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# SHAPE OPTIMIZATION OF ELASTIC AXISYMMETRIC PLATE ON AN ELASTIC FOUNDATION

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*Summary.* An elastic simply supported axisymmetric plate of given volume, fixed on an elastic foundation, is considered. The design variable is taken to be the thickness of the plate. The thickness and its partial derivatives of the first order are bounded.

The load consists of a concentrated force acting in the centre of the plate, forces concentrated on the circle, an axisymmetric load and the weight of the plate.

The cost functional is the norm in the weighted Sobolev space of the deflection curve of radius.

Existence of a solution of the optimization problem of the state problem is proved. Approximate problem is introduced and convergence of its solutions to that of the continuous problem is established.

*Keywords*: shape optimization, axisymmetric elliptic problems, elasticity

*AMS classification*: 73K10, 73K40, 73C99

## INTRODUCTION

This work concerns the optimal design of the circular axisymmetrically loaded te. The theme of this work stems from [5], where the optimal design of the beam is considered. This problem is two-dimensional.

We consider an elastic circular plate of variable thickness, which is fixed on an elastic foundation. The function of thickness is optimized, under the condition of constant volume of the plate, in the class of Lipschitz functions bounded simultaneously from below by a strictly positive constant and from above. Its own weight, a concentrated force acting at the centre, forces concentrated on the circle and the so-called rotational symmetrical load act on the plate.

In view of the fact that the body is loaded and supported axisymmetrically it is suitable to formulate the problem in polar coordinates $r$, $\theta$. Then the displacement, the deformation and the strain do not depend on the angle $\theta$ and we get a one-dimensional boundary problem. The bilinear form and the linear functional have a special form in polar coordinates, therefore we define the space of functions with finite energy as a weighted Sobolev space with different powers at different derivatives. The weight is such that in the books [7], [8] there are no theorems suitable for this weighted space. A certain characteristic is proved with help of an isometric isomorphism between the space of functions with finite energy and the original two-dimensional Hilbert space (see Lemma 1.1).

The cost functional is defined as the norm in the weighted Sobolev space of the deflection curve of radius.

In Section 1 we formulate an optimal design problem, transform it to polar coordinates and define a weak formulation of the problem in the one-dimensional space of functions with finite energy. Further we prove the existence of the solution.

In Section 2 we study the numerical approximation of the problem and the convergence of approximate solutions to the solution.

In Appendix we study the characteristics of the weighted space of functions with finite energy and the characteristics of the space of finite elements we use in numerical analysis.

## 1. Formulation of the problem and the existence theorem

### 1.1. Formulation of the problem.

Let us consider an elastic isotropic homogeneous axisymmetrical plate occupying the region

$$G = \left\{(x, y, z);\ (x, y) \in \Omega,\ -\frac{t}{2} < z < \frac{t}{2}\right\},$$

where

$$\Omega = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < R\}$$

and the thickness $t$ is constant.

We define a Cartesian coordinate system in the following way: the origin of the coordinate system is in the middle of the height of the cylinder, the axis $z$ corresponds to the height of the cylinder positively orientated upward, the axes $x$ and $y$ are parallel with the base of the cylinder.

We assume that the plate is simply supported along its circumference and the axisymmetric load acts in the direction of the $z$ axis. The orientation of the load is positive upward along the $z$-axis.

We use Kirchhoff's hypothesis about invariance of normal elements and look for approximate functions of displacement in the form

$$\begin{aligned} u_t &= -\frac{\partial w}{\partial x}\, z, \\ v_t &= -\frac{\partial w}{\partial y}\, z, \\ w_t &= w, \end{aligned}$$

where $u_t$, $v_t$, $w_t$ denote the components of displacement in the directions $x$, $y$, $z$ and $w\colon \Omega \to \mathbb{R}$ is an unknown function of two variables.

In the standard way (see e.g. the case of simply supported plate in Chapter 10.4.3 of [4]) we define the space

$$W = \{v \in W^{2,2}(\Omega)\,;\ v = 0 \text{ on } \partial\Omega\}.$$

Recall that $W^{2,2}(\Omega)$ is a Sobolev space with the norm $\|.\|_{2,2}$ defined by

$$\|u\|_{2,2} = \left( \int_\Omega \left( \sum_{|\alpha|=2} (D^\alpha u)^2 + u^2 \right) \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{2}}$$

(see e.g. [6] Theorem 30.4).

The bilinear form corresponding to Kirchhoff's hypothesis of a plate resting on a coherent elastic foundation with resistance of subgrade $a_0$ is defined on $W$ by the formula

$$\mathcal{A}(t; w, \varphi) = \int_\Omega \left( -M_{11}(w) \frac{\partial^2 \varphi}{\partial x^2} - 2M_{12}(w) \frac{\partial^2 \varphi}{\partial x \partial y} - M_{22}(w) \frac{\partial^2 \varphi}{\partial y^2} + a_0 w \varphi \right) \mathrm{d}x\, \mathrm{d}y,$$

where $a_0 > 0$ is a constant and

$$\begin{aligned} M_{11}(w) &= -D(t) \left( \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right), \\ M_{22}(w) &= -D(t) \left( \frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} \right), \\ M_{12}(w) &= -D(t)\, (1 - \sigma)\, \frac{\partial^2 w}{\partial x \partial y} \end{aligned}$$

denote the bending moments of the plate (see e.g. [4], Chapter 10.4.3),

$$D(t) = \frac{E\, t^3}{12\, (1 - \sigma^2)},$$

$E$ is Young's modulus of elasticity and $\sigma$ Poisson's constant ($E = \text{const.}$, $E > 0$, $\sigma = \text{const.}$, $0 < \sigma < \frac{1}{2}$).

We define a linear functional on $W$ as follows:

$$\langle F(t), \varphi \rangle_A = P_0 \varphi(0,0) + \sum_{i=1}^{k} P_i \int_{\gamma_i} \varphi \, d\gamma + \int_{\Omega} (f - pt) \varphi \, dx \, dy,$$

where $P_0 \in \mathbb{R}$ describes the concentrated force acting at the origin, $P_i \in \mathbb{R}$ describes the force concentrated on the circle $\gamma_i$ with centre at the origin and radius $R_i \in (0, R)$, $f \in L_1(\Omega)$ is the distributed load and $pt$ is the weight of the plate ($p = \text{const.}$, $p > 0$).

Now we can write the variational formulation of the state problem as

$$\mathcal{A}(t; w, \varphi) = \langle F(t), \varphi \rangle_A \quad \forall \varphi \in W, \tag{1.1}$$

where $w \in W$ is to be found.

