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Applications of Mathematics, Vol. 40 (1995), No. 2, 147–158

A SIMPLE MODEL OF THERMOELECTRIC OSCILLATIONS

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(Received October 22, 1993)

Summary. A system of ordinary differential equations modelling an electric circuit with a thermistor is considered. Qualitative properties of solution are studied, in particular, the existence and nonexistence of time-periodic solutions (the Hopf bifurcation).

Keywords: thermistor, time-periodic solution, Hopf bifurcation

AMS classification: 78A97, 34C25

1. INTRODUCTION

In the electric circuit of Figure 1 R_T is a special resistor called "thermistor" whose conductance σ is a function of the temperature u, R is an ordinary resistor and C a capacitor. We make the simplifying assumption that the temperature and the electric potential φ across the thermistor depend on time only. Let u_a be the room's temperature. By Newton's law of cooling we have, taking into account the Joule heating,

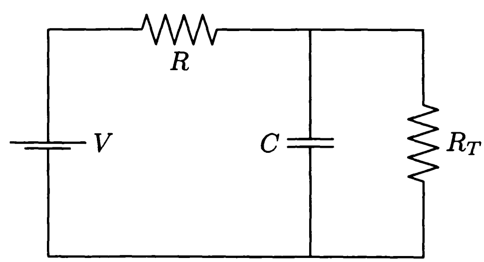

Fig. 1

$$H\frac{\mathrm{d}u}{\mathrm{d}t} = -k(u-u_a) + \sigma(u)\varphi^2 \tag{1.1}$$

where H and k are the incremental heat capacity and the incremental dissipation constant. Using Kirchhoff's principles we obtain

$$V = \varphi + \varrho\left(\varphi\sigma(u) + c\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) \tag{1.2}$$

where ϱ is the resistance of R, c the capacitance of C and V a fixed applied difference of potential. This circuit is interesting for two reasons: firstly, multiple states of equilibrium (typically three) can exist, secondly, low-frequency thermoelectric oscillations are observed experimentally. This effect gives rise to various practical applications, see e.g. [2]. In Section 2 we find conditions under which time periodic solutions cannot exist, in particular this will be the case when $\frac{d\sigma}{du} < 0$. Section 3 is devoted to the study of the local stability of stationary points and to the proof of the existence of periodic solutions using Hopf's bifurcation. Sections 4 and 5 deal with a more specific conductivity and with the corresponding bifurcation diagram. The physically crucial problem of the stability of periodic solutions is discussed in Section 5. Finally a more realistic, but more difficult, model is presented in the last part.

Let $s(y) \in C^2(\mathbb{R}^1)$ and suppose

$$s(y) \geqslant s_m > 0 \qquad \text{for all } y \geqslant 0. \tag{1.3}$$

We assume $\sigma(u)$ to be an empirically given conductivity of the form

$$\sigma(u) = \sigma_0 s\left(\frac{u - u_a}{u_a}\right). \tag{1.4}$$

Use will be made of the following dimensionless quantities

$$y = (u - u_a)/u_a, \quad x = \varphi/V, \quad c_1 = k\varrho c/H, \quad \tau = t/c\varrho,$$
$$\eta = (\sigma_0 \varrho c V^2)/H u_a, \quad \alpha = \varrho\sigma_0, \quad \beta = c_1/\eta.$$

System (1.1), (1.2) can be rewritten in dimensionless form as follows:

$$\dot{x} = F(x, y, \alpha) := 1 - x - \alpha x s(y) \tag{1.5}$$

$$\left(\dot{x} = \frac{dx}{d\tau}\right)$$

$$\dot{y} = G(x, y, \beta, \eta) := \eta(x^2 s(y) - \beta y). \tag{1.6}$$

By their physical meaning α, β and η must all be positive. Concerning the global behaviour of (1.5), (1.6) we have

Lemma 1.1. *There exists a constant $C = C(\beta, \eta)$ such that*

$$M = \left\{(x, y)\,;\, 0 < x < 1,\, 0 < y,\, \frac{x^2}{2} + \frac{\alpha y}{\eta} < C\right\}$$

forms an invariant absorbing set for the system (1.5), (1.6), *i.e. any solution* $(x(t), y(t))$ *of the problem reaches and remains inside* M *after a finite time.*

