Nguyen Van Ho; Nguyen Thi Hoa

Random n-ary sequence and mapping uniformly distributed

Applications of Mathematics, Vol. 40 (1995), No. 1, 33–46

RANDOM n-ARY SEQUENCE AND MAPPING UNIFORMLY DISTRIBUTED

Nguyen van Ho and Nguyen Thi Hoa, Hanoi

(Received April 28, 1993)

Summary. Višek [3] and Culpin [1] investigated infinite binary sequence $X = (X_1, X_2, \dots)$ with X_i taking values 0 or 1 at random. They investigated also real mappings $H(X)$ which have the uniform distribution on $[0;1]$ (notation $\mathscr{U}(0;1)$).
The problem for n-ary sequences is dealt with in this paper.

Keywords: Random n-ary sequences, uniform distribution

AMS classification: 60G99, 60F99

1. Introduction

Let $X = (X_1, X_2 \dots)$ be an infinite sequence of random variables taking values in

(1) $$\underline{K} = \{0; 1; 2; \dots; K\} \quad \text{for a given } K \in \mathbb{N} = \{1; 2; \dots\},$$

X is called a n-ary sequence.

If $X_1; X_2; \dots$ are independently identically distributed (i.i.d.), i.e.

(2) $$P(X_i = j) = p_j \geqslant 0,\ \forall j \in \underline{K},\ \sum_{j=0}^{K} p_j = 1,\ \forall i \in \mathbb{N},$$

$$P(X_{i_1} = j_1, \dots, X_{i_n} = j_n) = \prod_{s=1}^{n} p_{j_s},\ \forall n \in \mathbb{N}, j_s \in \underline{K},\ i_1 \neq \dots \neq i_n \in \mathbb{N},$$

the sequence is called multinomial. Denote

(3) $$\mathscr{X} = \{x = (x_1, x_2, \dots), x_i \in \underline{K},\ i \in \mathbb{N}\}.$$

An order relation $\leqslant$ in $\mathscr{X}$ and the distribution function (d.f.) $F(x)$ of X according to a law P will be defined. Conditions under which $F(X)$ is uniformly distributed will be studied. The results are given in Part 2, first for n-ary sequences, then for multinomial sequences and for Markov chains. For $K = 1$ these results reduce to those of Culpin in a more precise form: in Theorem 3 of Culpin [1] it suffices to require $F(x)$ to be increasing instead of strictly increasing and P to be continuous instead of positive continuous. For X being a real random variable this result is well-known, see e.g. [4], p. 34.

2. Results

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in \mathscr{X}$. Denote

$$x \equiv y \text{ iff } x_i = y_i, \ \forall i \in \mathbb{N},$$
$$x \sim y \text{ iff } \exists n \in \mathbb{N}\colon x_1 = y_1, \ldots x_{n-1} = y_{n-1},\ x_n = y_n - 1,$$
$$x_{n+1} = x_{n+2} = \ldots = K,\ y_{n+1} = y_{n+2} = \ldots = 0,$$

or equivalently, $x \sim y$ iff x, y are of the form

$$\text{(4)} \qquad x = (x_1, \ldots, x_{n-1}, y_n - 1, \overline{K}),\ \text{where } \overline{K} = (K, K, \ldots),$$
$$y = (x_1, \ldots, x_{n-1}, y_n, \overline{O}),\ \text{where } \overline{O} = (O, O, \ldots).$$

Define an order relation $\leqslant$ in $\mathscr{X}$ as follows:

$$\text{(5)} \quad x = y \iff \text{either } x \equiv y \text{ or } x \sim y$$
$$x < y \iff x \neq y \text{ and } x_1 = y_1, \ldots, x_{n-1} = y_{n-1},\ x_n < y_n \text{ for some } n \in \mathbb{N}.$$

It is easy to see that the ordering $\leqslant$ is linear, the set of pairs $x \sim y$ is denumerable and $\mathscr{X}$ is the continuum.

Lemma 1. *Let $x, y \in \mathscr{X}$, $x < y$. There exist $z' = (z_1, \ldots, z_r, \overline{O})$ and $z'' = (z_1, \ldots z_r, \overline{K}) \in \mathscr{X}$ for some $r \in \mathbb{N}$ such that*

$$x \leqslant z' < z'' \leqslant y.$$

Proof. Since $x = (x_1, x_2, \ldots) < y = (y_1, y_2, \ldots)$, there is $n \in \mathbb{N}$ such that

(i) either $x_1 = y_1, \ldots, x_{n-1} = y_{n-1},\ x_n \leqslant y_n - 2$,

(ii) or $x_1 = y_1, \ldots, x_{n-1} = y_{n-1},\ x_n = y_n - 1$ and for some $m \in \mathbb{N}$,

$$x_{n+1} = \ldots = x_{n+m-1} = K,\ y_{n+1} = \ldots = y_{n+m-1} = 0$$

and $x_{n+m} \leqslant K - 1$ or $y_{n+m} \geqslant 1$.

