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NONCOERCIVE HEMIVARIATIONAL INEQUALITY AND ITS APPLICATIONS IN NONCONVEX UNILATERAL MECHANICS

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Summary. This paper is devoted to the study of a class of hemivariational inequalities which was introduced by P. D. Panagiotopoulos [31] and later by Z. Naniewicz [22]. These variational formulations are natural nonconvex generalizations [15–17], [22–33] of the well-known variational inequalities. Several existence results are proved in [15]. In this paper, we are concerned with some related results and several applications.

Keywords: hemivariational inequalities, variational inequalities, abstract set-valued law in mechanics, star-shaped admissible sets

AMS classification: 49J52, 49J40

1. INTRODUCTION

This paper follows the work of Z. Naniewicz [22] concerning the study of constrained problems in a reflexive Banach space, in which the set of all admissible elements is not necessarily convex but fulfils some star-shaped property.

In [15] we proved some new abstract existence results for the following problem:

Problem P. Find $u \in C$ wuch that

$$\langle Au - f, v\rangle \geqslant 0, \quad \forall v \in T_C(u),$$

where the set C is assumed to be closed and star-shaped with respect to a certain ball and $T_C(u)$ denotes Clarke's tangent cone of C at $u \in C$, and where A is assumed to be a pseudomonotone operator. Problem P is a special case of a large class of variational problems called hemivariational inequalities which was introduced by

P. D. Panagiotopoulos [24–33] in order to study several mechanical problems connected to energy functionals which are neither convex nor differentiable. The first existence result concerning directly problem P is due to Z. Naniewicz [23]. However, the original theory of Z. Naniewicz requires a coercivity assumption on A. In our previous paper [15], we have extended the theory of Z. Naniewicz to problem P when A is no more coercive. In this paper, we will show that this improvement is useful for the study of some unilateral problems where the boundary conditions are insufficiently blocked up.

Moreover, if C is supposed to be convex, then problem P takes the form of the following variational inequality:

Problem P'. *Find $u \in C$ such that*

$$\langle Au - f, v - u\rangle \geqslant 0, \quad \forall v \in C,$$

and as a consequence of our study devoted to problem P, we will also obtain some new results concerning problem P'.

Abstract theory for constrained problems

Let X be a real reflexive Banach space. We will write "$\rightarrow$" and "$\rightharpoonup$" to denote respectively the strong convergence and the weak convergence. For a nonempty subset D of X, we write $\operatorname{int}(D)$ for the interior of D in X and $\operatorname{cl}(D)$ for the closure. For an operator $A\colon X \rightarrow X^*$ we write $\operatorname{Ker}(A)$ for the kernel, $R(A)$ for the range and $D(A)$ for the domain.

An operator $T\colon X \rightarrow 2^{X^*}$ is said to be *pseudomonotone* if

(i) the set Tu is nonempty, bounded, closed and convex for any $u \in X$;

(ii) T is upper semicontinuous from each finite dimensional subspace F of X to X^* equipped with the weak topology, i.e. to a given element $f \in F$ and a weak neighborhood V of $T(f)$ in X^* there exists a neighborhood U of f in F such that $T(u) \subset F$ for all $u \in U$;

(iii) if $u_n \rightharpoonup u$ and if $z_n \in T(u_n)$ is such that

$$\limsup\langle z_n, u_n - u\rangle \leqslant 0,$$

then for each $v \in V$ there exists $z(v) \in T(u)$ such that

$$\liminf\langle z_n, u_n - v\rangle \geqslant \langle z(v), u - v\rangle.$$

An operator $A\colon X \to X^*$ is said to have the *S^+-property* if $u_n \rightharpoonup u$ and $\limsup\langle Au_n, u_n - u\rangle \leqslant 0$ implies that $u_n \to u$.

An operator $A\colon D(A) \to X^*$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geqslant 0, \quad \forall x, y \in D(A),$$

and *maximal monotone* if it is monotone and the inequality

$$\langle f - Ax, z - x\rangle \geqslant 0, \quad \forall x \in D(A)$$

implies $z \in D(A)$ and $f = Az$.

The operator $A\colon D(A) \to X^*$ is said to be *hemicontinuous* at $x \in D(A)$ if $D(A)$ is convex and for any $y \in D(A)$ the map $t \to A((1-t)x + ty)$ is continuous from $[0,1]$ into the weak topology of X^*. We refer the reader to [34] for more details concerning these classes of operators.

Let $G\colon X \to \mathbb{R} \cup \{+\infty\}$ be a functional. Then its behavior at infinity can be described in terms of what is called the *recession function of* G [4], defined as follows:

$$\begin{aligned} G_\infty(x) : &= \lim_{v \to x} \inf_{t \to +\infty} \frac{G(tv)}{t} \\ &= \inf\{\lim \inf_{n \to +\infty} G(t_n v_n)/t_n \colon t_n \to \infty,\ v_n \to x\}. \end{aligned}$$

We recall that if G and H are two functionals defined on X with values in $(-\infty, +\infty]$, then $(G + H)_\infty \geqslant G_\infty + H_\infty$.