Proof. By direct inspection we find from (1.5), (1.6) that for any $(x(0), y(0)) \in \mathbb{R}^2$ there is $t_0 = t_0(|x(0)|, y(0)|)$ such that

$$0 < x(t) < 1 \quad 0 < y(t) \quad \text{for all} \quad t \geqslant t_0. \tag{1.7}$$

Now we can multiply (1.5) by $x(t)$ and (1.6) by $\frac{\alpha}{\eta}$ and, setting $z(t) = \frac{x^2}{2} + \frac{\alpha}{\eta}$, we obtain

$$\dot{z}(t) + \beta\eta z(t) = \frac{2x + (\beta\eta - 2)x^2}{2}. \tag{1.8}$$

Seeing that the right-hand side of (1.8) is bounded for $t \geqslant t_0$ and $\beta\eta > 0$, the standard decay estimates for first order O.D.E. yield

$$z(t) \leqslant C(\beta, \eta) \quad \text{for all} \quad t \geqslant t_1 \tag{1.9}$$

with $t_1 = t_1(|x(0)|, |y(0)|)$ which, along with (1.7), completes the proof. □

2. Nonexistence of periodic solutions

Lemma 2.1. *If*

$$\frac{\mathrm{d}s}{\mathrm{d}y} < \frac{1}{\eta}\big(1 + \alpha s(y) + \eta\beta\big) \quad \textit{for all} \quad y > 0, \tag{2.1}$$

then the only periodic solutions of system (1.5), (1.6) *are the stationary ones.*

Proof. By Lemma 1.1 all possible periodic solutions are contained in M. A direct computation shows that

$$F_x + G_y = -1 - \alpha s(y) + \eta x^2 \frac{\mathrm{d}s}{\mathrm{d}y} - \eta\beta.$$

Since $0 < x < 1$ on M, condition (2.1) implies $F_x + G_y < 0$. Hence, by the Bendixon criterion [1] no periodic solution can exist in M and therefore in the whole phase plane. □

Note that, by virtue of (1.3), the inequality (2.1) is satisfied in the physically relevant case $\mathrm{d}s/\mathrm{d}y < \beta$ for all $y \geqslant 0$.

The stationary points of system (1.5), (1.6) are given by

$$X(x,y,\alpha) := 1 - x - \alpha x s(y) = 0, \tag{3.1}$$

$$Y(x,y,\beta) := x^2 s(y) - \beta y = 0. \tag{3.2}$$

From (3.1) we have

$$x = \delta(y,\alpha) := \frac{1}{1+\alpha s(y)}. \tag{3.3}$$

Substituting into (3.2) we find

$$\beta = H(y,\alpha) := \frac{s(y)}{y\big(1+\alpha s(y)\big)^2}. \tag{3.4}$$

If α is fixed, the plot of (3.4) gives the bifurcation diagram for the solutions of (3.1), (3.2). Let

$$(x,y) = \left(\frac{1}{1+\alpha s(y)}, y\right) \tag{3.5}$$

be any stationary point of (1.5), (1.6) and $A(y,\alpha)$ the 2×2 matrix of the corresponding linearized system. Note that $X_x Y_\beta = \big(1+\alpha s(y)\big)y \neq 0$ if (x,y) is a stationary point. Calculating $\operatorname{Det} A$ and using (3.3), (3.4) we have

$$\eta H_y(y,\alpha) = -\left[\frac{\operatorname{Det} A}{X_x Y_\beta}\right]_{x=\delta(y,\alpha)\beta=H(y,\alpha)}. \tag{3.6}$$

In view of (3.6) we have that the curve of the plane y,α in which $\operatorname{Det} A(y,\alpha)$ vanishes coincides with the locus of points in which $H_y(y,\alpha)$ is zero. The equation of this curve is given by

$$\alpha = f(y) := \frac{1}{s(y)}\frac{y(\mathrm{d}s/\mathrm{d}y) - s(y)}{y(\mathrm{d}s/\mathrm{d}y) + s(y)}. \tag{3.7}$$

The following lemma gives a partial information concerning the stability of the stationary points.