The investigated plate is circular, therefore we transform the bilinear form $\mathcal{A}(t; w, \varphi)$ and the linear functional $\langle F(t), \varphi \rangle_A$ to polar coordinates by the transformation

$$x = r \cos \theta,$$
$$y = r \sin \theta.$$

Let the load $f$ be axisymmetric. Then we can assume that the functions $w$, $\varphi$ depend only on the radius $r$. We denote the derivative with respect to $r$ by primes (e.g. $\frac{\partial w}{\partial r} = w'$). For simplification we include the product $E \frac{2\pi}{12(1-\sigma^2)}$ in one constant and denote it again as $E$. Thus we obtain

$$\begin{aligned} A(t; w, \varphi) = E \int_0^R r\, t^3 w'' \varphi'' \, dr + \sigma E \int_0^R t^3 (w'' \varphi' + w' \varphi'') \, dr \\ + E \int_0^R \frac{1}{r} \, t^3 w' \varphi' \, dr + 2\pi a_0 \int_0^R r\, w \varphi \, dr \end{aligned}$$

and

$$\langle F(t), \varphi \rangle = P_0 \varphi(0) + 2\pi \sum_{i=1}^{k} P_i R_i \varphi(R_i) + 2\pi \int_0^R r (f - pt) \varphi \, dr.$$

We define the space of functions with finite energy as the weighted Sobolev space $W^{2,2}((0, R), \varrho_{(r)})$ (see [7] p. 10) with the norm

$$\|v\|_{2,2,\varrho_{(r)}} = \left( \int_0^R r (v'')^2 \, dr + \int_0^R \frac{1}{r} (v')^2 \, dr + \int_0^R r\, v^2 \, dr \right)^{\frac{1}{2}}.$$

We have to prove that the terms $A(t;w,\varphi)$ and $\langle F(t),\varphi\rangle$ are finite for functions $w,\varphi \in W^{2,2}((0,R),\varrho_{(r)})$. It is evident that the first, third and fourth integral in the term $A(t;w,\varphi)$ and the integral in the term $\langle F(t),\varphi\rangle$ are finite. It is necessary to prove that the second integral in the formula $A(t;w,\varphi)$ is finite as well.

If $w \in W^{2,2}((0,R),\varrho_{(r)})$ then $\sqrt{r}w'' \in L_2(0,R)$, in a similar way we obtain from $\varphi \in W^{2,2}((0,R),\varrho_{(r)})$ that if $\frac{1}{\sqrt{r}}\varphi' \in L_2(0,R)$ then $\int_0^R w''\varphi' \,\mathrm{d}r < \infty$ which means that $w''\varphi' \in L_1(0,R)$. In the same way we prove $\int_0^R w'\varphi'' \,\mathrm{d}r < \infty$.

We denote

$$\vartheta = \{v \in C^\infty([0,R])\,;\ \operatorname{supp} v' \cap \{0\} = \emptyset,\ v(R) = 0\}.$$

Let $V$ be the closure of the set $\vartheta$ in the space $W^{2,2}((0,R),\varrho_{(r)})$.

**Lemma 1.1.** *The set $V$ with the scalar product*

$$(u,v) = \int_0^R r\,u''v''\,\mathrm{d}r + \int_0^R \frac{1}{r}\,u'v'\,\mathrm{d}r + \int_0^R r\,uv\,\mathrm{d}r$$

*is a Hilbert space.*

Proof. Since $V$ is a closed subspace $W^{2,2}((0,R),\varrho_{(r)})$, it suffices to prove that $W^{2,2}((0,R),\varrho_{(r)})$ is complete.

Let $v_n$ be a Cauchy sequence in $W^{2,2}((0,R),\varrho_{(r)})$. Let $W^{2,2}_{\mathrm{sym}}(\Omega)$ be the subspace of $W^{2,2}(\Omega)$ of axisymmetric functions. We define a mapping

$$Z\colon W^{2,2}((0,R),\varrho_{(r)}) \to W^{2,2}_{\mathrm{sym}}(\Omega)$$

by the formula

$$Z\colon v \mapsto \hat{v},\ \text{where}\ \ \hat{v}(x(r,\theta),y(r,\theta)) = v(r) \quad \forall r \in [0,R] \quad \forall \theta \in [0,2\pi).$$

This mapping is an isomorphism (that is an injective linear mapping of $W^{2,2}((0,R),\varrho_{(r)})$ onto $W^{2,2}_{\mathrm{sym}}(\Omega)$). We have

$$\|v\|_{2,2,\varrho_{(r)}} = \frac{1}{\sqrt{2\pi}}\,\|Z(v)\|_{2,2},$$

thus $Z(v_n)$ is a Cauchy sequence in the space $W^{2,2}(\Omega)$. As this space is complete, there exists a $w \in W^{2,2}(\Omega)$ such that $\lim\limits_{n\to\infty} Z(v_n) = w$. The space $W^{2,2}_{\mathrm{sym}}(\Omega)$ is closed in $W^{2,2}(\Omega)$ (see Appendix, Lemma 3.1), therefore $w \in W^{2,2}_{\mathrm{sym}}(\Omega)$. The mapping $Z$ is an isomorphism; therefore there exists a $Z^{-1}(w) \in W^{2,2}((0,R),\varrho_{(r)})$ such that $\lim\limits_{n\to\infty} v_n = Z^{-1}(w)$ in the space $W^{2,2}((0,R),\varrho_{(r)})$. □

Now we assume that the plate thickness is not constant over the whole area $\Omega$. Let it be axisymmetric. Then we can assume that it is possible to express the thickness of the plate by a function $t \in U_{ad}$, where

$$U_{ad} = \Big\{ t \in C^{(0),1}([0,R])\,;\ |t'| \leqslant c_1,\ 0 < t_{\min} \leqslant t(r) \leqslant t_{\max},\quad r \in [0,R],$$
$$2\pi \int_0^R r\, t(r)\,\mathrm{d}r = c_2,\ c_1, c_2 > 0 \Big\}.$$

We will consider the following variational formulation:
for each $t \in U_{ad}$ find $w \in V$ such that

$$A(t; w, \varphi) = \langle F(t), \varphi\rangle \qquad \forall \varphi \in V. \tag{1.2}$$

In what follows we will show that the state problem (1.2) has a unique solution $w(t)$ for any $t \in U_{ad}$.

Now we define the cost functional

$$j(w) = \|w\|^2_{2,2,\varrho(r)}.$$

The functional $j$ means the "average size" of the deflection function and its second derivative in the original two-dimensional problem (1.1).

We will solve the **problem of the optimal design:** We define

$$J(t) = j(w(t)),$$

where $w(t)$ is the solution of problem (1.2) and look for optimal design $t^0 \in U_{ad}$, satisfying the condition

$$J(t^0) = \min_{t \in U_{ad}} J(t). \tag{1.3}$$

**1.2. Existence Theorem.**

We apply the abstract theorem on existence of an optimal design (see [3]). First we set out its assumptions.

Let $U$ be a Banach space of controls and $U_{ad}$ a set of admissible design variables. Assume that $U_{ad}$ is compact in $U$.

Let a Hilbert space $V$ be given with a norm $\|\cdot\|$. Consider a bilinear form $A(t;.,.)$ and a linear continuous functional $\langle F(t), \cdot\rangle$ on $V$, both depending on a parameter $t \in U$. Assume that there exist positive constants $\alpha_0$, $\alpha_1$ and a subset $U^0$, $U_{ad} \subset U^0 \subset U$, independent of $t$, $u$, $v$ and such that

$$A(t; u, v) \leqslant \alpha_1 \|u\|\,\|v\|, \tag{1.4}$$

$$A(t; u, u) \geqslant \alpha_0 \|u\|^2 \tag{1.5}$$

hold for all $t \in U^0$ and $u, v \in V$.

Moreover, assume that:

(1.6) if $t, t_n \in U^0$, $t_n \to t$ in $U$ and $u_n \rightharpoonup u$ (weakly) in $V$ for $n \to \infty$ then $A(t_n; u_n, v) \to A(t; u, v)$ for all $v \in V$;

(1.7) if $t, t_n \in U^0$, $t_n \to t$ in $U$, then $\langle F(t_n), v\rangle \to \langle F(t), v\rangle$ for all $v \in V$ .

there exists a positive constant $\gamma$, independent of $t$, $v$ and such that

(1.8) $$|\langle F(t), v\rangle| \leqslant \gamma\|v\|$$

holds for all $t \in U^0$ and $v \in V$.