Let $u_0 \in X$, we define the *recession function associated to a general operator* A: $X \to X^*$ *with respect to* u_0 by the formula

$$\underline{r}_{u_0,A}(x) := \lim_{v \to x} \inf_{t \to +\infty} \frac{\langle A(tv), tv - u_0\rangle}{t}.$$

If we set $G(x) := \langle Ax, x - u_0\rangle$, then clearly

$$\underline{r}_{u_0,A}(x) = G_\infty(x).$$

If $u_0 = 0$, then our definition reduces to the one introduced by H. Brezis and L. Nirenberg [5] in order to characterize the range of some nonlinear operators.

Let K be a subset of X, the *recession cone* of K is the closed cone

$$K_\infty := \mathrm{dom}[(\psi_K)_\infty = \{x \in X \colon (\psi_K)_\infty(x) < +\infty\},$$

where ψ_K denotes the indicator function of K. Equivalently, this amonts to saying that x belongs to K_∞ if and only if there exist sequences $\{t_n \mid n \in \mathbb{N}\}$ and $\{x_n \mid n \in \mathbb{N}\} \subset K$ such that $t_n \to +\infty$ and $t_n^{-1} x_n \to x$.

Let $C \subset X$ be a nonempty closed subset. We denote by

$$T_C(u) := \{k \in X : \forall u_n \in C, u_n \to u, \forall \lambda_n \downarrow 0, \\ \exists k_n \to k : u_n + \lambda_n k_n \in C\}$$

Clarke's tangent cone of C at u, by

$$N_C(u) := \{u^* \in X^* : \langle u^*, k\rangle \leqslant 0, \forall k \in T_C(u)\}$$

Clarke's normal cone to C at u, by

$$d_C(u) := \inf_{w \in C} \|u - w\|$$

the *distance function* of C, by

$$d_C^0(u,v) := \limsup_{\substack{y \to u \\ t \downarrow 0}} \frac{d_C(y+tv) - d_C(y)}{t}$$

the *generalized directional derivative in the direction* v *of* d_C *at* u, and by

$$\partial d_C(u) := \{w \in X^* : d_C^0(u,v) \geqslant \langle w, v\rangle, \forall v \in X\}$$

the *generalized gradient of Clarke of* d_C *at* u [10].

Let $B(u_0, \varrho)$ be a closed ball in X with center u_0 and radius $\varrho > 0$. We say that C is *star-shaped* with respect to $B(u_0, \varrho)$ [22] if

$$v \in C \Leftrightarrow \lambda v + (1-\lambda)w \in C, \forall \lambda \in [0,1], \forall w \in B(u_0, \varrho).$$

We call in the following lemma a basic result concerning the function distance of a star-shaped set which has been proved by Z. Naniewicz.

Lemma 2.1. [22; Z. Naniewicz]. *Let X be a real reflexive Banach space, C a nonempty closed subset of X. If C is star-shaped with respect to $B(u_0, \varrho)$ then*

1) $d_C^0(u, u_0 - u) \leqslant -d_C(u) - \varrho, \forall u \notin C,$
2) $d_C^0(u, u_0 - u) = 0, \forall u \in C.$

The following remarkable result concerning the pseudomonotonicity property of the generalized Clarke's gradient is also due to Z. Naniewicz.

Lemma 2.2 [22; Z. Naniewicz]. *Let X be a real reflexive Banach space. Let $f_i \colon X \to \mathbb{R}$ be a finite collection of locally Lipschitzian convex functions defined on X. Define $f \colon X \to \mathbb{R}$ as*

$$f(u) := \min\{f_i(u) : i = 1, \ldots, N\}, u \in X.$$

Let $A: X \to X^*$ be a maximal monotone operator with $D(A) = X$ satisfying the S^+-property. Then $A + \partial f$ is pseudomonotone.

The next result is an easy consequence of the previous one.

Lemma 2.3. *Let X be a real reflexive Banach space, $A: X \to X^*$ a maximal monotone operator with $D(A) = X$ satisfying the S^+-property. Let C be a subset of X which can be represented as the union of a finite collection of nonempty closed convex subsets C_j $(j = 1, \dots, N)$ of X, i.e. $C = \bigcup_{j=1}^{N} C_j$. We assume that $\int \Big(\bigcap_{j=1}^{N} C_j \Big) \neq \emptyset$. Then* i) *$C$ is star-shaped with respect to a certain ball and* ii) *for each $\lambda \geqslant 0$, $A + \lambda \cdot \partial d_C$ is pseudomonotone.*

Let us now introduce the following set of asymptotic directions:

$$R(A, f, C, u_0) := \{w \in X : \exists u_n \in C, \|u_n\| \to +\infty, w_n := u_n \cdot \|u_n\|^{-1} \rightharpoonup w$$

and

$$\langle Au_n, u_n - u_o \rangle \leqslant \langle f, u_n - u_o \rangle \}.$$

On this set, we introduce a compactness condition.