Lemma 3.1. *Let α, β and y satisfy* (3.4) *and assume $\alpha < f(y)$, $ys'(y) + s(y) > 0$. Then the corresponding stationary point* (3.5) *is a saddle point.*

Proof. Simply note that if $\alpha < f(y)$ we have $\operatorname{Det} A(y,\alpha) < 0$ by (3.6). $\square$

R e m a r k 3.1. The degree of the mapping (F, G) evaluated with respect to ∂M is one. Therefore the existence of a saddle point implies the existence of at least one other stationary point. If, in particular, $\operatorname{Det} A \neq 0$, there must be at least two different critical points either attracting or repelling.

To make further progress in the study of the local stability we need the trace, $\operatorname{Tr} A$, of A which is given by

$$\operatorname{Tr} A = \frac{\eta\big(y(\mathrm{d}s/\mathrm{d}y) - s(y)\big) - y\big(1+\alpha s(y)\big)^3}{y\big(1+\alpha s(y)\big)^2}. \tag{3.8}$$

Using the well-known Hopf's bifurcation theorem (see [4]), we can prove

Lemma 3.2. *Suppose $\bar{\alpha}, \bar{\beta}$ and $\bar{y}$ satisfy* (3.4) *and*

$$\bar{\eta} = \frac{\bar{y}\big(1+\bar{\alpha} s(\bar{y})\big)^3}{\bar{y}(\mathrm{d}s/\mathrm{d}y) - s(y)}. \tag{3.9}$$

Assume

$$\bar{\alpha} > f(\bar{y}) \tag{3.10}$$

and

$$\bar{y}(\mathrm{d}s/\mathrm{d}y) - s(\bar{y}) > 0. \tag{3.11}$$

Then there exist two continuous functions $\omega, \eta\colon (0, \xi_0) \to \mathbb{R}^+$, $\omega(\xi) \to \bar{\omega}$, $\eta(\xi) \to \bar{\eta}$ as $\xi \to 0_+$, and a branch of non-constant periodic solutions of system (1.5), (1.6) *(with $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$, $\eta = \eta(\xi)$) with period $T_\xi = 2\pi/\omega(\xi)$ for any $\xi \in (0, \xi_0)$.*

P r o o f. First of all, (3.11) implies $\bar{\eta} > 0$ and, therefore, the result is physically meaningful. The surface of the y, α, η space on which $\operatorname{Tr} A$ vanishes is (3.9). Condition (3.10) guarantees that $\operatorname{Det} A > 0$ by (3.6). Hence, $A(\bar{y}, \bar{\alpha})$ has two purely imaginary complex conjugate eigenvalues $\lambda = \pm \mathrm{i}\bar{\omega}$. By (3.11) we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \operatorname{Re} \lambda(\eta) > 0 \tag{3.12}$$

when η is given by (3.9). All hypotheses of Hopf's bifurcation theorem are satisfied and the lemma holds. $\square$

To prove the stability of these periodic solutions is a more difficult task. A result in this direction is presented in Section 5.

The bifurcation diagram (3.7) may have many shapes corresponding to the possible choices of $s(y)$. We examine in detail the following case which makes it possible to predict the three solutions situation which is typical of the circuit. Assume

$$\frac{d^2 s}{dy^2} \geqslant m > 0 \quad \text{for all} \quad y \geqslant 0, \tag{4.1}$$

$$\frac{ds}{dy}(0) > 0. \tag{4.2}$$

The function $f(y)$ defined in (3.9) vanishes when $y(\mathrm{d}s/\mathrm{d}y) - s(y) = 0$. By (1.3) and (4.1) there exists a unique $y_0 > 0$ such that $f(y_0) = 0$. Suppose $\mathrm{d}f/\mathrm{d}y$ to vanish only when $y = y_M$ with $y_M > y_0$ (see Figure 2). This implies $f(y) > 0$ if $y > y_0$ and $f(y) \to 0$ when $y \to \infty$. Define $\alpha_M = f(y_M)$. We distinguish the following two cases:

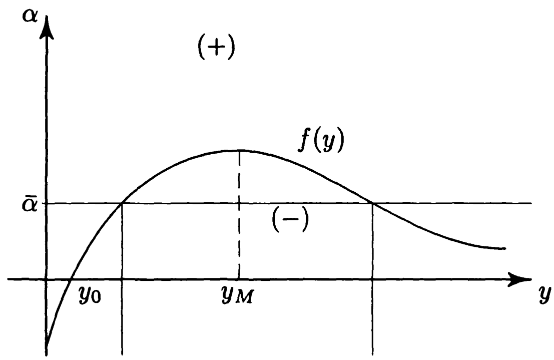

Fig. 2

Case (A). Let

$$0 < \alpha < \alpha_M. \tag{4.3}$$

The bifurcation diagram (3.4) has in this case the characteristic S shape of Figure 3. Moreover, $H(y,\alpha) \to \infty$ when $y \to 0+$ and $H(y,\alpha) \to 0$ when $y \to \infty$. Referring to Figure 2 we see that $\operatorname{Det} A$ is positive in region (+) and negative in region (−) by