We consider the following state problem: for $t \in U_{ad}$ find $u(t) \in V$ such that

(1.9) $$A(t; u(t), v) = \langle F(t), v\rangle \qquad \forall v \in V.$$

Under the assumptions (1.4), (1.5), (1.8), the state problem (1.9) is uniquely solvable for any $t \in U^0$.

Let a functional

$$j\colon (U \times V) \to \mathbb{R}$$

be given, which satisfies the condition

(1.10) $$t_n,\ t \in U^0,\ t_n \to t \text{ in } U,\ u_n \rightharpoonup u \text{ in } V \text{ (weakly)} \implies \liminf_{n\to\infty} j(t_n, u_n) \geqslant j(t, u).$$

Defining the cost functional as

$$J(t) = j(t, u(t)),$$

where $u(t)$ denotes the solution of (1.9), we may consider the following optimal design problem:

(1.11) $$\text{find } t^0 \in U_{ad} \text{ such that } J(t^0) \leqslant J(t) \quad \forall t \in U_{ad}.$$

**Theorem 1.1.** *Under the assumptions* (1.4)–(1.8) *and* (1.10), *the optimal design problem* (1.11) *has at least one solution.*

Proof. See [3], p. 270. □

The existence of the solution of our optimal design problem (1.3) is claimed in the following theorem.

**Theorem 1.2.** *The problem* (1.3) *has at least one solution.*

Proof. It is sufficient to verify the assumptions of Theorem 1.1. Let us introduce $U = C([0,R])$, $U^0 = \{t \in U;\, t_{\min} \leqslant t(r) \leqslant t_{\max}\ \forall r \in [0,R]\}$. The set $U_{ad}$ is bounded and closed in $C([0,R])$ and, moreover, consists of uniformly continuous functions. The theorem of Arzelà-Ascoli implies the compactness of $U_{ad}$ in $C([0,R])$.

The space $V$ is a Hilbert space (see Lemma 1.1).

We denote by $|\,.\,|_{k,2,\varrho}$ the $k$-th seminorm of the space $W^{2,2}((0,R);\varrho_{(r)})$, i.e.

$$|w|_{0,2,r} = \left(\int_0^R r\,w^2\,\mathrm{d}r\right)^{\frac12},$$

$$|w|_{1,2,\frac1r} = \left(\int_0^R \frac1r\,(w')^2\,\mathrm{d}r\right)^{\frac12},$$

$$|w|_{2,2,r} = \left(\int_0^R r\,(w'')^2\,\mathrm{d}r\right)^{\frac12}.$$

Then

$$\|w\|_{2,2,\varrho_{(r)}} = \left(|w|^2_{2,2,r} + |w|^2_{1,2,\frac1r} + |w|^2_{0,2,r}\right)^{\frac12}.$$

If $w,\varphi \in W^{2,2}((0,R),\varrho_{(r)})$ then

$$\begin{aligned}(1.12)\quad |A(t;w,\varphi)| &\leqslant K\Bigg(\int_0^R \left(|\sqrt{r}w''|\,|\sqrt{r}\varphi''|\right)\mathrm{d}r\\ &\quad+\int_0^R \left(|\sqrt{r}w''|\,\Big|\frac{1}{\sqrt r}\varphi'\Big| + \Big|\frac{1}{\sqrt r}w'\Big|\,|\sqrt{r}\varphi''|\right)\mathrm{d}r\\ &\quad+\int_0^R \left(\Big|\frac{1}{\sqrt r}w'\Big|\,\Big|\frac{1}{\sqrt r}\varphi'\Big|\right)\mathrm{d}r + \int_0^R \left(|\sqrt{r}w|\,|\sqrt{r}\varphi|\right)\mathrm{d}r\Bigg)\\ &\leqslant K(|w|_{2,2,r}|\varphi|_{2,2,r} + |w|_{2,2,r}|\varphi|_{1,2,\frac1r} + |w|_{1,2,\frac1r}|\varphi|_{2,2,r}\\ &\quad+|w|_{1,2,\frac1r}|\varphi|_{1,2,\frac1r} + |w|_{0,2,r}|\varphi|_{0,2,r})\\ &\leqslant 2K\,\|w\|_{2,2,\varrho_{(r)}}\|\varphi\|_{2,2,\varrho_{(r)}},\end{aligned}$$

where $K = \max(E\,t^3_{\max}, 2\pi\,a_0)$. In the last expression we have used the Hölder inequality and the following inequality for the scalar product in $\mathbb{R}^2$:

$$(|w|_{2,2,r} + |w|_{1,2,\frac1r}, |w|_{0,2,r})(|\varphi|_{2,2,r} + |\varphi|_{1,2,\frac1r}, |\varphi|_{0,2,r}) \leqslant 2\|w\|_{2,2,\varrho_{(r)}}\|\varphi\|_{2,2,\varrho_{(r)}}.$$

Further we have

$$(1.13)\ A(t;w,w) = E\int_0^R t^3\Big[r(w'')^2 + 2\sigma w''w' + \frac{1}{r}(w')^2\Big]\,dr + 2\pi a_0\int_0^R rw^2\,dr$$

$$= E\int_0^R t^3\Big[\Big(\sigma r(w'')^2 + 2\sigma w''w' + \frac{\sigma}{r}(w')^2\Big)$$

$$+ (1-\sigma)\Big(r(w'')^2 + \frac{1}{r}(w')^2\Big)\Big]\,dr + 2\pi a_0\int_0^R rw^2\,dr$$

$$\geqslant E\,t_{\min}^3(1-\sigma)\,[\,|w|^2_{2,2,r} + |w|^2_{1,2,\frac{1}{r}}\,] + 2\pi\,a_0\,|w|^2_{0,2,r}$$

$$\geqslant \min(E\,t_{\min}^3(1-\sigma),\ 2\pi\,a_0)\,\|w\|^2_{2,2,\varrho(r)}$$

and (1.4), (1.5) are fulfilled.

We shall now consider (1.6). We can write $t_n = t + e_n$, $e_n \in C([0,R])$. We know that $t_n \to t$ in $U$. Then for all $\varepsilon > 0$ there exists $n$ such that

$$\|e_n\|_{C([0,R])} \leqslant \varepsilon.$$

Let us choose $\varepsilon < \min(1, \frac{1}{2}t_{\min})$. Let $\{w_n\}$, $w_n \rightharpoonup w$ (weakly) be a sequence. The assumption of weak convergence implies boundedness of the sequence $\{w_n\}_{n=1}^{\infty}$ in $W^{2,2}((0,R),\varrho_{(r)})$.