Definition 2.1. We say that $R(A, f, C, u_0)$ is *asymptotically compact* (shortly "*a-compact*") if for each $w \in R(A, f, C, u_0)$, the sequences $\{w_n \mid n \in \mathbb{N}\}$ which appear in the definition of this set are strongly convergent to w, i.e. if $\{w_n ; n \in \mathbb{N}\}$ is a sequence such that $w_n := u_n \cdot \|u_n\|^{-1} \rightharpoonup w$, where $u_n \in C$, $\|u_n\| \to +\infty$ and $\langle Au_n, u_n - u_0 \rangle \leqslant \langle f, u_n - u_0 \rangle$, then $w_n \to w$ in X.

The use of recession sets and compactness conditions for the study of noncoercive problems is now classical. See for instance [2], [4], [5], [8], [12], [14], [20], [21] and [36] for similar approaches in the field of partial differential equations, variational inequalities and optimization problems.

In [15], we proved the following abstract existence result.

Theorem 2.1. *Suppose that the following hypotheses hold true:*

(H_1) *X is a real reflexive Banach space and C is a nonempty closed subset of X which is star-shaped with respect to a ball $B(u_0, \varrho)$, $\varrho > 0$;*

(H_2) *$A + \lambda \partial d_C$ is pseudomonotone for each $\lambda > 0$;*

(H_3) *A is bounded.*

If $R(A, f, C, u_0) = \emptyset$ then problem P has at least one solution.

Sketch of the proof. Fix $n \in \mathbb{N} \setminus \{0\}$ and let

$$B_k := \{x \in X : \|x\| \leqslant k\},$$

where $k \in \mathbb{N} \setminus \{0\}$ is chosen great enough so that $u_0 \in B_k$.

Let $j \geqslant k$ be given in $\mathbb{N}$. A well-known theorem in the theory of variational inequalities [6; Theorem 7.8] guarantees the existence of $u_{n,j} \in B_j$ such that (see [15] or [22] for more details)

$$n \cdot d_C^0(u_{n,j}, v - u_{n,j}) + \langle Au_{n,j} - f, v - u_{n,j} \rangle \geqslant 0, \quad \forall v \in B_j.$$

We claim that there exists $\theta = \theta(j) \in \mathbb{N} \setminus \{0\}$ such that $u_{\theta,j} \in C$. Indeed, suppose on the contrary that $u_{n,j} \notin C, \forall n \in \mathbb{N} \setminus \{0\}$. Then

$$\langle Au_{n,j} - f, u_o - u_{n,j} \rangle + n \cdot d_C^0(u_{n,j}, u_o - u_{n,j}) \geqslant 0$$

implies

$$\langle Au_{n,j} - f, u_o - u_{n,j} \rangle \geqslant n \cdot d_C(u_{n,j}) + n \cdot \varrho \geqslant n \cdot \varrho.$$

Thus

$$n \cdot \varrho \leqslant \|f\|_* \cdot (j + \|u_0\|) + \|A\|_* \cdot j \cdot (j + \|u_0\|).$$

This is a contradiction if n is large enough.

We prove that there exists $k' \in \mathbb{N}$, $k' \geqslant k$ such that $\|u_{\theta(k'),k'}\| < k'$. If not, $\|u_{\theta(i),i}\| = i$ for each $i \in N$, $i \geqslant k$. After relabeling if necessary, the sequence defined by $w_i := w_{\theta(i),i} = u_{\theta(i),i}/i$ satisfies $w_i \rightharpoonup w$, $u_i := u_{\theta(i),i} \in C$ and

$$\langle Au_i - f, u_i - u_0 \rangle \leqslant 0,$$

which means that $w \in R(A, f, C, u_0)$, a contradiction.

We have

$$\theta(k') \cdot d_C^0(u_{k'}, v - u_{k'}) + \langle Au_{k'} - f, v - u_{k'} \rangle \geqslant 0, \ \forall v \in B_{k'}.$$

Let $y \in X$ be given. There exists $\varepsilon > 0$ such that

$$u_{k'} + \varepsilon(y - u_{k'}) \in B_{k'}.$$

If we set $v := u_{k'} + \varepsilon(y - u_{k'})$ then we get

$$\theta(k') \cdot \varepsilon \cdot d_C^0(u_{k'}, y - u_{k'}) + \varepsilon \cdot \langle Au_{k'} - f, y - u_{k'} \rangle \geqslant 0.$$

This means that

$$\theta(k') \cdot d_C^0(u_{k'}, v) + \langle Au_{k'} - f, v\rangle \geqslant 0, \ \forall v \in X.$$