(3.6). Let y_1, y_2 ($y_1 < y_2$) be the local minimum and the local maximum of $H(y,\alpha)$

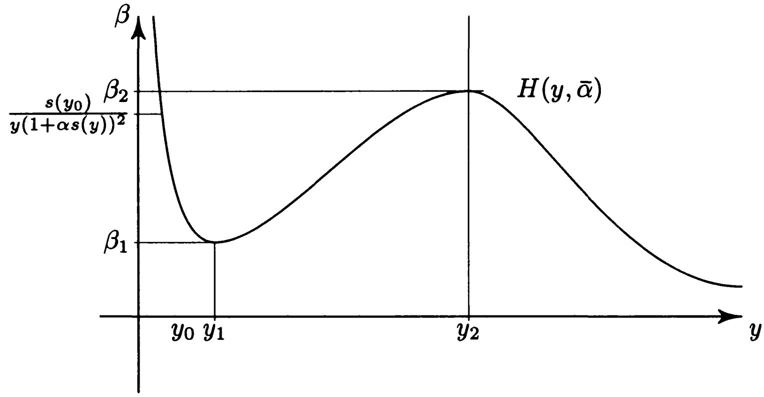

Fig. 3

and put $\beta_1 = H(y_1,\alpha)$, $\beta_2 = H(y_2,\alpha)$. Clearly $\beta_1 < \beta_2$ and $y_0 < y_1 < y_2$. To concentrate on a specific case, let us assume

$$\beta_2 > \frac{s(y_0)}{y_0\big(1+\alpha s(y_0)\big)^2}. \tag{4.4}$$

We consider the following four sub-cases.

($\mathrm{A_I}$) If

$$\beta > \beta_2 \tag{4.5}$$

then there is only one fixed point $(\bar{x}_1, \bar{y}_1)$ with

$$\bar{y}_1 < y_0. \tag{4.6}$$

This implies $\operatorname{Det} A(\bar{y}_1,\alpha) > 0$ and $\operatorname{Tr} A(\bar{y}_1,\alpha) < 0$. Thus $(\bar{x}_1, \bar{y}_1)$ is asymptotically stable for every η.

($\mathrm{A_{II}}$) If

$$\beta_2 > \beta > \frac{s(y_0)}{y_0\big(1+\alpha s(y_0)\big)^2} \tag{4.7}$$

then there are three fixed points $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_2, \bar{y}_2)$ and $(\bar{x}_3, \bar{y}_3)$ such that

$$\bar{y}_1 < y_0 < \bar{y}_2 < \bar{y}_3. \tag{4.8}$$

Since $\operatorname{Det} A(\bar{y}_1, \alpha) > 0$ and $\operatorname{Tr} A(\bar{y}_1, \alpha) < 0$, $(\bar{x}_1, \bar{y}_1)$ is asymptotically stable for all η. Moreover, $\operatorname{Det} A(\bar{y}_2, \alpha) < 0$ thus $(\bar{x}_2, \bar{y}_2)$ is a saddle point in accordance with Lemma 3.1. Finally, $\operatorname{Det} A(\bar{y}_3, \alpha) > 0$ and the sign of $\operatorname{Tr} A(\bar{y}_3, \alpha)$ (and therefore the stability) depends only on the numerical value of η.
(A_{III}) If

$$\frac{s(y_0)}{y_0\big(1+\alpha s(y_0)\big)^2} > \beta > \beta_1 \tag{4.9}$$

then there are again three stationary points, but now

$$y_0 < \bar{y}_1 < \bar{y}_2 < \bar{y}_3. \tag{4.10}$$

The stationary point $(\bar{x}_2, \bar{y}_2)$ is again a saddle point. Since $\operatorname{Det} A(\bar{x}_1, \bar{y}_1) > 0$ and $\operatorname{Det} A(\bar{x}_3, \bar{y}_3) > 0$, the stationary points $(\bar{x}_1, \bar{y}_1)$, $(\bar{x}_3, \bar{y}_3)$ can be asymptotically stable or not depending on the value of η.
(A_{IV}) When

