6.2) we get

$$\langle \mathcal{N}^\varepsilon \partial_t U_n^\varepsilon(t), U_n^\varepsilon(t) \rangle + \langle \mathcal{A} U_n^\varepsilon(t), U_n^\varepsilon(t) \rangle + \langle \mathcal{M}^\varepsilon(U_n^\varepsilon(t)), U_n^\varepsilon(t) \rangle = \langle G^\varepsilon, U_n^\varepsilon(t) \rangle.$$

Thus, by using Corollary 6.3, the symmetries of η_{ij} and μ_{ij}, the definition of $\mathcal{M}^\varepsilon$, and the property (vi) in the class $S_{\sharp,Y}$, we obtain

$$\frac{1}{2}\partial_t(E_n^\varepsilon(t), D_n^\varepsilon(t)) + \frac{1}{2}\partial_t(H_n^\varepsilon(t), B_n^\varepsilon(t)) + C\|E_n^\varepsilon(t)\|_{L^p(\Omega)^3}^p \leqslant \langle G^\varepsilon, U_n^\varepsilon(t) \rangle.$$

The remaining steps in the proof, which can be found in [18], are similar to the proof of Proposition 5.3 in [17]. □

Lemma 6.6. *The Galerkin problem* (6.2) *has a unique solution* U_n^ε *for every fixed* $n > 0$.

Proof. We note that the a priori estimate

$$\|U_n^\varepsilon\|_{\mathcal{L}_n} \leqslant C \quad \text{on } [0,T]$$

holds for all n and that the mapping

$$(U_n^\varepsilon, t) \mapsto \langle \mathcal{A}U_n^\varepsilon(t) + \mathcal{M}^\varepsilon(U_n^\varepsilon(t)), w_j\rangle$$

satisfies the Carathéodory condition on $\mathcal{L}_n \times [0,T]$, i.e.,
1) $t \mapsto \langle \mathcal{A}U_n^\varepsilon(t) + \mathcal{M}^\varepsilon(U_n^\varepsilon(t)), w_j\rangle$ is measurable on $[0,T]$.
2) $U_n^\varepsilon \mapsto \langle \mathcal{A}U_n^\varepsilon + \mathcal{M}^\varepsilon(U_n^\varepsilon), w_j\rangle$ is continuous on $\mathcal{L}_n$.

Moreover, we note that $\mathcal{N}^\varepsilon$ is linear bounded and positive definite and consequently has an inverse with the same properties. Since $\dim(\mathcal{L}_n) < \infty$ and all norms are equivalent on finite dimensional spaces, we have the estimate

$$\begin{aligned}(6.3)\qquad &|\langle \mathcal{A}U_n^\varepsilon(t) + \mathcal{M}^\varepsilon(U_n^\varepsilon(t)), w_j\rangle| \\ &\leqslant \big(\|\mathcal{A}U_n^\varepsilon(t)\|_{\mathcal{L}^{p',2}(\Omega)} + \|\mathcal{M}^\varepsilon(U_n^\varepsilon(t))\|_{\mathcal{L}^{p',2}(\Omega)}\big)\|w_j\|_{\mathcal{L}^{p,2}(\Omega)} \\ &\leqslant \big(\|U_n^\varepsilon(t)\|_V + C(1 + \|E_n^\varepsilon(t)\|^{p/p'}_{L^p(\Omega)^3})\big)\|w_j\|_{\mathcal{L}^{p,2}(\Omega)}\end{aligned}$$

for every (U_n^ε, t) in $\mathcal{L}_n \times [0,T]$ and $j = 1, \ldots, n$. We note that due to Carathéodory's theorem (cf. [5] or [19]) and the properties of $\mathcal{N}^\varepsilon$ the solution of (6.2) $U_n^\varepsilon\colon\ [0,T] \to \mathcal{L}_n$ is continuous and the derivative $\mathcal{N}^\varepsilon \partial_t U_n^\varepsilon$ exists for almost every $t \in [0,T]$. Hence

$$U_n^\varepsilon \in \mathcal{L}^{p,2}(I;\mathcal{L}_n),$$

i.e., $E_n^\varepsilon \in L^p(I;L^p(\Omega)^3)$ and $H_n^\varepsilon \in L^2(I;L^2(\Omega)^3)$. Moreover, by (6.3) we find that the function $t \mapsto \langle \mathcal{A}U_n^\varepsilon(t) + \mathcal{M}^\varepsilon(U_n^\varepsilon(t)), w_j\rangle$ belongs to $L^{p'}(I)$. Since $|\langle G^\varepsilon(t), w_j\rangle|^{p'} \leqslant \|G^\varepsilon(t)\|^{p'}_{\mathcal{L}^{p',2}(\Omega)}\|w_j\|^{p'}_{\mathcal{L}^{p,2}(\Omega)}$ and $G^\varepsilon \in X'$ we conclude, via the Galerkin equation (6.2) and the properties of $\mathcal{N}^\varepsilon$, that

$$\partial_t U_n^\varepsilon \in \mathcal{L}^{p',2}(I;\mathcal{L}_n).$$

Thus, the Galerkin problem (6.2) has a solution. Uniqueness of solution follows easily from the assumptions made by the arguments as those used in the a priori estimates. The proof is complete. □