We know that $u_{k'} \in C$ and thus

$$y \in T_C(u_{k'}) \Leftrightarrow d_C^0(u_{k'}, y) = 0.$$

Therefore

$$\langle Au_{k'} - f, v\rangle \geqslant 0, \ \forall y \in T_C(u_{k'}).$$

□

Corollary 2.1. *Suppose that assumptions* (H_1)–(H_3) *are satisfied. If*

(i) $R(A, f, C, u_0)$ *is a-compact,*

(ii) *there exists a nonempty subset* W *of* $X \setminus \{0\}$ *such that*

$$R(A, f, C, u_0) \subset W$$

and

$$\text{(C)} \qquad r_{uo,A}(w) > \langle f, w\rangle, \ \forall w \in W,$$

then problem P *has at least one solution.*

Proof. We claim that $R(A, f, C, u_0) = \emptyset$. Indeed, if we suppose the contrary then we can find a sequence $u_n \in C$ such that $\|u_n\| \to +\infty$, $w_n := u_n \cdot \|u_n\|^{-1} \rightharpoonup w$ and

$$\langle Au_n, u_n - u_0\rangle \leqslant \langle f, u_n - u_0\rangle.$$

The a-compactness of $R(A, f, C, u_0)$ implies that $w_n \to w$. Moreover, we have

$$\langle Au_n, u_n - u_0\rangle / t_n^2 \leqslant \langle f, w_n - u_0/t_n\rangle$$

so that

$$\underline{r}_{uo,A}(w) \leqslant \langle f, w\rangle,$$

which contradicts condition (C). □

Some useful properties of the set of asymptotic directions are now given in the following three propositions.

Proposition 2.1. *Let u_0 be given in X and f in X^*. If*
(i) *A satisfies the S^+-property;*
(ii) $\langle Ax, x\rangle \geqslant 0$, $\forall x \in X$;
(iii) *A is weakly continuous, i.e. $x_n \rightharpoonup x \Rightarrow Ax_n \rightharpoonup Ax$;*
(iv) *A is positively homogeneous.*
(v) *B is monotone on X. Then*

$$R(A+B, f, C, u_0) \text{ is a-compact} \tag{1}$$

and

$$R(A+B, f, C, u_0) \subset \{w \in C_\infty \setminus \{0\} \colon \langle Aw, w\rangle = 0\}. \tag{2}$$

Proof. Let $w \in R(A+B, f, C, u_0)$ be given. There exists $u_n \in C$ such that $t_n := \|u_n\| \to +\infty$, $w_n := u_n/t_n \rightharpoonup w$, and

$$\langle Au_n + Bu_n, u_n - u_0\rangle \leqslant \langle f, u_n - u_0\rangle. \tag{2.1}$$

Since B is monotone, (2.1) implies that

$$\langle Bu_0, u_n - u_0\rangle + \langle Au_n, u_n - u_0\rangle \leqslant \langle f, u_n - u_0\rangle. \tag{2.2}$$

Dividing (2.2) by t_n^2, we obtain

$$\langle Aw_n, w_n\rangle \leqslant \left\langle Aw_n, \frac{u_0}{t_n}\right\rangle + \frac{1}{t_n}\left\langle Bu_0, \frac{u_0}{t_n} - w_n\right\rangle + \left\langle \frac{f}{t_n}, w_n - \frac{u_0}{t_n}\right\rangle \tag{2.3}$$

and thus, using assumption (iii),

$$\limsup \langle Aw_n, w_n\rangle \leqslant 0. \tag{2.4}$$

We have

$$\limsup\langle Aw_n, w_n - w\rangle \leqslant \limsup\langle Aw_n, w_n\rangle + \limsup\langle Aw_n, -w\rangle.$$

Thus, using assumptions (ii) and (iii) we obtain

$$\limsup\langle Aw_n, w_n - w\rangle \leqslant \limsup\langle Aw_n, w_n\rangle.$$

This together with (2.4) imply that

$$\limsup \langle Aw_n, w_n - w\rangle \leqslant 0,$$

and thus, by assumption (i), the sequence w_n is strongly convergent to w, which proves the a-compactness of $R(A+B, f, C, u_0)$. Since $\|w_n\| = 1$ and $w_n \to w$, we have $\|w\| = 1$. On the other hand, since $w_n = u_n/t_n$, $t_n \to +\infty$ and $w_n \to w$, we obtain $w \in C_\infty$, so that $R(A+B, f, C, u_0) \subset C_\infty \setminus \{0\}$. Moreover, if we use (2.3) again, we get $\langle Aw, w\rangle \leqslant 0$. This together with assumption (ii) imply that $R(A+B, f, C, u_0) \subset \{w \in C_\infty \setminus \{0\} : \langle Aw, w\rangle = 0\}$. □

Remark 2.1. i) If A is bounded linear and coercive, i.e. there exists $\alpha > 0$ such that $\langle Au, u\rangle \geqslant \alpha \cdot \|u\|^2$, $\forall u \in X$, then A satisfies assumptions (i)–(iv). If in addition B is monotone then we can directly see that $R(A+B, f, C, u_0) = \emptyset$. ii) It is easy to see that Proposition 2.1 remains true if assumption (v) is replaced by

(v′) $$\langle Bx, x - u_0\rangle \geqslant 0, \ \forall x \in X.$$

Proposition 2.2. *Let u_0 be given in X and f in X^*. If*

(i) *A is bounded linear and semicoercive, i.e. there exists $\alpha > 0$ such that*

$$\langle Au, u\rangle \geqslant \alpha \cdot \|Pu\|^2, \ \forall u \in X,$$

with $P = I - Q$, where I denotes the identity mapping and Q denotes the orthogonal projection of X onto $\mathrm{Ker}(A + A^*)$ *(A^* is the adjoint operator of A);*