$$\beta_1 < \beta, \tag{4.11}$$

then there is only one fixed point $(\bar{x}, \bar{y})$ with $\operatorname{Det} A(\bar{y}, \alpha) > 0$. The sign of $\operatorname{Tr} A$ and therefore the stability depends only on η.
Case(B). If

$$\alpha_M < \alpha \tag{4.12}$$

then there is one and only one stationary point for all $\beta > 0$ and $\operatorname{Det} A(\bar{y}, \alpha) > 0$. The asymptotic stability depends on the value of η.

Remark 4.1. It is interesting to note that the existence of three nondegenerate stationary points such that one of them is a saddle, while the other one is a repellor, necessarily brings about the existence of at least one closed trajectory, either a homoclinic orbit or a periodic one.

As a consequence of the Poincaré-Bendixon theorem [1] we have

Lemma 5.1. *If the invariant absorbing set M guaranteed by Lemma 2.1 contains exactly one unstable stationary point $(\bar{x}, \bar{y})$, this point is surrounded by (at least one) stable periodic orbit.*

To apply the above Lemma we note that a sufficient condition for the existence of a unique stationary point $(\bar{x}, \bar{y})$ is given by

$$\frac{\mathrm{d}s}{\mathrm{d}y} > 0 \quad \text{for all} \quad y > 0, \tag{5.1}$$

$$\alpha s(0) \geqslant 1. \tag{5.2}$$

Assume further that there exists an interval (y_1, y_2) such that

$$y\frac{\mathrm{d}s}{\mathrm{d}y} - s(y) > 0 \quad \text{for all} \quad y \in (y_1, y_2). \tag{5.3}$$

From (3.4) it follows that

$$\beta = \frac{s(y)}{y\big(1 + \alpha s(y)\big)^2}. \tag{5.4}$$

Hence, there exists an interval (β_1, β_2) such that for any $\beta \in (\beta_1, \beta_2)$ we have $\bar{y} \in (y_1, y_2)$. Thus, recalling the expression for $\operatorname{Tr} A$, i.e. (3.8), there exists $\eta_0 = \eta_0(\alpha, \beta, s)$ having the following properties: if $\eta < \eta_0$ the unique stationary point is locally stable, if $\eta > \eta_0$ the stationary point is unstable and therefore by Lemma 5.1, M contains a stable limit cycle.

Under the assumptions of Lemma 3.2 it is also possible to prove the existence of periodic solutions using the following version of Hopf's bifurcation.

Theorem 5.1. *Assume the system*

$$\dot{x} = X(x, y, \eta) \quad \dot{y} = Y(x, y, \eta) \tag{5.5}$$

has a fixed point $(\bar{x}, \bar{y})$ for all values of the parameter η. Furthermore, suppose the eigenvalues of the linearized system $\lambda_1(\eta)$, $\lambda_2(\eta)$ are purely imaginary complex conjugate when $\eta = \eta_0$. If the real part of the eigenvalues satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\operatorname{Re}[\lambda(\eta)] > 0 \quad \textit{when} \quad \eta = \eta_0 \tag{5.6}$$

and $(\bar{x}, \bar{y})$ is asymptotically stable when $\eta = \eta_0$, then

a) *there exists $\eta_1 < \eta_0$ such that when $\eta \in (\eta_1, \eta_0)$, $(\bar{x}, \bar{y})$ is a stable focus;*

b) *there exists $\eta_2 > \eta_0$ such that when $\eta \in (\eta_0, \eta_2)$, $(\bar{x}, \bar{y})$ is an unstable focus surrounded by a stable limit cycle.*