Due to the reflexivity of the spaces $\mathcal{L}^{p,2}(I;\mathcal{L}^{p,2}(\Omega))$ and $L^{p'}(\Omega\times I)^6$ and the boundedness of sequences we can, by a diagonalization procedure, extract subsequences $\{U^\varepsilon_{n_k}\}$ (for fixed ε) which converge weakly, i.e.,

$$\begin{aligned} U^\varepsilon_{n_k} &\rightharpoonup U^\varepsilon && \text{weakly in } \mathcal{L}^{p,2}(I;\mathcal{L}^{p,2}(\Omega)),\\ \mathcal{A}U^\varepsilon_{n_k} &\rightharpoonup W^\varepsilon && \text{weakly in } \mathcal{L}^{p',2}(I;\mathcal{L}^{p',2}(\Omega))\\ \mathcal{M}^\varepsilon(U_{n_k})^\varepsilon &\rightharpoonup Z^\varepsilon && \text{weakly in } L^{p'}(\Omega\times I)^6 \end{aligned} \tag{6.4}$$

and

$$\mathcal{N}^\varepsilon U^\varepsilon_{n_k}(T) \rightharpoonup \mathcal{N}^\varepsilon g^\varepsilon \quad \text{weakly in } \mathcal{L}^{2,2}(\Omega)$$

as $n_k \to \infty$. It remains to identify the limits U^ε, W^ε, Z^ε and g^ε.

Lemma 6.7. *The limits in* (6.4) *satisfy the equation*

$$\mathcal{N}^\varepsilon \partial_t U^\varepsilon + \mathcal{M}^\varepsilon(U^\varepsilon) + \mathcal{A}U^\varepsilon = G^\varepsilon \tag{6.5}$$

almost everywhere in $\Omega\times I$. *Here* $U^\varepsilon \in W^1_{p,2}(I;V,\mathcal{L}^{2,2}(\Omega))$, $U^\varepsilon(0) = U^\varepsilon_0 = \{E^\varepsilon_0, H^\varepsilon_0\}$ *and* $U^\varepsilon(T) = g^\varepsilon$.

P r o o f. We note that $\langle \mathcal{A}U^\varepsilon_{n_k}, \varphi\rangle = -\langle U^\varepsilon_{n_k}, \mathcal{A}\varphi\rangle$ for every $\varphi \in D(\Omega)^6$. Using (6.4) we obtain that $\langle W^\varepsilon, \varphi\rangle = -\langle U^\varepsilon, \mathcal{A}\varphi\rangle$ for every $\varphi\in D(\Omega)^6$. Thus, by definition we find that $W^\varepsilon = \mathcal{A}U^\varepsilon$ in the generalized sense. Next, by using the integration by parts formula

$$\begin{aligned} (\mathcal{N}^\varepsilon g^\varepsilon, \psi(T)v) - (\mathcal{N}^\varepsilon U^\varepsilon_0, \psi(0)v) = \int_0^T &\langle G^\varepsilon(t) - Z^\varepsilon(t) - W^\varepsilon(t), \psi(t)v\rangle \\ &+ \langle \psi'(t)v, \mathcal{N}^\varepsilon U^\varepsilon(t)\rangle \,\mathrm{d}t, \end{aligned}$$

which holds for all $\psi \in C^\infty[0,T]$, $v\in V$ (cf. [19]), we get

$$\int_0^T \langle G^\varepsilon(t) - Z^\varepsilon(t) - \mathcal{A}U^\varepsilon(t), v\rangle \psi(t)\,\mathrm{d}t = -\int_0^T (\mathcal{N}^\varepsilon U^\varepsilon(t), v)\psi'(t)\,\mathrm{d}t$$

for all $\psi \in C_0^\infty(I)$, $v\in V$. This proves that $\mathcal{N}^\varepsilon\partial_t U^\varepsilon + Z^\varepsilon + \mathcal{A}U^\varepsilon = G^\varepsilon$ almost everywhere in $\Omega\times I$. Moreover, this implies that $\mathcal{N}^\varepsilon\partial_t U^\varepsilon \in X'$ which together with $U^\varepsilon \in X'$ gives $U^\varepsilon \in W^1_{p,2}(I;V,\mathcal{L}^{2,2}(\Omega))$. By integration by parts we get

$$\begin{aligned} (\mathcal{N}^\varepsilon U^\varepsilon(T), \psi(T)v) - (\mathcal{N}^\varepsilon U^\varepsilon(0), \psi(0)v) = \int_0^T &\langle \mathcal{N}^\varepsilon \partial_t U^\varepsilon(t), \psi(t)v\rangle \\ &+ \langle \psi'(t)v, \mathcal{N}^\varepsilon U^\varepsilon(t)\rangle\,\mathrm{d}t \end{aligned}$$

for all $\psi \in C^\infty[0,T]$, $v \in V$, which gives

$$(\mathcal{N}^\varepsilon U^\varepsilon(T), \psi(T)v) - (\mathcal{N}^\varepsilon U^\varepsilon(0), \psi(0)v) = (\mathcal{N}^\varepsilon g^\varepsilon, \psi(T)v) - (\mathcal{N}^\varepsilon U_0^\varepsilon, \psi(0)v)$$

for all $\psi \in C^\infty[0,T]$, $v \in V$. Now, by choosing $\psi(T) = 1$, $\psi(0) = 0$ and $\psi(T) = 0$, $\psi(0) = 1$, respectively, and using the density of V in $\mathcal{L}^{2,2}(\Omega)$ we find that $U^\varepsilon(T) = g^\varepsilon$ and $U^\varepsilon(0) = U_0^\varepsilon$. Further, integration by parts yields