(ii) $\dim\{\mathrm{Ker}(A + A^*)\} < +\infty$*;*

(iii) *B is monotone; or*

(iii)′ $\langle Bx, x - u_0\rangle \geqslant 0$, $\forall x \in X$, *then*

(1) $R(A+B, f, C, u_0)$ *is a-compact*

and

(2) $$R(A+B, f, C, u_0) \subset \mathrm{Ker}(A + A^*) \cap C_\infty \setminus \{0\}.$$

Proof. It is clear that assumptions (ii), (iii), (iv) and (v) (or (v′)) of Proposition 2.1 are satisfied. It remains to prove that A satisfies the S^+-property. Indeed, let $\{u_n ; n \in \mathbb{N}\}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } X,$$

and

$$\limsup \langle Au_n, u_n - u \rangle \leqslant 0.$$

We have

$$\begin{aligned}\alpha \cdot \limsup \|Pu_n - Pu\|^2 &\leqslant \limsup \langle Au_n - Au, u_n - u \rangle \\ &\leqslant \limsup \langle Au_n, u_n - u \rangle + \limsup \langle Au, u - u_n \rangle \\ &\leqslant 0,\end{aligned}$$

and thus $Pu_n \to Pu$.

Moreover, Q is bounded linear and thus weakly continuous. Therefore $Qu_n \to Qu$ since $\dim(\operatorname{Ker} A + A^*) < +\infty$. Thus $u_n = Pu_n + Qu_n \to Pu + Qu = u$. □

Proposition 2.3. *Let $X := X^{(1)} \times X^{(2)} \times \ldots \times X^{(N)}$ where each $X^{(\alpha)}$ is a real reflexive Banach space such that $X^{(\alpha)} \hookrightarrow L^2(\Omega^{(\alpha)})$ continuously ($\Omega^{(\alpha)}$ denotes an open set in $\mathbb{R}^n$) and let u_0 be given in X and f in X^*. Let $f^{(\alpha)}(x,z)\colon \Omega^{(\alpha)} \times \mathbb{R} \to \mathbb{R}$ be measurable in x and continuous in z. For a.e. $x \in \Omega^{(\alpha)}$ and all $z \in \mathbb{R}$, assume*

$$(\mathrm{F}_1^{(\alpha)}) \qquad |f^{(\alpha)}(x,z)| \leqslant a^{(\alpha)} \cdot |z| + b^{(\alpha)}(x), \;\; a^{(\alpha)} \in \mathbb{R}, b^{(\alpha)} \in L^2(\Omega^{(\alpha)})$$

and

$$(\mathrm{F}_2^{(\alpha)}) \quad z \cdot f^{(\alpha)}(x,z) \geqslant -c^{(\alpha)}(x) \cdot |z| - d^{(\alpha)}(x), \;\; c^{(\alpha)} \in L^2(\Omega^{(\alpha)}), d^{(\alpha)} \in L^1(\Omega^{(\alpha)}).$$

Let $A\colon X \to X^*$ be an operator satisfying assumptions (i)–(iv) *of Proposition* 2.1 *and let $B\colon X \to X^*$ be defined by*

$$\langle Bu, v \rangle := \sum_{\alpha=1}^{N} \int_{\Omega^{(\alpha)}} f^{(\alpha)}(x, u^{(\alpha)}) \cdot v^{(\alpha)} \,\mathrm{d}x,$$

$\forall u = (u^{(1)}, \ldots, u^{(N)})$, $v = (v^{(1)}, \ldots, v^{(N)}) \in X$.

Then

$$(1) \qquad R(A+B, f, C, u_0) \textit{ is a-compact}$$

and

$$(2) \qquad R(A+B, f, C, u_0) \subset \{w \in C_\infty \setminus \{0\} \mid \langle Aw, w \rangle = 0\}.$$

Proof. Let $w \in R(A+B, f, C, u_0)$. There exists $u_n \in X$ such that $t_n := \|u_n\| \to +\infty$, $w_n := u_n/t_n \rightharpoonup w$ and

$$\langle Aw_n, w_n \rangle + \Big\langle \frac{Bu_n}{t_n}, w_n \Big\rangle \leqslant \Big\langle Aw_n, \frac{u_0}{t_n} \Big\rangle + \frac{\langle Bu_n, u_n \rangle}{t_n^2} + \Big\langle \frac{f}{t_n}, w_n - \frac{u_0}{t_n} \Big\rangle,$$

which means that

$$\langle Aw_n, w_n\rangle \leqslant \sum_{\alpha=1}^{N} \int_{\Omega^{(\alpha)}} \frac{c^{(\alpha)}(x)}{t_n^2}|u_n^{(\alpha)}|\,\mathrm{d}x + \int_{\Omega^{(\alpha)}} \frac{d^{(\alpha)}(x)}{t_n^2}\,\mathrm{d}x + \left\langle Aw_n, \frac{u_0}{t_n}\right\rangle$$
$$+ a^{(\alpha)} \int_{\Omega^{(\alpha)}} \frac{|u_n^{(\alpha)}|}{t_n^2}|u_0^{(\alpha)}|\,\mathrm{d}x + \int_{\Omega} \frac{b^{(\alpha)}(x)}{t_n^2}|u_0^{(\alpha)}|\,\mathrm{d}x + \left\langle \frac{f}{t_n}, w_n - \frac{u_0}{t_n}\right\rangle.$$