A proof of Theorem 5.1 is to Chaffee and can be found in [3], Theorems (3.1), (3.B4). Let the hypotheses of Lemma 3.2 hold. To apply Theorem 5.1 it remains to check if the fixed point $(\bar{x}, \bar{y})$ is, in certain cases, asymptotically stable when η is given by (3.9). We can use (see [3]) the following algorithm:

a) translate the fixed point to the origin,

b) find a non-singular matrix M such that

$$M^{-1}AM = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

where A is the matrix of the linearized system and $\lambda = \pm i\omega$,

c) transform the system by the change of variable $\mathbf{x} = M\mathbf{y}$, $\mathbf{x} = (x, y)$, $\mathbf{y} = (u, v)$ into

$$\dot{u} = -\omega v + f(u, v) \quad \dot{v} = \omega u + g(u, v)$$

with $f(0,0) = g(0,0) = 0$, $Df(0,0) = Dg(0,0) = 0$,

d) calculate the index

$$\begin{aligned} a = {} & f_{uuu} + f_{uuv} + g_{uuv} + g_{vvv} \\ & + (1/\omega)[f_{uu}(f_{uu} + f_{vv}) + g_{uu}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}] \end{aligned}$$

where all partial derivatives are computed at $(u, v) = (0, 0)$. If $a < 0$, $(\bar{x}, \bar{y})$ is asymptotically stable and the conclusion of Theorem 5.1 follows. The calculations involved in computing a in the present case are rather tedious but straightforward. A computer program using the language for symbolic manipulation MAXIMA has been written to obtain an explicit expression of a in terms of α, β. This formula is however too massive to be reported here. We simply use it in the following

E x a m p l e. Let $s(y) = \exp(y)$, $\alpha = 1$ and $\beta = 15$. We find $(\bar{x}, \bar{y}) = (0.23305, 1.19142)$ and $\eta_0 = 149.471$. Using our MAXIMA code we find $a = -528\,634.05897$. The periodic solution given by Lemma 3.2 is therefore stable.

To get a more accurate description we can treat the thermistor as a three-dimensional body, represented by a bounded domain Ω of $\mathbb{R}^3$. The metallic, and therefore equipotential electrodes of the thermistor are assumed to be two disjoint surfaces S and $\bar{S}$ such that $\partial\Omega = S \cup \bar{S}$. If $\mathbf{J}$ and $\mathbf{E}$ are the current density and the electric field in Ω we have, by Ohm's law,

$$\mathbf{J} = \sigma(u)\mathbf{E}. \tag{6.1}$$

If $\varphi(x,t), x \in \Omega$ is the electric potential inside Ω, we can write, assuming quasistatic conditions,

$$\mathbf{E} = -\nabla\varphi. \tag{6.2}$$

By the law of conservation of charge we have

$$\nabla \cdot \mathbf{J} = 0. \tag{6.3}$$

Let $i(t)$ be the current crossing R. We have

$$i(t) = c\dot{\Phi}(t) + \int_S \sigma(u)\frac{\mathrm{d}\varphi}{\mathrm{d}n}\,\mathrm{d}S \tag{6.4}$$

where $\mathbf{n}$ is the unit vector normal to S and

$$\varphi = \Phi(t) \quad \text{on } S, \quad \varphi = 0 \quad \text{on } \bar{S}. \tag{6.5}$$

Hence

$$V = \varrho i(t) + \Phi(t). \tag{6.6}$$

Assuming the usual heat equation to be valid and inserting (6.1) and (6.2) into (6.3), we arrive at the following problem:

To find a period T and three T-periodic functions $\varphi(x,t)$, $u(x,t)$ and $\Phi(t)$ such that

$$\nabla \cdot (\sigma(u)\nabla\psi) = 0 \tag{6.7}$$

$$\varphi = \Phi(t) \quad \text{on } S, \quad \varphi = 0 \quad \text{on } \bar{S}. \tag{6.8}$$

$$V = \varrho c\dot{\Phi}(t) + \varrho\int_S \sigma(u)\frac{\mathrm{d}\varphi}{\mathrm{d}n}\,\mathrm{d}S + \Phi(t) \tag{6.9}$$

$$u_t = a_1\Delta u + a_2\sigma(u)|\nabla\varphi|^2 \tag{6.10}$$

$$u = 0 \quad x \in \partial\Omega \tag{6.11}$$

where $a_2 = 1/dc_v$ (d mass density, c_v specific heat), whereas $a_1 = \kappa/a_2$ (κ thermal conductivity) is the diffusivity. However, the application of the theory of Hopf's bifurcation to problem (6.7)–(6.11) seems to present serious difficulties.

Acknowledgement. It is pleasure to thank S. Steffè for his help in preparing the computer programs.*

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* The work of E. F. was supported by the Grant No. 119102 of the Grant Agency of AV ČR.