We can write

$$E\int_0^R r\,t_n^3 w_n''\varphi''\,dr = E\int_0^R r(t^3 + 3t^2e_n + 3te_n^2 + e_n^3)w_n''\varphi''\,dr,$$

$$\sigma E\int_0^R t_n^3(w_n''\varphi' + w_n'\varphi'')\,dr = \sigma E\int_0^R (t^3 + 3t^2e_n + 3te_n^2 + e_n^3)(w_n''\varphi' + w_n'\varphi'')\,dr,$$

$$E\int_0^R \frac{1}{r}t_n^3 w_n'\varphi'\,dr = E\int_0^R \frac{1}{r}(t^3 + 3t^2e_n + 3te_n^2 + e_n^3)w_n'\varphi'\,dr.$$

The inequalities

$$\Big|E\int_0^R r(3t^2e_n + 3te_n^2 + e_n^3)w_n''\varphi''\,dr\Big| \leqslant 3E\max(3t_{\max}^2, 3t_{\max}, 1)\varepsilon|w_n|_{2,2,r}|\varphi|_{2,2,r},$$

$$\Big|\sigma E\int_0^R (3t^2e_n + 3te_n^2 + e_n^3)(w_n''\varphi' + w_n'\varphi'')\,dr\Big|$$

$$\leqslant 3\sigma E\max(3t_{\max}^2, 3t_{\max}, 1)\varepsilon(|w_n|_{2,2,r}|\varphi|_{1,2,\frac{1}{r}} + |w_n|_{1,2,\frac{1}{r}}|\varphi|_{2,2,r}),$$

$$\Big|E\int_0^R \frac{1}{r}(3t^2e_n + 3te_n^2 + e_n^3)w_n'\varphi'\,dr\Big| \leqslant 3E\max(3t_{\max}^2, 3t_{\max}, 1)\varepsilon|w_n|_{1,2,\frac{1}{r}}|\varphi|_{1,2,\frac{1}{r}}$$

imply that

$$(1.14)\qquad |A(t_n;w_n,\varphi) - A(t;w_n,\varphi)| \leqslant C\varepsilon\|w_n\|_{2,2,\varrho(r)}\|\varphi\|_{2,2,\varrho(r)},$$

where the constant $C$ is independent of $n$. The sequence $\{w_n\}_{n=1}^{\infty}$ is bounded and $0 < \varepsilon < \min(1, \frac{1}{2}t_{\min})$ is arbitrary. This is why it suffices to show that

$$A(t; w_n, \varphi) \to A(t; w, \varphi) \qquad \text{for } n \to \infty.$$

Let $V'$ denote the dual space to $V$. Further, (1.12) implies

$$A(t; w_n, \varphi) \leqslant 2\max(E\, t_{\max}^3, 2\pi\, a_0)\|w_n\|_{2,2,\varrho(r)} \|\varphi\|_{2,2,\varrho(r)}.$$

We get $A(t; ., \varphi) \in V'$ for all $\varphi \in V$ and $\lim_{n\to\infty} A(t; w_n, \varphi) = A(t; w, \varphi)$ holds for all $\varphi \in V$ as we wanted to prove.

To prove (1.7) let us estimate

$$\begin{aligned} (1.15)\qquad |\langle F(t_n), \varphi\rangle - \langle F(t), \varphi\rangle| &\leqslant 2\pi p \int_0^R |r(t_n - t)\varphi| \,\mathrm{d}r \\ &\leqslant 2\pi p \Big( \int_0^R r\varphi^2 \,\mathrm{d}r \Big)^{\frac{1}{2}} \Big( \int_0^R r(t_n - t)^2 \,\mathrm{d}r \Big)^{\frac{1}{2}} \\ &\leqslant \sqrt{2}\pi pR |\varphi|_{0,2,r} \|t_n - t\|_{C([0,R])}. \end{aligned}$$

The inequality

$$|\langle F(t), \varphi\rangle| \leqslant 2\pi \left( |P_0|/2\pi + \sum_{i=1}^{k} |P_i| R_i + R^2 p\, t_{\max} + R\, \|f\|_{L_1(0,R)} \right) \|\varphi\|_{C([0,R])}$$

and the continuous embedding $W^{2,2}((0,R); \varrho_{(r)}) \hookrightarrow C([0,R])$ (see Appendix, Lemma 3.2) give (1.8). The linearity of the functional $\langle F(t), .\rangle$ together with (1.8) give its continuity. We have to verify (1.10), but this is simple because the square of the norm $\|w\|_{2,2,\varrho(r)}$ is a weak lower semicontinuous functional. □

## 2. Approximation of the problem

We use the following four assumptions for approximations of the problem:

(I) Let $N(h)$ be an integer and $\tau_h$ a partition of the interval $[0, R]$ into $N(h)$ subintervals $\Delta_j = [X_{j-1}, X_j]$ of the length $h$; $j = 1, 2, \ldots, N(h)$; $X_0 = 0$, $X_{N(h)} = R$. In the subsequent text we shall consider only $h$ that lead to a uniform partition of the interval.

Let $P_k(\Delta)$ be the set of polynomials the order of which is at most $k$. We define

$$U_{ad}^h = \{t \in U_{ad};\, t|_{\Delta_j} \in P_1(\Delta_j)\ \forall j\ \}, V_h = \{v \in V;\ v|_{\Delta_j} \in P_3(\Delta_j)\ \forall j\ \}.$$

Note that if the polynomial is $ar^3 + br^2 + cr + d \in V_h$, where $a$, $b$, $c$, $d$ are constants, then $c = 0$ (in the opposite case $|ar^3 + br^2 + cr + d|_{1,2,\frac{1}{r}} = \infty$). It means $v'(0_+) = 0$ in $V_h$.

(II) Instead of $A(t_h; w_h, \varphi_h)$ we shall use the form

$$A_h(t_h; w_h, \varphi_h) = \sum_{j=1}^{N(h)} \left[ Et_h^3(\xi_j) \left( \int_{\Delta_j} r w_h'' \varphi_h'' \, dr + \sigma \int_{\Delta_j} (w_h'' \varphi_h' + w_h' \varphi_h'') \, dr + \int_{\Delta_j} \frac{1}{r} w_h' \varphi_h' \, dr \right) + 2\pi a_0 \int_{\Delta_j} r \, w_h \varphi_h \, dr \right],$$

where $\xi_j = \frac{1}{2}(X_{j-1} + X_j)$, $t_h \in U_{ad}^h$, $w_h, \varphi_h \in V_h$.

(III) Assume that there exist open subintervals $D_l$ such that $\bigcup_{l=1}^{l_0} \overline{D}_l = [0, R]$, $D_l \cap D_m = \emptyset$, $l \neq m$, $l_0 \geqslant 1$ is finite, and for any $l$ the function $f$ is extensible from $D_l$ onto $\overline{D}_l$ in such a way that the extension $\tilde{f} \in C^1(\overline{D}_l)$. Let $\tau_h^*$ denote the mesh $\tau_h$ refined by the points $Y_l = \overline{D}_l \cap \overline{D}_{l+1}, l = 1, \ldots, l_0 - 1$.

(IV) The approximation of $\langle F(t_h), \varphi_h \rangle$ will have the form

$$\langle F(t_h), \varphi_h \rangle_h = P_0 \varphi_h(0) + 2\pi \sum_{i=1}^{k} P_i R_i \varphi_h(R_i) - 2\pi p \int_0^R r t_h \varphi_h \, dr + \left\{ 2\pi \int_0^R r f \varphi_h \, dr \right\}_{h^*},$$

where $t_h \in U_{ad}^h$, $R_i$ are the points of acting of the lone forces and $\{.\}_{h^*}$ denotes the approximate value of the integral, obtained by means of the trapezoidal rule on the mesh $\tau_h^*$.