The embedding $X^{(\alpha)} \hookrightarrow L^2(\Omega^{(\alpha)})$ is continuous and there exists $C_\alpha > 0$ such that $|u|_\alpha \leqslant C_\alpha \cdot \|u\|_\alpha$, $\forall u \in X^{(\alpha)}$ ($\|\cdot\|_\alpha$ denotes the $X^{(\alpha)}$-norm and $|\cdot|_\alpha$ the $L^2(\Omega^{(\alpha)})$-norm). Then it is easy to compute positive constants σ and τ such that

$$\langle Aw_n, w_n\rangle \leqslant \left\langle Aw_n, \frac{u_0}{t_n}\right\rangle + \left\langle \frac{f}{t_n}, w_n - \frac{u_0}{t_n}\right\rangle + \sigma/t_n + \tau/t_n^2.$$

Thus

$$\limsup \langle Aw_n, w_n\rangle \leqslant 0,$$

and we complete the proof as in Proposition 2.1. □

Remark 2.2. If assumptions $(\mathrm{F}_1^{(\alpha)})$ and $(\mathrm{F}_2^{(\alpha)})$ are satisfied, then the functions

$$(f^{(\alpha)})_\infty(x) := \liminf\{f^{(\alpha)}(x,u) \mid u \to +\infty\},$$
$$(f^{(\alpha)})^\infty(x) := \limsup\{f^{(\alpha)}(x,u) \mid u \to -\infty\}$$

are well defined and if $X^{(\alpha)} \hookrightarrow L^2(\Omega^{(\alpha)})$ continuously, then by using [5; Proposition ii.4], it is easy to see that

$$\underline{r}_{0,B}(\omega) \geqslant \sum_{\alpha=1}^{N} \int_{\Omega^{+(\alpha)}(w)} (f^{(\alpha)})_\infty(x) \cdot w^{(\alpha)}(x)\,\mathrm{d}x$$
$$+ \int_{\Omega^{-(\alpha)}(w)} (f^{(\alpha)})^\infty(x) \cdot w^{(\alpha)}(x)\,\mathrm{d}x, \ \forall w \in X,$$

where

$$\Omega^{+(\alpha)}(w) := \{x \in \Omega^{(\alpha)} \colon w^{(\alpha)}(x) > 0\}$$

and

$$\Omega^{-(\alpha)}(w) := \{x \in \Omega^{(\alpha)} \colon w^{(\alpha)}(x) < 0\}.$$

Throughout the rest of this paper, we will suppose that the following assumption is satisfied.

(H_1') X is a real reflexive Banach space and C is a nonempty closed subset of X which can be represented as the union of a finite collection of nonempty closed convex subsets C_j ($j = 1, \ldots, N$) of X, i.e.

$$C = \bigcup_{j=1}^{N} C_j.$$

Moreover, we assume that

$$\operatorname{int}\Big(\bigcap_{j=1}^{N} C_j\Big) \neq \emptyset.$$

Then C is star-shaped with respect to a certain ball in X with center $u_0 \in C$ and radius $\varrho > 0$.

We are now able to establish four basic results which will be referred to in the following two sections when we will be concerned with concrete applications. Theorem 3.1 and 3.3 concern nonlinear perturbations of coercive linear hemivariational inequalities while Theorems 3.2 and 3.4 are applicable to nonlinear perturbations of semicoercive linear hemivariational inequalities.

Theorem 3.1. *Suppose that assumption* (H_1') *is satisfied. If*

(i) $A\colon X \to X^*$ *is bounded linear and coercive;*

(ii) $B\colon X \to X^*$ *is bounded, hemicontinuous and monotone or*

(ii′) $B\colon X \to X^*$ *is bounded, pseudomonotone and*

$$\langle Bx, x - u_0\rangle \geqslant 0, \ \forall x \in X,$$

then for each $g \in X^*$ *there exists* $u \in C$ *such that*

$$\langle Au + Bu - g, v\rangle \geqslant 0, \ \forall v \in T_C(u).$$

P r o o f. All assumptions required by Proposition 2.1 are satisfied and thus $R(A+B, g, C, u_0) \subset W := C_\infty \cap \operatorname{Ker}(A+A^*) \setminus \{0\}$. Since A is coercive, $\operatorname{Ker}(A+A^*) = \{0\}$. Thus W is empty and $R(A+B, g, C, u_0)$ is empty, too. In order to get our result, it is sufficient to prove that all assumptions of Theorem 2.1 are satisfied. By assumption (i), A is a maximal monotone operator which satisfies the S^+-property. Thus, using Lemma 2.3 together with assumption (H_1'), we obtain the pseudomonotonicity of $A+\lambda\partial d_C$ for each $\lambda > 0$. If B is monotone and hemicontinuous, it is pseudomonotone too and in each case, assumption (H_2) is satisfied. The conditions required by (H_1) and (H_3) are direct consequences of our assumptions. The conclusion follows. □