In case (i) one can choose $r = n$ and

$$z' = (x_1, \dots, x_{n-1}, x_n + 1, \bar{O}),$$
$$z'' = (x_1, \dots, x_{n-1}, x_n + 1, \bar{K}).$$

In case (ii), if $x_{n+m} \leqslant K - 1$, one can put $r = n + m$, and

$$z' = (x_1, \dots, x_n, \underbrace{K, \dots, K}_{m}, \bar{O}),$$
$$z'' = (x_1, \dots, x_n, \underbrace{K, \dots, K}_{m}, \bar{K}) = (x_1, x_2, \dots, x_n, \bar{K}),$$

or if $y_{n+m} \geqslant 1$, one puts $r = n + m$ and

$$z' = (x_1, \dots, x_{n-1}, x_n + 1, \underbrace{O, \dots, O}_{m}, \bar{O}) = (x_1, \dots, x_{n-1}, x_n + 1, \bar{O}),$$
$$z'' = (x_1, \dots, x_{n-1}, x_n + 1, \underbrace{O, \dots, O}_{m}, \bar{K}).$$

□

Definition 1. A mapping F of $\mathscr{X}$ into $[0; 1]$ is called unique, increasing or continuous iff the following condition (i), (ii) or (iii) is satisfied, respectively:

(6) (i) $x \sim y \implies F(x) = F(y)$,

(ii) $x \leqslant y \implies F(x) \leqslant F(y)$,

(iii) $F(x_1, x_2, \dots, x_n, y_{n+1}, y_{n+2}, \dots) \to F(x_1, x_2, \dots)$ as $n \to \infty$.

Remark 1. If F is increasing, then F is continuous iff for every $x = (x_1, x_2, \dots) \in \mathscr{X}$

(7) $F(x_1, \dots, x_n, \bar{K})$ and $F(x_1, \dots, x_n, \bar{O}) \to F(x)$ as $n \to \infty$,
or equivalently, $F(x_1, \dots, x_n, \bar{K}) - F(x_1, \dots, x_n, \bar{O}) \to 0$ as $n \to \infty$.

Definition 2. F is said to have "Property D" iff it is unique, increasing, continuous and $F(\bar{O}) = 0$, $F(\bar{K}) = 1$.

Theorem 1. *Let F be a mapping of $\mathscr{X}$ into $[0; 1]$. F has "Property D" iff it is of the form*

(8) $$F(x) = \sum_{n=1}^{\infty} \sum_{j=0}^{x_n - 1} f_n(x_1, \dots, x_{n-1}, j), \; x = (x_1, x_2, \dots) \in \mathscr{X},$$

where we use the convention $\sum_{j=0}^{-1} a_j = 0$, *and the* f_n*'s defined on* $\underline{K}^n$ *satisfy*

$$(9) \qquad \text{(i)} \quad f_n \geqslant 0,$$

$$\text{(ii)} \quad \sum_{j=0}^{K} f_n(x_1, \ldots, x_{n-1}, j) = f_{n-1}(x_1, \ldots, x_{n-1}), \text{ where } f_0 = 1,$$

$$\text{(iii)} \quad f_n(x_1, \ldots, x_n) \to 0 \text{ as } n \to \infty.$$

The f_n'*s are uniquely determined from* F *by*

$$(10) \qquad f_n(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, \overline{K}) - F(x_1, \ldots, x_n, \overline{O}).$$

Proof. Let F have "Property D". Defining f_n by (10), one has

$$\begin{aligned}
&\sum_{n=1}^{N} \sum_{j=0}^{x_n - 1} f_n(x_1, \ldots, x_{n-1}, j) \\
&= \sum_{n=1}^{N} \sum_{j=0}^{x_n - 1} \{F(x_1, \ldots, x_{n-1}, j, \overline{K}) - F(x_1, \ldots, x_{n-1}, j, \overline{O})\} \\
&= \sum_{n=1}^{N} \sum_{j=0}^{x_n - 1} \{F(x_1, \ldots, x_{n-1}, j+1, \overline{O}) - F(x_1, \ldots, x_{n-1}, j, \overline{O})\} \\
&= \sum_{n=1}^{N} \{F(x_1, \ldots, x_n, \overline{O}) - F(x_1, \ldots, x_{n-1}, \overline{O})\} \\
&= F(x_1, \ldots, x_N, \overline{O}) - F(\overline{O}) = F(x_1, \ldots, x_N, \overline{O}) \to F(x) \quad \text{as } N \to \infty.
\end{aligned}$$