$$\begin{aligned}\frac{1}{2}\big((\mathcal{N}^\varepsilon U_n^\varepsilon(T), U_n^\varepsilon(T)) &- (\mathcal{N}^\varepsilon U_n^\varepsilon(0), U_n^\varepsilon(0))\big) \\ &= \int_0^T \langle \mathcal{N}^\varepsilon \partial_t U_n^\varepsilon(t), U_n^\varepsilon(t)\rangle \, \mathrm{d}t \\ &= \langle G^\varepsilon - \mathcal{A}U_n^\varepsilon - \mathcal{M}^\varepsilon(U_n^\varepsilon), U_n^\varepsilon\rangle_{\mathcal{L}^{p,2}(I;\mathcal{L}^{p,2}(\Omega))}.\end{aligned}$$

Using the properties of the operator $\mathcal{A}$ (see Corollary 6.3) we find that

$$\begin{aligned}\langle \mathcal{M}^\varepsilon(U_n^\varepsilon), U_n^\varepsilon\rangle_{\mathcal{L}^{p,2}(I;\mathcal{L}^{p,2}(\Omega))} &= \langle G^\varepsilon, U_n^\varepsilon\rangle_{\mathcal{L}^{p,2}(I;\mathcal{L}^{p,2}(\Omega))} \\ &\quad + \frac{1}{2}\big((\mathcal{N}^\varepsilon U_n^\varepsilon(0), U_n^\varepsilon(0)) - (\mathcal{N}^\varepsilon U_n^\varepsilon(T), U_n^\varepsilon(T))\big).\end{aligned}$$

This means that

$$\begin{aligned}&\int_0^T \big(\mathcal{M}_1^\varepsilon(E_n^\varepsilon(t)), E_n^\varepsilon(t)\big) \, \mathrm{d}t \\ &= \int_0^T (F^\varepsilon(t), E_n^\varepsilon(t)) \, \mathrm{d}t + \frac{1}{2}\big((\eta^\varepsilon E_n^\varepsilon(0), E_n^\varepsilon(0)) + (\mu^\varepsilon H_n^\varepsilon(0), H_n^\varepsilon(0)) \\ &\quad - (\eta^\varepsilon E_n^\varepsilon(T), E_n^\varepsilon(T)) - (\mu^\varepsilon H_n^\varepsilon(T), H_n^\varepsilon(T))\big).\end{aligned}$$

Since $H_n^\varepsilon(0) \to H^\varepsilon(0)$ and $E_n^\varepsilon(0) \to E^\varepsilon(0)$ strongly in $L^2(\Omega)^3$ and $E_n^\varepsilon \rightharpoonup E^\varepsilon$ weakly in $L^p(\Omega \times I)^3$ we get

$$\lim_{n\to\infty} \int_0^T (F^\varepsilon(t), E_n^\varepsilon(t)) \, \mathrm{d}t = \int_0^T (F^\varepsilon(t), E^\varepsilon(t)) \, \mathrm{d}t.$$

Further, since $E_n^\varepsilon(T) \rightharpoonup E^\varepsilon(T)$, $H_n^\varepsilon(T) \rightharpoonup H^\varepsilon(T)$ weakly in $L^p(\Omega)^3$ and $L^2(\Omega)^3$, respectively,

$$\begin{aligned}\liminf_{n\to\infty} \|E_n^\varepsilon(T)\|_{L^p(\Omega)^3} &\geqslant \|E^\varepsilon(T)\|_{L^p(\Omega)^3} \\ \liminf_{n\to\infty} \|H_n^\varepsilon(T)\|_{L^2(\Omega)^3} &\geqslant \|H^\varepsilon(T)\|_{L^2(\Omega)^3}\end{aligned}$$

by the weakly sequential lower semicontinuity of the norms in $L^p(\Omega)^3$ and $L^2(\Omega)^3$. This implies that

$$
\begin{aligned}
\limsup_{n\to\infty} & \int_0^T \langle \mathcal{M}_1^\varepsilon(E_n^\varepsilon(t)), E_n^\varepsilon(t)\rangle \,\mathrm{d}t \\
&\leqslant \int_0^T \langle F^\varepsilon(t), E^\varepsilon(t)\rangle \,\mathrm{d}t + \frac{1}{2}\big((\eta^\varepsilon E^\varepsilon(0), E^\varepsilon(0)) + (\mu^\varepsilon H^\varepsilon(0), H^\varepsilon(0)) \\
&\quad - (\eta^\varepsilon E^\varepsilon(T), E^\varepsilon(T)) - (\mu^\varepsilon H^\varepsilon(T), H^\varepsilon(T))\big) \\
&= \int_0^T \langle Z_1^\varepsilon(t), E^\varepsilon(t)\rangle \,\mathrm{d}t,
\end{aligned}
$$

i.e.,

$$\limsup_{n\to\infty} \int_0^T \langle \mathcal{M}_1^\varepsilon(E_n^\varepsilon(t)), E_n^\varepsilon(t)\rangle \,\mathrm{d}t \leqslant \int_0^T \langle Z_1^\varepsilon(t), E^\varepsilon(t)\rangle \,\mathrm{d}t.$$

Since $\mathcal{M}_1^\varepsilon$ is uniformly monotone and continuous,

$$Z_1^\varepsilon = \mathcal{M}_1^\varepsilon(E^\varepsilon).$$

Thus, $\mathcal{M}^\varepsilon(U^\varepsilon) = Z^\varepsilon$ holds according to the definition of the operator $\mathcal{M}^\varepsilon$ and the fact that $Z = \{Z_1, 0\}$. The proof is complete. □

We have proved the existence of a solution, next we prove that this solution is unique.