The approximations of the bilinear form and the functional are suitable as the next lemma shows:

**Lemma 2.1.** *Let the assumptions* (I)–(IV) *be satisfied. Then*

1) *For all* $t_n \in U_{ad}^h$ *and* $w_h, \varphi_h \in W^{2,2}((0,R), \varrho_{(r)})$ *the following holds:*

$$|A_h(t_h; w_h, \varphi_h) - A(t_h; w_h, \varphi_h)| \leqslant Ch \|w_h\|_{2,2,\varrho_{(r)}} \|\varphi_h\|_{2,2,\varrho_{(r)}}, \tag{2.1}$$

*where the constant $C$ is independent of $h$.*

2) *For all* $\varphi_h \in W^{2,2}((0,R), \varrho_{(r)})$ *the following holds:*

$$|\langle F(t_h), \varphi_h \rangle_h - \langle F(t_h), \varphi_h \rangle| \leqslant C(f) h^{\frac{1}{2}} \|\varphi_h\|_{2,2,\varrho_{(r)}}, \tag{2.2}$$

*where the constant $C(f)$ is independent of $h$ but depends on $R$.*

Proof. 1)

$$|A_h(t_h;w_h,\varphi_h)-A(t_h;w_h,\varphi_h)|\leqslant E\sum_{j=1}^{N(h)}\Big(\int_{\Delta_j}|rw_h''\varphi_h''[t_h^3(\xi_j)-t_h^3(r)]|\,\mathrm{d}r$$
$$+\sigma\int_{\Delta_j}|[w_h''\varphi_h'+w_h'\varphi_h''][t_h^3(\xi_j)-t_h^3(r)]|\,\mathrm{d}r+\int_{\Delta_j}\Big|\frac{1}{r}w_h'\varphi_h'[t_h^3(\xi_j)-t_h^3(r)]\Big|\,\mathrm{d}r\Big)$$
$$\leqslant E\sum_{j=1}^{N(h)}Ch\Big(\int_{\Delta_j}|rw_h''\varphi_h''|\;dr+\sigma\int_{\Delta_j}|w_h''\varphi_h'+w_h'\varphi_h''|\,\mathrm{d}r+\int_{\Delta_j}|\frac{1}{r}w_h'\varphi_h'|\,\mathrm{d}r\Big)$$
$$\leqslant 2ECh\|w_h\|_{2,2,\varrho(r)}\|\varphi_h\|_{2,2,\varrho(r)}.$$

The main step in the proof of the inequality is the estimate

$$|t_h^3(\xi_j)-t_h^3(r)|\leqslant\frac{3}{2}\|t_h^2t_h'\|_{C(\Delta_j)}h\leqslant\frac{3}{2}t_{\max}^2C_1h=Ch,\ r\in\Delta_j.$$

2) Denote $q=rf\varphi_h$. Let $\Delta_j^*=[x_{j-1},x_j]$, where $x_{j-1},x_j$ denote points of the mesh $\tau_h^*$. Then

$$(2.3)\ |\langle F(t_h),\varphi_h\rangle_h-\langle F(t_h),\varphi_h\rangle|\leqslant\Big|2\pi\int_0^R q\;dr-\Big\{2\pi\int_0^R q\,\mathrm{d}r\Big\}_{h^*}\Big|$$
$$\leqslant 2\pi\sum_{j=1}^{N^*(h)}\Big|\int_{\Delta_j^*}q\,\mathrm{d}r-\Big\{\int_{\Delta_j^*}q\,\mathrm{d}r\Big\}_{h^*}\Big|$$
$$\leqslant 2\pi\sum_{j=1}^{N^*(h)}\int_{\Delta_j^*}\Big|q(r)-\frac{1}{2}(q(x_{j-1})+q(x_j))\Big|\,\mathrm{d}r,$$

where $N^*(h)$ is the number of subintervals of the refined mesh $\tau_h^*$.

We have

$$(2.4)\qquad \Big|q(r)-\frac{1}{2}(q(x_{j-1})+q(x_j))\Big|\leqslant\Big|\frac{1}{2}\int_{x_{j-1}}^{r}q'\,\mathrm{d}r-\frac{1}{2}\int_r^{x_j}q'\,\mathrm{d}r\Big|$$
$$\leqslant\int_{x_{j-1}}^{x_j}|1q'|\,\mathrm{d}r\leqslant h^{\frac{1}{2}}\|q'\|_{L_2(\Delta_j^*)}.$$

Further,

$$\|q'\|_{L_2(\Delta_j^*)}=\|(f+rf')\varphi_h+fr\varphi_h'\|_{L_2(\Delta_j^*)}$$
$$\leqslant h\|f+rf'\|_{C(\Delta_j^*)}\|\varphi_h\|_{C(\Delta_j^*)}+\|f\|_{C(\Delta_j^*)}\|r\varphi_h'\|_{L_2(\Delta_j^*)}.$$

If $R \leqslant 1$, then

$$\|r\varphi_h'\|^2_{L_2(\Delta_j^*)} \leqslant \int_0^R r^2(\varphi_h')^2\,\mathrm{d}r \leqslant \int_0^R \frac{1}{r}(\varphi_h')^2\,\mathrm{d}r \leqslant \|\varphi_h\|^2_{2,2,\varrho(r)}.$$

If $R > 1$ , then

$$\begin{aligned}\|r\varphi_h'\|^2_{L_2(\Delta_j^*)} &\leqslant \int_0^1 r^2(\varphi_h')^2\,\mathrm{d}r + \int_1^R r^2(\varphi_h')^2\,\mathrm{d}r \\ &\leqslant \int_0^1 \frac{1}{r}(\varphi_h')^2\,\mathrm{d}r + R^3\int_1^R \frac{1}{R}(\varphi_h')^2\,\mathrm{d}r \\ &\leqslant \int_0^1 \frac{1}{r}(\varphi_h')^2\,\mathrm{d}r + R^3\int_1^R \frac{1}{r}(\varphi_h')^2\,\mathrm{d}r \\ &\leqslant R^3\int_0^R \frac{1}{r}(\varphi_h')^2\,\mathrm{d}r \leqslant R^3\|\varphi_h\|^2_{2,2,\varrho(r)}.\end{aligned}$$

Then altogether

$$\|r\varphi_h'\|_{L_2(\Delta_j^*)} \leqslant \max(1, R^{\frac{3}{2}})\|\varphi_h\|_{2,2,\varrho(r)}.$$

Using the continuous embedding $W^{2,2}((0,R),\varrho(r)) \hookrightarrow C([0,R])$ (see Appendix, Lemma 3.2) we obtain

$$\|\varphi_h\|_{C(\Delta_j^*)} \leqslant C\|\varphi_h\|_{2,2,\varrho(r)},$$

hence

$$\|q'\|_{L_2(\Delta_j^*)} \leqslant C(f)\|\varphi_h\|_{2,2,\varrho(r)}. \tag{2.5}$$

The estimate holds even for $\Delta_j^* = [X_{j-1}, Y_l]$ (or $\Delta_j^* = [Y_l, X_j]$ or $\Delta_j^* = [Y_l, Y_{l+1}]$), since meas $\Delta_j^* \leqslant h$ . The proof is completed by the inequality

$$\begin{aligned}|\langle F(t_h),\varphi_h\rangle - \langle F(t_h),\varphi_h\rangle_h| &\leqslant 2\pi\sum_{j=1}^{N^*(h)} h^{\frac{1}{2}}C(f)\|\varphi_h\|_{2,2,\varrho(r)}\int_{\Delta_j^*} 1\,\mathrm{d}r \\ &\leqslant 2\pi RC(f)h^{\frac{1}{2}}\|\varphi_h\|_{2,2,\varrho(r)} = \tilde{C}(f)h^{\frac{1}{2}}\|\varphi_h\|_{2,2,\varrho(r)},\end{aligned}$$

where we have used estimates (2.3), (2.4) and (2.5). □

Since the subspace $V_h$ is finite-dimensional, $V_h$ is a closed subspace of $V$. The bilinear form $A_h(t_h;.,.)$ is bounded and $V_h$-elliptic on $V_h \times V_h$ (the proof is similar to that of (1.12) and (1.13)) and the functional $\langle F(t_h), .\rangle_h$ on $V_h$ both comply with the assumptions of the Lax-Milgram Theorem. The problem

$$A_h(t_h; w_h(t_h), \varphi_h) = \langle F(t_h), \varphi_h\rangle_h \qquad \forall \varphi_h \in V_h \tag{2.6}$$

is uniquely solvable for any fixed $t_h \in U_{ad}^h$.