This proves (8). Conditions (9) (i) and (iii) follow from (10), (6) and (7). One gets (9) (ii) by direct calculation:

$$\begin{aligned}
\sum_{j=0}^{K} f_n(x_1, \ldots, x_{n-1}, j) &= \sum_{j=0}^{K} \{F(x_1, \ldots, x_{n-1}, j, \overline{K}) - F(x_1, \ldots, x_{n-1}, j, \overline{O})\} \\
&= F(x_1, \ldots, x_{n-1}, \overline{K}) - F(x_1, \ldots, x_{n-1}, K, \overline{O}) \\
&\quad + \sum_{j=0}^{K-1} \{F(x_1, \ldots, x_{n-1}, j, \overline{K}) - F(x_1, \ldots, x_{n-1}, j, \overline{O})\} \\
&= F(x_1, \ldots, x_{n-1}, \overline{K}) - F(x_1, \ldots, x_{n-1}, K, \overline{O}) \\
&\quad + \sum_{j=0}^{K-1} \{F(x_1, \ldots, x_{n-1}, j+1, \overline{O}) - F(x_1, \ldots, x_{n-1}, j, \overline{O})\} \\
&= F(x_1, \ldots, x_{n-1}, \overline{K}) - F(x_1, \ldots, x_{n-1}, \overline{O}) \\
&= f_{n-1}(x_1, \ldots, x_{n-1}).
\end{aligned}$$

Suppose now that F is of the form (8) with f_n satisfying (9). The conventions $\sum_0^{-1} a_j = 0$ and $f_0 = 1$ imply $F(\overline{O}) = 0$ and $F(\overline{K}) = 1$.

Let $x \sim y$, i.e. x, y are of the form (4). Then $F(y) - F(x) = f_n(x_1, \ldots, x_n) + A - B$, where

$$A = \sum_{s=n+1}^{\infty} \sum_{j=0}^{y_s-1} f_s(y_1, \ldots, y_{s-1}, j) = 0, \quad \text{since } \sum_0^{-1} = 0,$$

and

$$\begin{aligned} B &= \sum_{s=n+1}^{\infty} \sum_{j=0}^{x_s-1} f_s(x_1, \ldots, x_{s-1}, j) \\ &= \sum_{s=n+1}^{\infty} \Big\{ \sum_{j=0}^{K} f_s(x_1, \ldots, x_{s-1}, j) - f_s(x_1, \ldots, x_{s-1}, K) \Big\}, \\ &\qquad \text{since for } s \geqslant n+1,\ x_s = K, \\ &= \sum_{s=n+1}^{\infty} \{ f_{s-1}(x_1, \ldots, x_{s-1}) - f_s(x_1, \ldots, x_{s-1}, K) \} \\ &= f_n(x_1, \ldots, x_n). \end{aligned}$$

This implies that $F(y) - F(x) = 0$, i.e. F is unique. Let $x \not\sim y$, $x \leqslant y$. Then $x_1 = y_1, \ldots, x_{n-1} = y_{n-1}$, $x_n \leqslant y_n - 1$ for some $n \in \mathbb{N}$, and

$$F(y) - F(x) = \sum_{j=x_n}^{y_n-1} f_n(x_1, \ldots, x_{n-1}, j) + A - B, \text{ where}$$

$$\begin{aligned} A &= \sum_{s=n+1}^{\infty} \sum_{j=0}^{y_s-1} f_s(y_1, \ldots, y_{s-1}, j) \geqslant 0 \\ B &= \sum_{s=n+1}^{\infty} \sum_{j=0}^{x_s-1} f_s(x_1, \ldots, x_{s-1}, j) \\ &\leqslant \sum_{s=n+1}^{\infty} \sum_{j=0}^{K} f_s(x_1, \ldots, x_{s-1}, j) = f_n(x_1, \ldots, x_n). \end{aligned}$$

Thus

$$F(y) - F(x) \geqslant \sum_{j=x_n}^{y_n-1} f_n(x_1, \ldots, x_{n-1}, j) - f_n(x_1, \ldots, x_n) \geqslant 0,$$

i.e. F is increasing.