Lemma 6.8. *The equation* (6.5) *has a unique solution.*

Proof. Let U_1^ε and U_2^ε be two solutions. By inserting these solutions into (6.5) and subtracting we get

$$\mathcal{N}^\varepsilon \partial_t (U_1^\varepsilon - U_2^\varepsilon) + \mathcal{A}U_1^\varepsilon - \mathcal{A}U_2^\varepsilon + \mathcal{M}^\varepsilon(U_1^\varepsilon) - \mathcal{M}^\varepsilon(U_2^\varepsilon) = 0. \tag{6.6}$$

Multiplying (6.6) by $\Delta U^\varepsilon = U_1^\varepsilon - U_2^\varepsilon$, integrating with respect to x over Ω and using Corollary 6.3 and the uniform monotonicity of $\mathcal{M}^\varepsilon$ yields

$$\frac{1}{2}\partial_t\big(\mathcal{N}^\varepsilon \Delta U^\varepsilon(t), \Delta U^\varepsilon(t)\big) + C\|\Delta E^\varepsilon(t)\|^p_{L^p(\Omega)^3} \leqslant 0.$$

Therefore, integrating with respect to time and using Gronwall's inequality, we find that $\Delta U^\varepsilon(x,t) = 0$ almost everywhere in $\Omega \times I$. This contradiction proves that the limit system has a unique solution and we conclude that the total sequence converges to the limit system, not only a subsequence of the solutions to the Galerkin equations. Furthermore, the limit of the Galerkin solutions solves the original system, i.e., the existence of a unique solution is proved. □

Now we will present a priori estimates for the field $\{E^\varepsilon, H^\varepsilon\}$.

Proposition 6.9. *If $F^\varepsilon, \partial_t F^\varepsilon \in L^\infty(I; L^{p'}(\Omega)^3)$ and $\partial_t^2 F^\varepsilon \in L^{p'}(\Omega \times I)^3$ are bounded then*

$$E^\varepsilon \in L^\infty(I; L^2(\Omega)^3) \cap L^p(I; L^p(\Omega)^3)$$

and

$$H^\varepsilon \in L^\infty(I; L^2(\Omega)^3)$$

with bounded norms. Moreover, $\sigma^\varepsilon(E^\varepsilon), \partial_t E^\varepsilon, \operatorname{rot} H^\varepsilon \in L^{p'}(I; L^{p'}(\Omega)^3)$ and $\partial_t H^\varepsilon$, $\operatorname{rot} E^\varepsilon \in L^\infty(I; L^2(\Omega)^3)$ are all bounded.

P r o o f. The proof is analogous to that of the a priori estimates for the Galerkin equation (6.2) in Lemma 6.5 and is therefore omitted. A complete proof can also be found in [18]. □

The uniqueness and existence results in Theorem 3.6 are proved next.

P r o o f of Theorem 3.6. We note that the constitutive relations for the two-scale limit system have the same properties as for the ε-system. Therefore, the solutions will satisfy the same kind of a priori estimates and the existence of solutions can be proved by the use of the Galerkin method as in the ε-dependent system. In the local problems we change variables, e.g. in (3.2) let $u(x,y,t) = E_i(x,t)y_i + \varphi(x,y,t)$, and change the boundary conditions according to this. The existence of solutions to the global homogenized problem follows from the fact that the ε-dependent system has a unique solution for each $\varepsilon > 0$ and that this sequence of solutions has a subsequence which converges weakly to a limit which satisfies the homogenized system.

Uniqueness is proved by assuming, as usual, that there exist two solutions to the two-scale limit system $E^1(x,t) + D_y\varphi^1(x,y,t)$, $E^2(x,t) + D_y\varphi^2(x,y,t)$, $H^1(x,t) + D_y\Phi^1(x,y,t)$ and $H^2(x,t) + D_y\Phi^2(x,y,t)$. By inserting these solutions into (3.1) and subtracting the respective equations for the two sets of solutions we find

$$\begin{aligned}(6.7)\quad \int_0^T\!\!\int_\Omega\int_Y &\big[\eta_{ij}(x,y)\partial_t(E_j^1(x,t) + \partial_{y_j}\varphi^1(x,y,t) \\ &- E_j^2(x,t) - \partial_{y_j}\varphi^2(x,y,t)) + \sigma_i(x,y,E^1(x,t) + D_y\varphi^1(x,y,t)) \\ &- \sigma_i(x,y,E^2(x,t) + D_y\varphi^2(x,y,t))\big] v_i(x,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\!\!\int_\Omega &(\operatorname{rot} H^1(x,t) - \operatorname{rot} H^2(x,t))_i\, v_i(x,t)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and

$$\int_0^T\int_\Omega\int_Y \mu_{ij}\,(x,y)\,\partial_t\big(H_j^1(x,t)+\partial_{y_j}\Phi^1(x,y,t)-H_j^2(x,t) - \partial_{y_j}\Phi^2(x,y,t)\big)v_i(x,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega -(\operatorname{rot}E^1(x,t)-\operatorname{rot}E^2(x,t))_i\,v_i(x,t)\,\mathrm{d}x\,\mathrm{d}t, \tag{6.8}$$

for all v in $L^p(I;\mathcal{L}^{p,2}_{\mathrm{rot}_0}(\Omega))$ and $L^2(I;\mathcal{L}^{2,p'}_{\mathrm{rot}}(\Omega))$, respectively. We note that $E^1(x,0)-E^2(x,0)=0$ and $H^1(x,0)-H^2(x,0)=0$. Analogous relations can be derived from the local problems (3.2) and (3.3).