Theorem 3.2. *Let X be a real Hilbert space and suppose that assumptions* (H_1') *is satisfied. Let g be given in X^*. If*

(i) $A\colon X \to X^*$ *is bounded linear and semicoercive;*
(ii) $\dim\{\mathrm{Ker}(A + A^*)\} < +\infty$;
(iii) $u_0 \in C \cap \mathrm{Ker}(A)$;
(iv) $\langle g, w\rangle < 0,\ \forall w \in C_\infty \cap \mathrm{Ker}(A + A^*) \setminus \{0\}$;
(v) $B\colon X \to X^*$ *is bounded, hemicontinuous, monotone and* $u_0 \in \mathrm{Ker}\, B$;
or
(v′) $B\colon X \to X^*$ *is bounded, pseudomonotone and*

$$\langle Bx, x - u_0\rangle \geqslant 0,\ \forall x \in X,$$

then there exists $u \in C$ such that

$$\langle Au + Bu - g, v\rangle \geqslant 0,\ \forall v \in T_C(u).$$

Proof. By Proposition 2.2, $R(A + B, g, C, u_0)$ is a-compact and $R(A + B, g, C, u_0) \subset W$ with $W = C_\infty \cap \mathrm{Ker}(A + A^*) \setminus \{0\}$. As in the proof of Theorem 3.1, it is easy to show that conditions (H_1)–(H_3) are satisfied. If we prove that

$$\underline{r}_{u_0,A+B}(w) > \langle g, w\rangle,\ \forall w \in W, \tag{3.1}$$

then we can apply Corollary 2.1 to get our result.

Case 1. Suppose that assumption (v) is satisfied. We have

$$\frac{\langle A(tx), tx - u_0\rangle}{t} + \frac{\langle B(tx), tx - u_0\rangle}{t} \geqslant -\langle Ax, u_0\rangle + \frac{\langle B(u_0), tx - u_0\rangle}{t} = 0.$$

The inequality follows from the monotonicity of B, and the equality is due to the fact that $u_0 \in \mathrm{Ker}(A) \cap \mathrm{Ker}(B)$ and $\mathrm{Ker}(A) = \mathrm{Ker}(A^*)$ since A is positive. Therefore

$$\underline{r}_{u_0,A+B}(w) \geqslant 0,\ \forall w \in X. \tag{3.2}$$

Using (3.2) together with assumption (iv), we get (3.1).

Case 2. Suppose that assumption (v′) is satisfied. Then

$$\frac{\langle A(tx), tx - u_0\rangle}{t} + \frac{\langle B(tx), tx - u_0\rangle}{t} \geqslant -\langle Ax, u_0\rangle = 0.$$

Thus $\underline{r}_{uo,A+B}(w) \geqslant 0$, $\forall w \in X$ and we may conclude as before. □

Remark 3.1. If A is symmetric then $W \subset C_\infty \cap \mathrm{Ker}(A) \setminus \{0\}$ so that

$$\underline{r}_{uo,A+B}(w) \geqslant 0, \ \forall w \in W$$

even if assumption (iii) is not satisfied.

Theorem 3.3. *Let $X := X^{(1)} \times X^{(2)} \times \ldots \times X^{(N)}$ where each $X^{(\alpha)}$ is a real reflexive Banach space such that $X^{(\alpha)} \hookrightarrow L^2(\Omega^{(\alpha)})$ compactly, and suppose that assumption (H_1') is satisfied. If*

(i) *$A\colon X \to X^*$ is bounded linear and coercive;*

(ii) *$f^{(\alpha)}(x,u)\colon \Omega^{(\alpha)} \times \mathbb{R} \to \mathbb{R}$ is measurable in x, continuous in u and satisfies assumptions $(F_1^{(\alpha)})$ and $(F_2^{(\alpha)})$ $(\alpha = 1,\ldots,N)$, then for each $g \in X^*$ there exists $u \in C$ such that*

$$\langle Au, v\rangle + \sum_{\alpha=1}^{N} \int_{\Omega^{(\alpha)}} f^{(\alpha)}(x, u^{(\alpha)}) \cdot v^{(\alpha)}\,\mathrm{d}x \geqslant \langle g, v\rangle, \ \forall v \in T_C(u).$$

Proof. Let $B\colon X \to X^*$ be defined by

$$\langle Bu, v\rangle := \sum_{\alpha=1}^{N} \int_{\Omega^{(\alpha)}} f^{(\alpha)}(x, u^{(\alpha)}) \cdot v^{(\alpha)}\,\mathrm{d}x, \ \forall u, v \in X.$$