For $x \in \mathscr{X}$, $n \in \mathbb{N}$ one has

$$\begin{aligned} F(x_1, \ldots, x_n, \overline{K}) - F(x_1, \ldots, x_n, \overline{O}) &= \\ &= \sum_{s=n+1}^{\infty} \sum_{j=0}^{K-1} f_s(x_1, \ldots, x_n, \underbrace{K, \ldots, K}_{s-n-1}, j) \\ &= \sum_{s=n+1}^{\infty} \{f_{s-1}(x_1, \ldots, x_n, \underbrace{K, \ldots, K}_{s-n-1}) - f_s(x_1, \ldots, x_n, \underbrace{K, \ldots, K}_{s-n})\} \\ &= f_n(x_1, \ldots, x_n). \end{aligned}$$

This proves (10), and the continuity of F follows by (9) (iii). □

Corollary 1. *Let F have "Property D". F is strictly increasing iff*

$$f_n(x_1, \ldots, x_n) > 0, \ \forall n \in \mathbb{N}, \ \forall x \in \mathscr{X}.$$

Proof. It follows from Lemma 1 and (10). □

Theorem 2. *F has "Property D" iff F is an increasing mapping of $\mathscr{X}$ onto* $[0; 1]$.

Proof. Let F be an increasing mapping of $\mathscr{X}$ onto $[0; 1]$. Clearly, $F(\overline{O}) = O$, $F(\overline{K}) = 1$.

Suppose there exist $x, y \in \mathscr{X}$ such that $x \sim y$ and $F(x) < F(y)$. There must be a $z \in \mathscr{X} : F(x) < F(z) < F(y)$. Then $x \leqslant z \leqslant y$ and $z \not\equiv x$, $z \not\equiv y$, which implies, by the definition (4), that $z \nsim x$, $z \nsim y$. Thus $x < z < y$. Hence $x < y$. This contradiction proves the uniqueness of F. It remains to verify the continuity of F.

For $x = (x_1, x_2, \ldots) \in \mathscr{X}$, let us denote

$$x'_{(m)} = (x_1, \ldots, x_m, \overline{O}), \quad x''_{(m)} = (x_1, \ldots, x_m, \overline{K}), \quad m \in \mathbb{N}.$$

Then $x'_{(m)} \leqslant x \leqslant x''_{(m)}$ and $F(x'_{(m)}) \leqslant F(x) \leqslant F(x''_{(m)})$. Since $x'_{(m)}$ $(x''_{(m)})$ is increasing (decreasing) with m, there exist a' and $a'' \in [0; 1]$, such that

$$F(x'_{(m)}) \nearrow a' \leqslant F(x) \quad \text{and} \quad F(x''_{(m)}) \searrow a'' \geqslant F(x).$$

If $a' < F(x)$ there would be $y \in \mathscr{X}$ such that $a' < F(y) < F(x)$. Thus, $y < x$. Therefore $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$, $y_n < x_n$ for some $n \in \mathbb{N}$. Hence, for $m \geqslant n$, $y \leqslant x'_{(m)}$ and

$$F(y) \leqslant F(x'_{(m)}) \leqslant a', \quad \text{i.e. } F(y) \leqslant a'.$$

This contradiction yields that $a' = F(x)$. In the same way, $a'' = F(x)$. This implies (7).

Suppose now that F has "Property D". By Theorem 1, F is of the form (8) with f_n satisfying (9). For a given $t \in [0;1]$ we will determine two sequences $x = (x_1, x_2, \dots) \in \mathscr{X}$ and $(v_0, v_1, v_2, \dots)$ such that

$$t = v_0 \geqslant v_1 \geqslant v_2 \geqslant \dots,$$
$$0 \leqslant v_n \leqslant f_n(x_1, \dots, x_n), \quad \forall n \in \mathbb{N},$$

in the following way:

$$v_0 = t,$$
$$x_1 = \max\Big\{i \colon i \in \underline{K},\ \sum_{j=0}^{i-1} f_1(j) \leqslant v_0 = t \leqslant \sum_{j=0}^{i} f_1(j)\Big\},$$
$$v_1 = v_0 - \sum_{j=0}^{x_1-1} f_1(j),$$
$$\vdots$$
$$x_n = \max\Big\{i \colon i \in \underline{K}, \sum_{j=0}^{i-1} f_n(x_1, \dots, x_{n-1}, j) \leqslant v_{n-1} \leqslant \sum_{j=0}^{i} f_n(x_1, \dots, x_{n-1}, j)\Big\},$$
$$v_n = v_{n-1} - \sum_{j=0}^{x_n-1} f_n(x_1, \dots, x_{n-1}, j)$$
$$\leqslant \sum_{j=0}^{x_n} f_n(x_1, \dots, x_{n-1}, j) - \sum_{j=0}^{x_n-1} f_n(x_1, \dots, x_{n-1}, j) = f_n(x_1, \dots, x_n).$$