In the next step we let $v=\Delta E(x,t)=E^1(x,t)-E^2(x,t)$ in (6.7) and $v=\Delta H(x,t)=H^1(x,t)-H^2(x,t)$ in (6.8), $v_2=\Delta\varphi(x,y,t)=\varphi^1(x,y,t)-\varphi^2(x,y,t)$, $v_2=\Delta\Phi(x,y,t)=\Phi^1(x,y,t)-\Phi^2(x,y,t)$ in the equations corresponding to (3.2) and (3.3), respectively. After an integration over $\Omega\times I$ we get the following relation when we take the sum of the local and global equations:

$$\begin{aligned}\int_0^T\int_\Omega\int_Y &\big[\eta_{ij}(x,y)\partial_t(\Delta E_j(x,t)+\partial_j\Delta\varphi(x,y,t))\\ &+\sigma_i(x,y,E^1(x,t)+D_y\varphi^1(x,y,t))-\sigma_i(x,y,E^2(x,t)+D_y\varphi^2(x,y,t))\big]\\ &\times[\Delta E_i(x,t)+\partial_{y_i}\Delta\varphi(x,y,t)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_\Omega\int_Y \mu_{ij}(x,y)\partial_t\big(\Delta H_j(x,t)+\partial_j\Delta\Phi(x,y,t)\big)\\ &\times[\Delta H_i(x,t)+\partial_{y_i}\Delta\Phi(x,y,t)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

Now, by using Gronwall's inequality and the initial conditions we find that $\Delta E(x,t)+D_y\Delta\varphi(x,y,t)=0$ and $\Delta H(x,t)+D_y\Delta\Phi(x,y,t)=0$ almost everywhere in $\Omega\times Y\times I$. This contradicts the assumption of existence of two solutions and the proof is complete. □

Proofs of the homogenization and corrector results

Proof of Theorem 3.4. We will carry out the details only for the local problem (3.2) because (3.3) is proved as in [17]. First we note that we can extract subsequences which converge in two-scale sense due to the a priori estimates. We use the fact that the homogenized system has a unique solution and conclude that the whole sequence of solutions converges to the homogenized system.

By taking the divergence of the first equation in (1.1) we get

$$\partial_{x_i}\Big(\eta_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\partial_t E_j^\varepsilon+\sigma_i\Big(x,\frac{x}{\varepsilon},E^\varepsilon\Big)\Big)=\partial_{x_i}F_i^\varepsilon. \tag{7.1}$$

By assumption, $\partial_{x_i} F_i^\varepsilon$ is bounded in $L^{p'}(\Omega \times I)$. Moreover, multiplying (7.1) with $\varepsilon v_1 v_2$, $v_1 \in D(\Omega)$, $v_2 \in W_\sharp^{1,p}(Y)$, integrating by parts and applying Proposition 5.4, Lemma 5.1 and Lemma 2.8 in [17] we obtain the following local problem in a weak formulation if we take the limit of a subsequence:

$$\int_\Omega \int_Y \big(\eta_{ij}(x,y)\partial_t[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)] + J_i^0(x,y,t)\big) v_1(x)\partial_{y_i} v_2(y)\,\mathrm{d}y\,\mathrm{d}x = 0.$$

□

Further, by using $v(x,t) = v_1(x)b(t)$, $b \in C_0^\infty(I)$, as test functions, we get the following weak formulation of the first equation in (1.1):

$$\begin{aligned}\int_0^T\!\!\int_\Omega \eta_{ij}\Big(x,\frac{x}{\varepsilon}\Big)\partial_t E_j^\varepsilon(x,t) + \sigma_i\Big(x,\frac{x}{\varepsilon},E^\varepsilon(x,t)\Big) v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\!\!\int_\Omega [\operatorname{rot} H^\varepsilon(x,t) + F^\varepsilon(x,t)]_i v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore, using the admissibility of $v_1(x)$, $\eta_{ij}\big(x,\frac{x}{\varepsilon}\big)v_1(x)$ and Proposition 5.2, Lemma 5.1 and Lemma 2.8 in [17] once again, we get the following equation when we take the limit of a subsequence:

$$\begin{aligned}\int_0^T\!\!\int_\Omega\!\int_Y \big(\eta_{ij}(x,y)\partial_t[E_j(x,t) + \partial_{y_j}\varphi(x,y,t)] + J^0(x,y,t)\big) \\ \times v_1(x)b(t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\!\!\int_\Omega [(\operatorname{rot} H(x,t))_i + F_i(x,t)]v_1(x)b(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.2}$$