**Lemma 2.2.** *Let the assumptions* (I)–(IV) *hold. Furthermore, let a sequence* $\{t_h\}, t_h \in U_{ad}^h$ *converge to a function* $t \in U_{ad}$ *uniformly on the interval* $[0, R]$ *for* $h \to 0_+$. *Finally, let* $w(t)$ *be the solution of* (1.2) *and let* $w_h(t_h)$ *be the solution of* (2.6). *Then*

$$\|w(t) - w_h(t_h)\|_{2,2,\varrho(r)} \to 0, \quad h \to 0_+.$$

Proof. It follows from Lemma 2.1 and from the proof of (1.8) in Theorem 1.2 that

$$|\langle F(t_h), \varphi_h\rangle_h - \langle F(t_h), \varphi_h\rangle + \langle F(t_h), \varphi_h\rangle| \leqslant (C(f)h^{\frac{1}{2}} + C)\|\varphi_h\|_{2,2,\varrho(r)}$$

for all $\varphi_h \in V_h$. Denote $w_h \equiv w_h(t_h)$. Analogously to (1.13) we show that

$$\begin{aligned} A_h(t_h; w_h, w_h) \geqslant Et_{\min}^3 \sum_{j=1}^{N(h)} \bigg\{ \int_{\Delta_j} &\Big(\sigma r (w_h'')^2 + 2\sigma w_h'' w_h' + \frac{\sigma}{r}(w_h')^2\Big) \\ &+ (1-\sigma)\Big(r(w_h'')^2 + \frac{1}{r}(w_h')^2\Big)\,\mathrm{d}r\bigg\} \\ &+ \sum_{j=1}^{N(h)} 2\pi a_0 \int_{\Delta_j} r w_h^2 \,\mathrm{d}r \\ &\geqslant \min(Et_{\min}^3(1-\sigma), 2\pi a_0)\|w_h\|_{2,2,\varrho(r)}^2. \end{aligned}$$

We choose $\alpha > 0$ such that $\alpha \leqslant \min(Et_{\min}^3(1-\sigma), 2\pi a_0)$, then we can write

$$\alpha\|w_h\|_{2,2,\varrho(r)}^2 \leqslant A_h(t_h; w_h, w_h) = \langle F(t_h), w_h\rangle_h \leqslant (C(f)h^{\frac{1}{2}} + C)\|w_h\|_{2,2,\varrho(r)}.$$

Then

$$\|w_h\|_{2,2,\varrho(r)} \leqslant \frac{C(f)h^{\frac{1}{2}} + C}{\alpha}.$$

The sequence $\{w_h\}$ is bounded, hence there exists weakly convergent subsequence and it can be denoted $\{w_h\}$ again. The space $V$ is convex and closed, i.e. weakly closed and $w_h \rightharpoonup w^* \in V$ (weakly) in $V$. For $\varphi \in V$ let $\{\varphi_h\}, \varphi_h \in V_h$ be the sequence from Theorem 3.1 (Appendix). The sequence $\{\varphi_h\}, h \to 0_+$ is bounded in $W^{2,2}((0,R), \varrho_{(r)})$ because $\|\varphi_h - \varphi\|_{2,2,\varrho(r)} \to 0, h \to 0_+$. We show that $w^* = w(t)$ by using the following estimates. From (1.14) we get

$$|A(t_h; w_h, \varphi_h) - A(t; w_h, \varphi_h)| \to 0, \quad h \to 0_+. \tag{2.7}$$

Further (cf. (1.12)),

$$|A(t;w_h,\varphi_h)-A(t;w_h,\varphi)|\leqslant 2\max(Et^3_{\max},2\pi a_0)\|w_h\|_{2,2,\varrho(r)} \times\|\varphi_h-\varphi\|_{2,2,\varrho(r)}\to 0,\quad h\to 0_+. \tag{2.8}$$

From the weak convergence and continuity of the linear functional $A(t;.,\varphi)$ (see the proof of Theorem 1.2) we get

$$|A(t;w_h,\varphi)-A(t;w^*,\varphi)|\to 0,\quad h\to 0_+. \tag{2.9}$$

Combining (2.1), (2.7), (2.8), (2.9) we have

$$|A_h(t_h;w_h,\varphi_h)-A(t;w^*,\varphi)|\to 0,\quad h\to 0_+. \tag{2.10}$$

Similarly, from the estimate (1.15) we have

$$|\langle F(t_h),\varphi_h\rangle-\langle F(t),\varphi_h\rangle|\to 0,\quad h\to 0_+. \tag{2.11}$$

From the continuity of the functional $\langle F(t),.\rangle$ (see the proof of Theorem 1.2) we have

$$|\langle F(t),\varphi_h\rangle-\langle F(t),\varphi\rangle|\to 0,\quad h\to 0_+ \tag{2.12}$$

and

$$|\langle F(t_h),\varphi_h\rangle_h-\langle F(t),\varphi\rangle|\to 0,\quad h\to 0_+, \tag{2.13}$$

by virtue of (2.2), (2.11), (2.12). We know that the equation (2.6) holds. Passing to the limit with $h\to 0_+$ and using (2.10), (2.13) we conclude that

$$A(t;w^*,\varphi)=\langle F(t),\varphi\rangle\quad\forall\varphi\in V.$$

The uniqueness of the solution $w(t)$ yields the equality $w(t)=w^*$ and the weak convergence of the original sequence $\{w_h\}$ to the function $w(t)$. It remains to prove the strong convergence. The sequence $\{\|w_h\|_{2,2,\varrho(r)}\}$ is bounded. The limit (2.11) and the continuity of $\langle F(t),.\rangle$ imply

$$|\langle F(t_h),w_h\rangle-\langle F(t),w(t)\rangle|\leqslant|\langle F(t_h),w_h\rangle-\langle F(t),w_h\rangle| +|\langle F(t),w_h\rangle-\langle F(t),w(t)\rangle|\to 0,\quad h\to 0_+.$$

Combining the last estimate and (2.6), (1.2), (2.2) we may write

$$|A_h(t_h;w_h,w_h)-A(t;w(t),w(t))|\leqslant|\langle F(t_h),w_h\rangle_h-\langle F(t_h),w_h\rangle| +|\langle F(t_h),w_h\rangle-\langle F(t),w(t)\rangle|\to 0,\quad h\to 0_+.$$

Then

$$|A(t;w_h,w_h) - A(t;w(t),w(t))| \leqslant |A(t;w_h,w_h) - A_h(t_h;w_h,w_h)| + |A_h(t_h;w_h,w_h) - A(t;w(t),w(t))| \to 0, \quad h \to 0_+.$$

The first term on the right hand side has the zero limit (from (2.1), (2.7)) and the second term has the zero limit as proved above.