Then

$$t = v_0 = v_N + \sum_{n=1}^{N} (v_{n-1} - v_n)$$
$$= v_N + \sum_{n=1}^{N} \sum_{j=0}^{x_n-1} f_n(x_1, \dots, x_{n-1}, j), \quad N \in \mathbb{N},$$

where $v_N \leqslant f_N(x_1, \dots, x_n) \to 0$ as $N \to \infty$. Therefore

$$t = \sum_{n=1}^{\infty} \sum_{j=0}^{x_n-1} f_n(x_1, \dots, x_{n-1}, j) = F(x).$$

This proves that F is a mapping of $\mathscr{X}$ onto $[0;1]$. □

Law and distribution function of X.

Let $X = (X_1, X_2, \ldots)$ be an infinite n-ary sequence with X_i taking values in $\underline{K}$, $i \in \mathbb{N}$. Let P be a probability law of X. A law P of X is given iff there is a system $\mathscr{P}$ of probabilities $P(X_{i_1} = x_{i_1}, \ldots, X_{i_m} = x_{i_m})$, $\forall m \in \mathbb{N}$, $\forall i_1 \neq \ldots \neq i_m \in \mathbb{N}$, $\forall x = (x_1, x_2, \ldots) \in \mathscr{X}$, satisfying the well-known consistency conditions which imply (9) (i)–(ii) with f_n defined by

$$f_n(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n). \tag{11}$$

Conversely, from a family of $P(X_1 = x_1, \ldots, X_n = x_n)$, $\forall n \in \mathbb{N}$, $\forall x \in \mathscr{X}$ satisfying (9) (i)–(ii) one can get the system $\mathscr{P}$ satisfying the consistency conditions by putting, for $i_1 \neq \ldots \neq i_m \in \mathbb{N}$,

$$P(X_{i_1} = x_{i_1}, \ldots, X_{i_m} = x_{i_m}) = \sum_{x_{j_1}, \ldots, x_{j_{n-m}} \in \underline{K}} P(X_1 = x_1, \ldots, X_n = x_n),$$

where $n = \max(i_1, \ldots, i_m)$, $\{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$, i.e. a law P of X is given.

Definition 3. A law P of X is said to be continuous iff

$$P(X = x) = \lim_{n \to \infty} P(X_1 = x_1, \ldots, X_n = x_n) = 0,$$

i.e. iff the f_n's defined by (11) satisfy (9) (iii).

Since the f_n's satisfy (9) (i)–(ii) as mentioned above, Definition 3 is equivalent to

Definition 3*. P is continuous iff the f_n's defined from (11) satisfy (9) (i)–(iii).

Definition 4. The mapping $F\colon \mathscr{X} \to [0;1]$ defined from

$$F(x) = P(X < x), \quad x \in \mathscr{X} \tag{12}$$

is called the distribution function of X according to the law P, (abbr.: d.f. of $X|P$).

Remark 2. For the case of a continuous P,

$$F(x) = P(X < x) = P(X \leqslant x). \tag{13}$$

Definition 5. A law P of X is called positive iff the system $\mathscr{P}$ is positive, i.e.

$$P(X_{i_1} = x_{i_1}, \ldots, X_{i_m} = x_{i_m}) > 0, \quad \forall m \in \mathbb{N},\ \forall i_1 \neq \ldots \neq i_m \in \mathbb{N},\ \forall x \in \mathscr{X}. \tag{14}$$

Theorem 3. *Let* $F\colon \mathscr{X} \to [0;1]$.

(i) *F has "Property D" iff F is d.f. of X according to a continuous law P. F and P are determined uniquely from each other:*

$$(15)\ F(x) = \sum_{n=1}^{\infty} \sum_{j=0}^{x_n - 1} P(X_1 = x_1, \dots, X_{n-1} = x_{n-1}, j), \quad x = (x_1, x_2, \dots) \in \mathscr{X},$$

$$(16)\ P(X_1 = x_1, \dots, X_n = x_n) = F(\bar{x}) - F(\underline{x}),$$

where

$$\underline{x} = (x_1, \dots, x_n, \overline{O}),$$
$$\bar{x} = (x_1, \dots, x_n, \overline{K}), \quad n \in \mathbb{N},\ x \in \mathscr{X}.$$

(ii) *Moreover, for F and P as in part* (i), *F is strictly increasing iff P is positive.*

Proof. (i) Let P be a continuous law of X. Since

$$\{X < x\} \subset \sum_{n=1}^{\infty} \sum_{j=0}^{x_n - 1} \{X_1 = x_1, \dots, X_{n-1} = x_{n-1}, X_n = j\} \subset \{X \leqslant x\},$$

with the convention $\sum_0^{-1}\{.\} = \emptyset$, one gets (15) by virtue of (13), i.e. F is of the form (8) with f_n defined from (11) satisfying (9). Thus, F has "Property D" by Theorem 1.