We will use the Evans technique of perturbed test functions to identify the limit function J^0 (cf. [1] and the references given there). Define test functions

$$v_i^\varepsilon(x,t) = E_i^k(x,t) + \varepsilon\partial_{x_i}\varphi^1\Big(x,\frac{x}{\varepsilon},t\Big) + s\varphi_i^2\Big(x,\frac{x}{\varepsilon},t\Big)$$

and

$$u_i^\varepsilon(x,t) = H_i^k(x,t) + \varepsilon\partial_{x_i}\Phi^1\Big(x,\frac{x}{\varepsilon},t\Big) + s\Phi_i^2\Big(x,\frac{x}{\varepsilon},t\Big),$$

where $E^k, H^k \in D(\Omega \times I)^3$ converge strongly to $E \in L^p(I; L^p(\Omega)^3)$ and $H \in L^2(I; L^2(\Omega)^3)$, respectively, as $k \to \infty$. Here $\varphi^1, \Phi^1 \in D(\Omega \times I; C_\sharp^\infty(Y))$ and $\varphi^2, \Phi^2 \in D(\Omega \times I; C_\sharp^\infty(Y))^3$. Moreover, we note that v_i^ε and u_i^ε are admissible test functions which two-scale converge to $E_i^k(x,t) + \partial_{y_i}\varphi^1(x,y,t) + s\varphi_i^2(x,y,t)$ and

$H_i^k(x,t) + \partial_{y_i}\Phi^1(x,y,t) + s\Phi_i^2(x,y,t)$, respectively. The monotonicity of σ and the positive definiteness of η and μ yield

$$
\begin{aligned}
0 \leqslant\ & \int_0^T\!\!\int_\Omega \Big(\sigma_i\Big(x, \frac{x}{\varepsilon}, E^\varepsilon\Big) - \sigma_i\Big(x, \frac{x}{\varepsilon}, v^\varepsilon\Big)\Big)(E_i^\varepsilon - v_i^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2}\int_0^T\!\!\int_\Omega \partial_t\Big(\eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big)(E_j^\varepsilon - v_j^\varepsilon)(E_i^\varepsilon - v_i^\varepsilon)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2}\int_\Omega \eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big)(E_j^\varepsilon(0) - v_j^\varepsilon(0))(E_i^\varepsilon(0) - v_i^\varepsilon(0))\,\mathrm{d}x \\
&+ \frac{1}{2}\int_0^T\!\!\int_\Omega \partial_t\Big(\mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big)(H_j^\varepsilon - u_j^\varepsilon)(H_i^\varepsilon - u_i^\varepsilon)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2}\int_\Omega \mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big)(H_j^\varepsilon(0) - u_j^\varepsilon(0))(H_i^\varepsilon(0) - u_i^\varepsilon(0))\,\mathrm{d}x \\
=\ & \int_0^T\!\!\int_\Omega \Big[\underbrace{(J_i^\varepsilon E_i^\varepsilon)}_{1} - \underbrace{\Big(\sigma_i\Big(x, \frac{x}{\varepsilon}, v^\varepsilon\Big)E_i^\varepsilon\Big)}_{2} - \underbrace{(J_i^\varepsilon v_i^\varepsilon)}_{3} + \underbrace{\Big(\sigma_i\Big(x, \frac{x}{\varepsilon}, v^\varepsilon\Big)v_i^\varepsilon\Big)}_{4}\Big]\,\mathrm{d}x\,\mathrm{d}t \\
&\underbrace{+ \frac{1}{2}\int_0^T \partial_t \int_\Omega \Big(\eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big)E_j^\varepsilon E_i^\varepsilon + \mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big)H_j^\varepsilon H_i^\varepsilon\Big)\,\mathrm{d}x\,\mathrm{d}t}_{5} \\
&\underbrace{- \frac{1}{2}\int_0^T \partial_t \int_\Omega \eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big)E_j^\varepsilon v_i^\varepsilon + \mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big)H_j^\varepsilon u_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t}_{6} \\
&\underbrace{+ \frac{1}{2}\int_0^T \partial_t \int_\Omega \eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big)v_j^\varepsilon v_i^\varepsilon + \mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big)u_j^\varepsilon u_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t}_{7} \\
&+ \frac{1}{2}\underbrace{\int_\Omega \eta_{ij}\Big(x, \frac{x}{\varepsilon}\Big)(E_j^\varepsilon(0) - v_j^\varepsilon(0))(E_i^\varepsilon(0) - v_i^\varepsilon(0))\,\mathrm{d}x}_{8} \\
&+ \frac{1}{2}\underbrace{\int_\Omega \mu_{ij}\Big(x, \frac{x}{\varepsilon}\Big)(H_j^\varepsilon(0) - u_j^\varepsilon(0))(H_i^\varepsilon(0) - u_i^\varepsilon(0))\,\mathrm{d}x}_{9}.
\end{aligned}
$$