In the norm $\|w\|_A = [A(t;w,w)]^{\frac{1}{2}}$ which is equivalent to $\|\cdot\|_{2,2,\varrho_{(r)}}$ (see (1.4), (1.5)), it means $\|w_h\|_A \to \|w(t)\|_A$, $h \to 0_+$. We define the scalar product $(.,.)_A = A(t;.,.)$ on $V$. We complete the proof by the estimate

$$\begin{aligned}\alpha\|w_h - w(t)\|^2_{2,2,\varrho_{(r)}} &\leqslant \|w_h - w(t)\|^2_A = (w_h - w(t), w_h - w(t))_A \\ &= \|w_h\|^2_A + \|w(t)\|^2_A - 2(w(t),w_h)_A \to 0, \quad h \to 0_+.\end{aligned}$$

We have used the weak convergence $w_h \rightharpoonup w(t)$, the convergence of the norm $\|w_h\|^2_A$ and the continuity of the linear functional $(w(t),.)_A$. □

**Definition 2.1.** Let the approximate optimal design problem $\mathcal{P}_h$ be defined in the following way: find $t^0_h \in U^h_{ad}$ such that

$$J(t^0_h) = j(w_h(t^0_h)) = \min_{t_h \in U^h_{ad}} J(t_h), \qquad (\mathcal{P}_h)$$

where $w_h(t_h)$ solves (2.6).

**Lemma 2.3.** *The problem $\mathcal{P}_h$ has at least one solution for any sufficiently small and positive $h$.*

Proof. We employ Theorem 1.1 for $A_h$ and $\langle F(t_h),.\rangle_h$ with $h$ fixed. Let us choose $U = C([0,R])$, $V = V_h$, $U^0 = \{t \in U;\, 0 < t_{\min} \leqslant t(r) \leqslant t_{\max}\ \forall r \in [0,R]\}$.

The set $U^h_{ad} \subset U_{ad}$ is closed. Then $U^h_{ad} \subset U$ is a compact set and the form $A_h$ and the functional $\langle F(t_h),.\rangle_h$ fulfil (1.4), (1.5) and (1.8) (see the proof of Lemma 2.2).

Let us verify (1.6). Let us assume $t, t_n \in U^0, t_n \to t$ in $U$ and $w_n \rightharpoonup w$ (weakly) in $V_h$ for $n \to \infty$. The dimension of the space $V_h$ is finite, therefore the convergence $w_n \to w \in V_h$ in $W^{2,2}((0,R),\varrho_{(r)})$ is strong. Then

$$|A_h(t_n;w_n,\varphi) - A_h(t;w,\varphi)| \leqslant |A_h(t_n;w_n,\varphi) - A_h(t;w_n,\varphi)| + |A_h(t;w_n,\varphi) - A_h(t;w,\varphi)| \to 0, \quad n \to \infty \quad \forall \varphi \in V_h.$$

Here the convergence of the first term is obtained from $t_n \to t$ in $U$ and the convergence of the second term follows by using an inequality analogous to (1.12) and by $w_n \to w$ in $W^{2,2}((0,R),\varrho_{(r)})$.

The condition (1.7) is a consequence of the equality

$$|\langle F(t_n),\varphi\rangle_h - \langle F(t),\varphi\rangle_h| = \left|2\pi p\int_0^R r(t-t_n)\varphi\,\mathrm{d}r\right|.$$

Finally, the condition (1.10) is fulfilled in virtue of the proof of Theorem 1.2 since $V_h \subset V$. □

**Lemma 2.4.** *Assume that a sequence $\{t_h\}, t_h \in U_{ad}^h$, converges to a function $t \in U_{ad}$ uniformly on the interval $[0,R]$ for $h \to 0_+$. Then*

$$\lim_{h\to 0_+} J(t_h) = J(t).$$

Proof. It is an immediate consequence of Lemma 2.2, since

$$\begin{aligned}&\big|\|w_h(t_h)\|^2_{2,2,\varrho(r)} - \|w(t)\|^2_{2,2,\varrho(r)}\big|\\ &\qquad\leqslant (\|w_h(t_h)\|_{2,2,\varrho(r)} + \|w(t)\|_{2,2,\varrho(r)})\|w_h(t_h)-w(t)\|_{2,2,\varrho(r)},\end{aligned}$$

where the sequence $\{\|w_h(t_h)\|_{2,2,\varrho(r)}\}$ is bounded (see the proof of Lemma 2.2). □

**Theorem 2.1.** *Let $\{t_h^0\}$, $h \to 0_+$, be a sequence of solutions of the approximate problems $\mathcal{P}_h$. Then there exists a subsequence $\{t_{\bar h}^0\}$ such that for $\bar h \to 0_+, t_{\bar h}^0 \to t^0$ in $C([0,R])$, $w_{\bar h}\,(t_{\bar h}^0) \to w(t^0)$ in $W^{2,2}((0,R),\varrho_{(r)})$, where $t^0 \in U_{ad}$ is the solution of the optimization problem (1.3) and $w(t^0) \in V$ is the corresponding solution of* (1.2).

Proof. We use the idea of the proof from [1]. Let $\eta \in U_{ad}$ be an arbitrary function. There exists a sequence $\{\eta_h\}, \eta_h \in U_{ad}^h$, such that $\eta_h \to \eta$ in $C([0,R])$ for $\to 0_+$. Let us denote by $w_h(\eta_h)$ the solution of (2.6), where $t_h$ is replaced by $\eta_h$. ince $U_{ad}$ is compact in $C([0,R])$, there exists a subsequence $\{t_{\bar h}^0\} \subset \{t_h^0\}$ such that $t_{\bar h}^0 \rightrightarrows t^0$ (uniformly) on $[0,R]$ for $\bar h \to 0_+$ so that $t^0 \in U_{ad}$.

We arrive at the inequality $J(t_{\bar h}^0) \leqslant J(\eta_{\bar h})$. Passing to the limit with $h \to 0_+$ and applying Lemma 2.4, we get $J(t^0) \leqslant J(\eta)$. Hence $t^0$ is a solution of the problem (1.3).

The remaining part of the assertion is essentially Lemma 2.2. □

## 3. Appendix

**Lemma 3.1.** *The space of axisymmetric functions $W^{2,2}_{\text{sym}}(\Omega)$ is closed in $W^{2,2}(\Omega)$.*

Proof. Let $\{v_n\} \subset W^{2,2}_{\text{sym}}(\Omega)$ be a sequence such that $v_n \to v$ in $W^{2,2}(\Omega)$. We know that $v_n \in W^{2,2}_{\text{sym}}(\Omega)$ if and only if $\frac{\partial v_n}{\partial \theta} = 0$ almost everywhere. It suffices to prove that $\frac{\partial v}{\partial \theta} = 0$ almost everywhere.