Let now F have "Property D". By Theorem 1, F is of the form (8) with f_n satisfying (9) and (10). Defining a family of $P(X_1 = x_1, \dots, X_n = x_n)$ by (11) which yields a system $\mathscr{P}$ and then a continuous law P, one gets (15) and (16) from (8) and (10), respectively. F is the d.f. of $X|P$ by the first part of the proof.

(ii) This is a consequence of Corollary 1. □

Corollary 2. *Let P be a continuous law of* $X = (X_1, X_2, \dots)$. *Then*

$$(17) \qquad P\{X_1 = x_1, \dots, X_n = x_n\} = P\{\underline{x} \leqslant X \leqslant \bar{x}\}$$

with $\underline{x}$, $\bar{x}$ *defined in Theorem* 3.

Proof. Let F denote the d.f. of $X|P$. It is easily seen that

$$(18)\ \{X_1 = x_1, \dots, X_n = x_n\} = \{X_1 = x_1, \dots, X_n = x_n,\ 0 \leqslant X_{n+i} \leqslant K,\ i \in \mathbb{N}\}$$
$$\subset \{\underline{x} \leqslant X \leqslant \bar{x}\}.$$

Thus,

$$P\{X_1 = x_1, \dots, X_n = x_n\} \leqslant P\{\underline{x} \leqslant X \leqslant \bar{x}\} = F(\bar{x}) - F(\underline{x})$$

by (13). This fact and (16) prove (17). □

Theorem 4. *Let F be an increasing mapping of $\mathscr{X}$ into $[0;1]$. Let P be a continuous law of $X = (X_1, X_2, \ldots)$. Then $F(X)\mathscr{L}\mathscr{U}(0;1)$ under P iff $F(x)$ is the d.f. of $X|P$.*

Proof. Let F be the d.f. of $X|P$, where P is continuous. By Theorems 2 and 3, F has "Property D" and maps $\mathscr{X}$ onto $[0;1]$. Then

$$\forall t \in [0;1],\ F^{-1}(t) = \{x;\ x \in \mathscr{X},\ F(x) = t\} \neq \emptyset.$$

Denote $x^t = \sup F^{-1}(t)$, where the supremum is taken according to the ordering $\leqslant$ defined in (5). Since F has "Property D" and P is continuous, one obtains

$$\begin{aligned}&\{F(X) \leqslant t\} = \{X \leqslant x^t\},\\ &P\{F(X) \leqslant t\} = P\{X \leqslant x^t\} = P\{X < x^t\} = F(x^t) = t,\end{aligned}$$

which shows that $F(X)\mathscr{L}\mathscr{U}(0;1)$ under P.

Conversely, let $F(X)\mathscr{L}\mathscr{U}(0;1)$ under the continuous law P. Let λ be the Lebesgue measure on $(\mathbb{R}, \mathscr{B})$. One has

$$\lambda\{[0;1] \setminus F(\mathscr{X})\} = 1 - \lambda\{F(\mathscr{X})\} = 1 - P\{F(X) \in F(\mathscr{X})\} = 1 - 1 = 0.$$

Thus $F(\mathscr{X})$ is everywhere dense in $[0;1]$. Therefore,

$$\begin{aligned}F(\bar{O}) &= \inf F(\mathscr{X}) = 0,\\ F(\bar{K}) &= \sup F(\mathscr{X}) = 1.\end{aligned}$$

Hence $0 \in F(\mathscr{X})$, $1 \in F(\mathscr{X})$.