Using Corollary 6.3 we find that the sum of the integrals corresponding to 1 and 5 can be identified with $\int_0^T\!\!\int_\Omega F_i^\varepsilon E_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t$ and, thus, by passing to the weak $L^p(I; L^p(\Omega)^3)$-limit these terms converge to

$$\int_0^T\!\!\int_\Omega F_i(x,t)E_i(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, by using the admissibility of v^ε, u^ε, $\sigma\big(x, \frac{x}{\varepsilon}, v^\varepsilon\big)$, $\mu_{ij}\big(x, \frac{x}{\varepsilon}\big)u_j^\varepsilon$, $\mu_{ij}\big(x, \frac{x}{\varepsilon}\big)u_i^\varepsilon$, $\eta_{ij}\big(x, \frac{x}{\varepsilon}\big)v_j^\varepsilon$ and $\eta_{ij}\big(x, \frac{x}{\varepsilon}\big)v_i^\varepsilon$, the integrals corresponding to 2, 3, and 4 converge to

$$
\begin{aligned}
&-\int_0^T\!\!\int_\Omega\int_Y \sigma_i(x,y,E^k(x,t)+D_y\varphi^1(x,y,t)+s\varphi^2(x,y,t))\\
&\qquad\qquad \times (E_i(x,t)+\partial_{y_i}\varphi(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,\\
&-\int_0^T\!\!\int_\Omega\int_Y J_i^0(x,y,t)(E_i^k(x,t)+\partial_{y_i}\varphi^1(x,y,t)+s\varphi_i^2(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega\int_Y \sigma_i(x,y,E^k(x,t)+D_y\varphi^1(x,y,t)+s\varphi^2(x,y,t))\\
&\qquad\qquad \times (E_i^k(x,t)+\partial_{y_i}\varphi^1(x,y,t)+s\varphi_i^2(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

respectively, when $\varepsilon \to 0$. The limits of 6 and 7 are evaluated similarly. Further, we find that

$$
8 \to \int_\Omega\int_Y \eta_{ij}(x,y)s^2\varphi_j^2(x,y,0)\varphi_i^2(x,y,0)\,\mathrm{d}y\,\mathrm{d}x
$$

and

$$
9 \to \int_\Omega\int_Y \mu_{ij}(x,y)s^2\Phi_j^2(x,y,0)\Phi_i^2(x,y,0)\,\mathrm{d}y\,\mathrm{d}x,
$$

by the admissibility of $E^\varepsilon(x,0)$ and $H^\varepsilon(x,0)$. We now send E^k and H^k strongly to E in $L^p(I;L^p(\Omega)^3)$ and H in $L^2(I;L^2(\Omega)^3)$, respectively, when $k\to\infty$. Further, we let $D_y\varphi^1 \to D_y\varphi$, $D_y\Phi^1 \to D_y\Phi$ strongly in $L^p(\Omega\times Y\times I)^3$ and $L^2(\Omega\times Y\times I)^3$, respectively, and use Corollary 6.3 to obtain

$$
\begin{aligned}
\int_\Omega\bigg(F_i(x,t)E_i(x,t) - \int_Y \big(\eta_{ij}(x,y)\partial_t[E_j+\partial_{y_j}\varphi(x,y,t)] + J_i^0(x,y,t)\big)E_i(x,t)\\
-\mu_{ij}(x,y)\partial_t[H_j(x,t)+\partial_{y_j}\Phi(x,y,t)]H_i(x,t)\bigg)\,\mathrm{d}y\,\mathrm{d}x = 0.
\end{aligned}
$$

We also find that

$$
\sigma(x,y,E^k+D_y\varphi^1+s\varphi^2) \to \sigma(x,y,E+D_y\varphi+s\varphi^2)
$$

and

$$
\begin{aligned}
&\sigma_i(x,y,E^k+D_y\varphi^1+s\varphi^2)(E_i^k+\partial_{y_i}\varphi^1+s\varphi_i^2)\\
&\qquad \to \sigma_i(x,y,E+D_y\varphi+s\varphi^2)(E_i+\partial_{y_i}\varphi+s\varphi_i^2)
\end{aligned}
$$

a.e. in $\Omega \times Y \times I$. Assumption (v) implies that

$$|\sigma(x, y, E^k + D_y\varphi^1 + s\varphi^2)| \leqslant C(1 + |E^k + D_y\varphi^1 + s\varphi^2|^{p-1})$$

and

$$\begin{aligned}|\sigma_i(x, y, E^k + D_y\varphi^1 + s\varphi^2)(E_i^k + \partial_{y_i}\varphi^1 + s\varphi_i^2)| \\ \leqslant C(|E^k + D_y\varphi^1 + s\varphi^2| + |E^k + D_y\varphi^1 + s\varphi^2|^p).\end{aligned}$$

Since

$$\begin{aligned}&\int_0^T\int_\Omega\int_Y (1 + |E^k + D_y\varphi^1 + s\varphi^2|^{p-1})\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \to \int_0^T\int_\Omega\int_Y (1 + |E + D_y\varphi + s\varphi^2|^{p-1})\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\int_\Omega\int_Y (|E^k + D_y\varphi^1 + s\varphi^2| + |E^k + D_y\varphi^1 + s\varphi^2|^p)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \to \int_0^T\int_\Omega\int_Y (|E + D_y\varphi + s\varphi^2| + |E + D_y\varphi + s\varphi^2|^p)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

when $k \to \infty$, by assumption, we find by Lebesgue's dominated convergence theorem

$$\begin{aligned}&\int_0^T\int_\Omega\int_Y \sigma_i\big(x, y, E^k(x,t) + D_y\varphi^1(x,y,t) + s\varphi^2(x,y,t)\big) \\ &\qquad \times \big(E_i^k(x,t) + \partial_{y_i}\varphi^1(x,y,t) + s\varphi_i^2(x,y,t)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \to \int_0^T\int_\Omega\int_Y \sigma_i\big(x, y, E(x,t) + D_y\varphi(x,y,t) + s\varphi^2(x,y,t)\big) \\ &\qquad \times \big(E_i(x,t) + \partial_{y_i}\varphi(x,y,t) + s\varphi_i^2(x,y,t)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