We have

$$\frac{\partial x}{\partial \theta} = -r \sin\theta = -y,$$
$$\frac{\partial y}{\partial \theta} = r\cos\theta = x.$$

We can write

$$\begin{aligned}\int_0^R \int_0^{2\pi} \left|\frac{\partial v}{\partial \theta}\right| d\theta \, \mathrm{d}r &= \int_0^R \int_0^{2\pi} \left|\frac{\partial v_n}{\partial \theta} - \frac{\partial v}{\partial \theta}\right| d\theta \, \mathrm{d}r \\ &= \int_\Omega \left| \left( -\frac{\partial v}{\partial x} y + \frac{\partial v}{\partial y} x + \frac{\partial v_n}{\partial x} y - \frac{\partial v_n}{\partial y} x \right) (x^2+y^2)^{-\frac{1}{2}} \right| \mathrm{d}x \, \mathrm{d}y \\ &\leqslant K_1 \int_\Omega \left| \frac{\partial v_n}{\partial y} - \frac{\partial v}{\partial y}\right| \mathrm{d}x\,\mathrm{d}y + K_2 \int_\Omega \left|\frac{\partial v_n}{\partial x} - \frac{\partial v}{\partial x}\right| \mathrm{d}x\,\mathrm{d}y \\ &\leqslant K_3 \left( \int_\Omega \left|\frac{\partial v_n}{\partial y} - \frac{\partial v}{\partial y}\right|^2 \mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{2}} + K_4 \left( \int_\Omega \left|\frac{\partial v_n}{\partial x} - \frac{\partial v}{\partial x}\right|^2 \mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{2}} \\ &\leqslant K_5 \|v_n - v\|_{2,2} \to 0, \quad n \to \infty,\end{aligned}$$

where we have used the fact that $\Omega$ is bounded and the Hölder inequality.

Then $\int_\Omega \left|\frac{\partial v}{\partial \theta}\right| \mathrm{d}x\,\mathrm{d}y = 0$ which means that $\frac{\partial v}{\partial \theta} = 0$ almost everywhere. □

We denote the space of $\frac{1}{2}$-Hölder continuous functions on the interval $[0, R]$ by $C^{(0),1/2}([0, R])$.

**Lemma 3.2.** *We have that $W^{2,2}((0,R), \varrho_{(r)}) \hookrightarrow C^{(0),1/2}([0,R])$, i.e. the space $W^{2,2}((0,R), \varrho_{(r)})$ is continuously embedded into $C^{(0),1/2}([0,R])$.*

Proof. We use the idea of the proof of Lemma 9.1.2 from [10] p. 280. We can assume that $R > 1$ (in opposite case both estimates would be without constants $R$ and $\sqrt{R}$). From the estimate

$$\begin{aligned}\int_0^R (w')^2 \, \mathrm{d}r &= \int_0^1 (w')^2 \, \mathrm{d}r + \int_1^R (w')^2 \, \mathrm{d}r \leqslant \int_0^1 \frac{1}{r}(w')^2 \, \mathrm{d}r + R \int_1^R \frac{1}{r}(w')^2 \, \mathrm{d}r \\ &\leqslant R \int_0^R \frac{1}{r}(w')^2 \, \mathrm{d}r < \infty\end{aligned}$$

we get $w' \in L_2(0,R)$.

We define $U = U(r)$ for $r \in [0,R]$ by the formula

$$U(r) = \int_0^r w'(t)\,\mathrm{d}t.$$

The integral is finite because $w' \in L_2(0,R)$, so $w' \in L_1(0,R)$. The function $U(r)$ is absolutely continuous in $[0,R]$ and

$$w(r) = U(r) + \text{const} \quad \text{a.e. in } [0,\mathrm{R}].$$

From the Schwarz inequality we get

$$\begin{aligned}|w(\alpha) - w(\beta)| &= \left|\int_\beta^\alpha w'(r)\,\mathrm{d}r\right| \leqslant \left(\int_\beta^\alpha (w'(r))^2\,\mathrm{d}r\right)^{\frac{1}{2}} |\alpha-\beta|^{\frac{1}{2}} \\ &\leqslant \left(R\int_\beta^\alpha \frac{1}{r}(w'(r))^2\,\mathrm{d}r\right)^{\frac{1}{2}} |\alpha-\beta|^{\frac{1}{2}} \leqslant \sqrt{R}\|w\|_{2,2,\varrho(r)}|\alpha-\beta|^{\frac{1}{2}}\end{aligned}$$

for all $\alpha, \beta \in [0,R]$, $\alpha > \beta$. This estimate yields the assertion of the lemma. □

**Theorem 3.1.** *Let $\varphi \in V$. Then there exists a sequence $\{\varphi_h\}$, $\varphi_h \in V_h$ such that*

$$\|\varphi_h - \varphi\|_{2,2,\varrho(r)} \to 0, \quad h \to 0_+.$$

Proof. For any $\varepsilon > 0$ there exists $v_\varepsilon \in \vartheta$ ($\vartheta$ is dense in $V$) such that $\|\varphi - v_\varepsilon\|_{2,2,\varrho(r)} < \varepsilon$. Denote the Hermite cube interpolation of the function $v_\varepsilon$ in the mesh $\tau_h$ by $\varphi_{h\varepsilon}$. Obviously $\varphi_{h\varepsilon} \in V_h$ for sufficiently small $h < h_0(\varepsilon)$. We know that $\operatorname{supp} v'_\varepsilon \cap \{0\} = \emptyset$, hence there exists the biggest $R_0$ such that $\operatorname{supp} v'_\varepsilon \subset [R_0, R]$.

Denote the "integer part" of $n$ by $\lfloor \frac{R_0}{h} \rfloor$. For sufficiently small $h$ there is $n > 1$, $R_0 < 2nh$, $0 < R_0 = \min\limits_{r \in \operatorname{supp} v_\varepsilon} r$. Since $v'_\varepsilon = 0$ in the interval $[0, nh]$, then $v_\varepsilon = \text{const.}$ and the cubic interpolation $\varphi_{h\varepsilon}$ is exact on the interval $[0, nh]$ (i.e. $\varphi_{h\varepsilon} - v_\varepsilon = 0$ on $[0, \frac{R_0}{2}]$).

Denote $u_\varepsilon = \varphi_{h\varepsilon} - v_\varepsilon$. Then

$$\begin{aligned}\|u_\varepsilon\|^2_{2,2,\varrho(r)} &= \int_0^R \left[r(u''_\varepsilon)^2 + \frac{1}{r}(u'_\varepsilon)^2 + ru^2_\varepsilon\right]\mathrm{d}r \\ &= \int_0^R \left[r(u''_\varepsilon)^2 + \frac{1}{r}(u'_\varepsilon)^2 + ru^2_\varepsilon\right]\mathrm{d}r \\ &\leqslant \max\left(R, \frac{2}{R_0}\right)\|u_\varepsilon\|^2_{2,2} \leqslant Ch^4|v_\varepsilon|^2_{4,2},\end{aligned}$$

where $|\cdot|_{4,2}$ denotes the fourth seminorm of the space $W^{4,2}(0,R)$, i.e.

$$|u|_{4,2} = \left( \int_0^R (u^{(4)})^2 \, \mathrm{d}r \right)^{\frac{1}{2}}.$$

In the last estimate we have used Theorem 3.1.4 from [2] (the Hermite interpolation is the mapping $\Pi\colon W^{4,2}(0,R) \to W^{2,2}(0,R)$ such that

$$\forall p \in P_3([0,R])\colon \Pi p = p).$$

Then

$$\|\varphi - \varphi_{h\varepsilon}\|_{2,2,\varrho(r)} \leqslant \|\varphi - v_\varepsilon\|_{2,2,\varrho(r)} + \|u_\varepsilon\|_{2,2,\varrho(r)} \leqslant \varepsilon + Ch^2 |v_\varepsilon|_{4,2} \leqslant 2\varepsilon.$$

The last estimate holds for sufficiently small $h$. □

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