For $t \in (0;1)$ there exist $\{a_n\}$ and $\{b_n\} \subset F(\mathscr{X})$ such that

$$\begin{aligned}&a_1 < a_2 < \ldots, \qquad \lim a_n = t,\\ &b_1 > b_2 > \ldots, \qquad \lim b_n = t.\end{aligned}$$

Then there exist $\{x^n\}$ and $\{y^n\} \subset \mathscr{X}$ such that

$$\begin{aligned}&x^1 \leqslant x^2 \leqslant \ldots \qquad F(x^n) = a_n,\ \forall n \in \mathbb{N},\\ &y^1 \geqslant y^2 \geqslant \ldots \qquad F(y^n) = b_n,\ \forall n \in \mathbb{N}.\end{aligned}$$

Denote $x = \sup\{x^n\}$, $y = \inf\{y^n\}$. Hence $x, y \in \mathscr{X}$, $F(x) = F(y) = t$, i.e. $t \in F(\mathscr{X})$. This proves that F maps $\mathscr{X}$ onto $[0;1]$. By Theorems 2 and 3, F has "Property D" and it is a d.f. of X according to a continuous law, say Q, which is determined from

$$Q(X_1 = x_1, \ldots, X_n = x_n) = F(\bar{x}) - F(\underline{x}). \tag{19}$$

It remains to prove that $Q = P$, or equivalently, to show that

$$Q(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_1, \dots, X_n = x_n),\ \forall n \in \mathbb{N},\ \forall x \in \mathscr{X}. \tag{20}$$

From (18) and $\{\underline{x} \leqslant X \leqslant \bar{x}\} \subset \{F(\underline{x}) \leqslant F(X) \leqslant F(\bar{x})\}$ one gets $P(X_1 = x_1, \dots, X_n = x_n) \leqslant F(\bar{x}) - F(\underline{x})$, since $F(X)\mathscr{L}\mathscr{U}(0;1)$ under P. Thus, by (19),

$$P(X_1 = x_1, \dots, X_n = x_n) \leqslant Q(X_1 = x_1, \dots, X_n = x_n). \tag{21}$$

On the other hand,

$$\{X < \underline{x}\} \subset \{F(X) \leqslant F(\underline{x})\} \quad \text{and} \quad \{X > \bar{x}\} \subset \{F(X) \geqslant F(\bar{x})\}$$

imply

$$P(X < \underline{x}) \leqslant F(\underline{x}) \quad \text{and} \quad P(X > \bar{x}) \leqslant 1 - F(\bar{x})$$
$$\text{or} \quad P(X \leqslant \bar{x}) \geqslant F(\bar{x}),$$

which yields

$$P(X \leqslant \bar{x}) - P(X < \underline{x}) \geqslant F(\bar{x}) - F(\underline{x}),$$

or, by Corollary 2 and (19),

$$P(X_1 = x_1, \dots, X_n = x_n) \geqslant Q(X_1 = x_1, \dots, X_n = x_n). \tag{22}$$

The desired result (20) is obtained from (21) and (22). □

Corollary 3. *Let P be a continuous law of $X = (X_1, X_2, \dots)$. The only decreasing mapping $G\colon \mathscr{X} \to [0;1]$ such that $G(X)\mathscr{L}\mathscr{U}(0;1)$ under P is determined from*

$$G(x) = \sum_{n=1}^{\infty} \sum_{j=x_n+1}^{K} P(X_1 = x_1, \dots, X_{n-1} = x_{n-1}, j), \quad \sum_{K+1}^{K} \doteq 0. \tag{23}$$

P r o o f. Let F be the d.f. of $X|P$. By Theorem 4, F is the only increasing mapping such that $F(X)\mathscr{L}\mathscr{U}(0;1)$ under P. Thus $1 - F(x)$ is the only decreasing mapping such that $1 - F(X)\mathscr{L}\mathscr{U}(0;1)$ under P. By Theorem 3 $F(x)$ is of the form (15). Thus $1 - F(x)$ is defined by (23). □

R e m a r k 3. Consider

(24) $$M(x) = \sum_{n=1}^{\infty} \frac{x_n}{(K+1)^n}, \quad x = (x_1, x_2, \dots) \in \mathscr{X}.$$

M is of the form (8) with

$$f_n(x_1, \dots, x_n) = \frac{1}{(K+1)^n} > 0, \quad n \in \mathbb{N},\ x \in \mathscr{X},$$

satisfying (9). Thus M has "Property D". Moreover, it is strictly increasing by Corollary 1. Also, $M(X)\mathscr{L}\mathscr{U}(0;1)$ only under P such that

$$P(X_1 = x_1, \dots, X_n = x_n) = f_n(x_1, \dots, x_n) = \frac{1}{(K+1)^n} > 0, \quad n \in \mathbb{N},\ x \in \mathscr{X},$$

i.e. $X = (X_1, X_2, \dots)$ is an i.i.d. sequence with

(25) $$P(X_i = j) = \frac{1}{K+1}, \quad j \in \underline{K},\ i \in \mathbb{N}.$$