as $k \to \infty$. Further, by using the local problems (3.2), (3.3) and the homogenized equation (7.2) we find that the sum of all the limits reduces to

$$\begin{aligned}&\int_0^T\int_\Omega\int_Y \big(\sigma(x, y, E(x,t) + D_y\varphi(x,y,t) + s\varphi^2(x,y,t)) - J^0(x,y,t), s\varphi^2(x,y,t)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\Omega\int_Y \eta_{ij}(x,y)s^2\varphi_j^2(x,y,0)\varphi_i^2(x,y,0)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_0^T \partial_t \int_\Omega\int_Y \eta_{ij}(x,y)s^2\varphi_j^2(x,y,t)\varphi_i^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\Omega\int_Y \mu_{ij}(x,y)s^2\Phi_j^2(x,y,0)\Phi_i^2(x,y,0)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_0^T \partial_t \int_\Omega\int_Y \mu_{ij}(x,y)s^2\Phi_j^2(x,y,t)\Phi_i^2(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \geqslant 0\end{aligned}$$

for all $s \in \mathbb{R}_+$ and $\varphi^2 \in D(\Omega \times I; C^\infty_\sharp(Y))^3$. Finally, dividing by s and sending $s \to 0$, we get

$$\int_0^T\!\!\int_\Omega\int_Y \bigl(\sigma(x,y,E(x,t)+D_y\varphi(x,y,t)) - J^0(x,y,t), \varphi^2(x,y,t)\bigr)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \geqslant 0$$

for all $\varphi^2 \in D(\Omega \times I; C^\infty_\sharp(Y))^3$. This implies

$$J^0(x,y,t) = \sigma\bigl(x,y,E(x,t)+D_y\varphi(x,y,t)\bigr)$$

almost everywhere in $\Omega \times Y \times I$. We conclude that the whole sequence of solutions converges to the solution of the homogenized system since the homogenized system has a unique solution. The proof is complete. □

P r o o f of Theorem 3.5. We will only prove (b). The proof of (a) is similar. Set $v^{\varepsilon,\delta} = E^\delta(x,t) + D_y\varphi^\delta\bigl(x,\frac{x}{\varepsilon},t\bigr)$ and $u^{\varepsilon,\delta} = H^\delta(x,t) + D_y\Phi^\delta\bigl(x,\frac{x}{\varepsilon},t\bigr)$. The monotonicity of σ (property (vi) in the class $S_{\sharp,Y}$) and the positive definiteness of η and μ yield

$$\begin{aligned}
0 &\leqslant C(\|E^\varepsilon - v^{\varepsilon,\delta}\|^p_{L^p(I\times\Omega)^3} + \|H^\varepsilon(T) - u^{\varepsilon,\delta}(T)\|^2_{L^2(\Omega)^3})\\
&\leqslant \int_0^T\!\!\int_\Omega \Bigl(\sigma_i\Bigl(x,\frac{x}{\varepsilon},E^\varepsilon\Bigr) - \sigma_i\Bigl(x,\frac{x}{\varepsilon},v^{\varepsilon,\delta}\Bigr)\Bigr)(E^\varepsilon_i - v^{\varepsilon,\delta}_i)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_0^T\!\!\int_\Omega \partial_t\Bigl(\eta_{ij}\Bigl(x,\frac{x}{\varepsilon}\Bigr)(E^\varepsilon_j - v^{\varepsilon,\delta}_j)(E^\varepsilon_i - v^{\varepsilon,\delta}_i)\Bigr)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_\Omega \eta_{ij}\Bigl(x,\frac{x}{\varepsilon}\Bigr)(E^\varepsilon_j(0) - v^{\varepsilon,\delta}_j(0))(E^\varepsilon_i(0) - v^{\varepsilon,\delta}_i(0))v_1(x)\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_0^T\!\!\int_\Omega \partial_t\Bigl(\mu_{ij}\Bigl(x,\frac{x}{\varepsilon}\Bigr)(H^\varepsilon_j - u^{\varepsilon,\delta}_j)(H^\varepsilon_i - u^{\varepsilon,\delta}_i)\Bigr)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_\Omega \mu_{ij}\Bigl(x,\frac{x}{\varepsilon}\Bigr)(H^\varepsilon_j(0) - u^{\varepsilon,\delta}_j(0))(H^\varepsilon_i(0) - u^{\varepsilon,\delta}_i(0))v_1(x)\,\mathrm{d}x.
\end{aligned}$$

The remaining steps in the proof are similar to the corresponding proof of Theorem 3.3 in [17], taking into account that $J^0(x,y,t) = \sigma(x,y,E(x,t)+D_y\varphi(x,y,t))$, and is therefore omitted. A complete proof can be found in [18]. □

R e m a r k 7.1. In the main homogenization theorem (Theorem 3.4) and in the proof of Theorem 3.3 the convergence is weak in $W^1_{p,2}(I;V,\mathcal{L}^{2,2}(\Omega))$. As a consequence of the a priori estimates, the theorems holds also in the weak*-convergence sense in $(L^\infty(I;L^2(\Omega)^3) \cap L^p(I;L^p(\Omega)^3)) \times L^\infty(I;L^2(\Omega)^3)$. But for the sake of simplifying the notation, we have, after some hesitation, carried out the theory in the usual weak convergence sense.

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