APPLICATION TO n-ARY SEQUENCES

Corollary 4. *Let $X = (X_1, X_2, \dots)$ be an independent sequence such that*

(26) $$P(X_i = j) = p_{ij} \geqslant 0,\ j \in \underline{K},\ \sum_{j \in \underline{K}} p_{ij} = 1,\ i \in \mathbb{N}.$$

Moreover, let

(27) $$\exists \alpha \in (0;1),\ \exists N \in \mathbb{N} \text{ such that}$$
$$0 \leqslant p_{ij} \leqslant 1 - \alpha,\ \forall j \in \underline{K},\ \forall i \geqslant N.$$

Then

(i) *the d.f. of $X|P$ is determined from*

(28) $$F(x) = \sum_{n=1}^{\infty} \left\{ \left(\prod_{i=1}^{n-1} p_{ix_i} \right) \sum_{j=0}^{x_n - 1} p_{nj} \right\},$$
$$x \in \mathscr{X}, \text{ where } \prod_1^0 \doteq 1,\ \sum_0^{-1} \doteq 0;$$

(ii) $F(X)\mathscr{L}\mathscr{U}(0;1)$ *under* P;

(iii) *the additional assumption* $p_{ij} > 0$, $\forall i \in \mathbb{N}$, $\forall j \in \underline{K}$, *ensures the positivity of* P *as well as the strict increasing of* F.

Proof. Note that

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_{ix_i}, \quad n \in \mathbb{N},\ x \in \mathscr{X}$$

and the f_n's defined by (11) satisfy (9). □

Remark 4. For X being an i.i.d. sequence, i.e., $p_{ij} = p_j$, $\forall i \in \mathbb{N}$, $\forall j \in \underline{K}$, the condition (27) is replaced by

$$0 \leqslant p_j < 1, \quad j \in \underline{K}. \tag{27*}$$

Application to Markov chains

Corollary 5. *Let* $X = (X_1, X_2, \ldots)$ *be a Markov chain with a finite state space* $\underline{E} = \{E_0, E_1, \ldots, E_K\}$ *which is identically denoted by* $\underline{K} = \{0, 1, \ldots, K\}$. *Let* $\pi_0 = \{p_0, p_1, \ldots, p_k)$ *be the initial probabilities and let* $\pi = (p_{ij})$, $i, j \in \underline{K}$ *be the matrix of transition probabilities:* $p_i \geqslant 0$, $p_0 + \ldots + p_K = 1$, $p_{ij} \geqslant 0$, $\sum_{j \in \underline{K}} p_{ij} = 1$, $i \in \underline{K}$.
Suppose that

$$0 \leqslant p_{ij} < 1, \quad i \in \underline{K},\ j \in \underline{K}. \tag{29}$$

Then

(i) *the distribution function of* X *is determined by*

$$F(x) = \sum_{n=1}^{\infty} \left\{ \left(p_{x_1} \prod_{i=1}^{n-2} p_{x_i, x_{i+1}} \right) \sum_{j=0}^{x_n - 1} p_{x_{n-1}, j} \right\}, \tag{30}$$

$$x \in \mathscr{X}, \text{ where } \prod_1^{-1} \doteq 1,\ \prod_1^{0} \doteq 1,\ \sum_0^{-1} \doteq 0;$$

(ii) $F(X)\mathscr{L}\mathscr{U}(0;1)$;

(iii) *moreover, if* $0 < p_i < 1$, $0 < p_{ij} < 1$, $i, j \in \underline{K}$, *the law* P *is positive and* F *is strictly increasing.*

Proof. Since $P(X_1 = x_1, \ldots, X_n = x_n) = p_{x_1} \cdot p_{x_1, x_2} \cdots p_{x_{n-1}, x_n}$, $n \in \mathbb{N}$, $x \in \mathscr{X}$, and the f_n's defined from (11) satisfy (9) provided (29) holds. □

References

[1] *D. Culpin*: Distribution of random binary sequence. Aplikace matematiky *25* (1980), 408–416.

[2] *I.I. Gichman, A.V. Skorochod*: Introduction to the theory of random processes. Moskva, 1977.

[3] *J.A. Višek*: On properties of binary random numbers. Aplikace matematiky *19* (1974), 375–385.

[4] *J. Hájek, Z. Šidák*: Theory of rank tests. Prague, 1967.

[5] *W. Feller*: An introduction to probability theory and its applications. New York, 1971.

[6] *D. Dacunha-Castelle, M. Duflo*: Probalilités et statistiques. Paris, 1983.

Author's address: *Nguyen van Ho*, Department of Mathematics, Polytechnic Institute of Hanoi, Hanoi, Vietnam.