Using Proposition 2.3, it is easy to see that $R(A+B, g, C, u_0)$ is empty, and it remains to prove that all assumptions of Theorem 2.1 are satisfied. Using Lemma 2.3 together with assumption (H_1'), we obtain the pseudomonotonicity of $A + \lambda\partial d_C$ for each $\lambda > 0$. The compact embedding $X^{(\alpha)} \hookrightarrow L^2(\Omega^{(\alpha)})$ together with assumption $(\mathrm{F}_1^{(\alpha)})$ $(\alpha = 1,\ldots,N)$ guarantees that B is completely continuous and thus bounded and pseudomotone, too. Therefore, condition (H_2) of Theorem 2.1 is satisfied. As already mentioned, the conditions required by (H_1) and (H_3) are direct consequences of our assumptions. □

Theorem 3.4. *Let $X := X^{(1)} \times X^{(2)} \times \ldots \times X^{(N)}$ where each $X^{(\alpha)}$ is a real Hilbert space such that $X^{(\alpha)} \hookrightarrow L^2(\Omega^{(\alpha)})$ compactly, and suppose that assumption (H_1') is satisfied with $u_0 = 0$. Let g be given in X^*. If*

(i) *$A\colon X \to X^*$ is bounded linear and semicoercive;*
(ii) $\dim\{\mathrm{Ker}(A + A^*)\} < +\infty$;
(iii) *$f^{(\alpha)}(x,z)\colon \Omega^{(\alpha)} \times \mathbb{R} \to \mathbb{R}$ is measurable in x, continuous in z and satisfies assumptions $(F_1^{(\alpha)})$ and $(F_2^{(\alpha)})$ $(\alpha = 1,\dots,N)$;*
(iv) $\sum\limits_{\alpha=1}^{N} \int_{\Omega^{+(\alpha)}(e)} (f^{(\alpha)})_\infty(x) \cdot e^{(\alpha)}\,\mathrm{d}x + \int_{\Omega^{-(\alpha)}(e)} (f^{(\alpha)})^\infty(x) \cdot e^{(\alpha)}\,\mathrm{d}x > \langle g, e\rangle, \forall e \in C_\infty \cap (\mathrm{Ker}\, A + A^*) \setminus \{0\}$,

then there exists $u \in C$ such that

$$\langle Au, v\rangle + \sum_{\alpha=1}^{N} \int_{\Omega^{(\alpha)}} f^{(\alpha)}(x, u^{(\alpha)}) \cdot v^{(\alpha)}\,\mathrm{d}x \geqslant \langle g, v\rangle,\ \forall v \in T_C(u).$$

Proof. By Proposition 2.3, $R(A + B, g, C, 0)$ is a-compact and $R(A + B, g, C, 0) \subset W$ with $W = C_\infty \cap \mathrm{Ker}(A + A^*) \setminus \{0\}$. As in the proof of Theorem 3.3, it is easy to prove that conditions (H$_1$)–(H$_3$) are satisfied. To be able to apply Corollary 2.1, it remains to prove that

$$\underline{r}_{0,A+B}(e) > \langle g, e\rangle,\ \forall e \in W. \tag{3.3}$$

By Remark 2.2, we have

$$\underline{r}_{0,B}(w) \geqslant \sum_{\alpha=1}^{N} \int_{\Omega^{+(\alpha)}(w)} (f^{(\alpha)})_\infty(x) \cdot w^{(\alpha)}(x)\,\mathrm{d}x + \int_{\Omega^{-(\alpha)}(w)} (f^{(\alpha)})^\infty(x) \cdot w^{(\alpha)}(x)\,\mathrm{d}x,\ \forall w \in X, \tag{3.4}$$

where

$$\Omega^{+(\alpha)}(w) := \{x \in \Omega^{(\alpha)} \mid w^{(\alpha)}(x) > 0\}$$

and

$$\Omega^{-(\alpha)}(w) := \{x \in \Omega^{(\alpha)} \mid w^{(\alpha)}(x) < 0\}.$$

Moreover

$$\underline{r}_{A+B,0}(w) \geqslant \underline{r}_{A,0}(w) + \underline{r}_{B,0}(w) \geqslant \underline{r}_{B,0}(w),$$

and thus assumption (iv) together with (3.4) imply (3.3). □

Remarks 3.2. i) Condition (iv) in Theorem 3.2 and condition (iv) in Theorem 3.4 (compatibility condition) have been used in various directions by quite a number of authors. See for instance [2], [12], [14] and [36] fot the study of noncoercive variational inequalities. We refer to [4] and the references cited therein for the study of noncoercive variational problems. See also [3], [5] and [19] where further references concerning the study of noncoercive partial differential equations may be found.

ii) It is worthwhile to notice that if C is convex then the existence result given by Theorem 2.1 can be proved without using the fact that C has a nonempty interior (see [2; Lemma 1.4 and Remark 3.5]). Therefore if C is convex then all results of this section remain true if assumption (H_1') is replaced by the following one:

(H_1'') X is a real reflexive Banach space and C is a nonempty closed convex subset of X.

4. Constrained equilibrium of a material point

As an example, in order to illustrate Problem P we consider a material point with mass m which is constrained to remain in a closed subset C of $\mathbb{R}^3$ which is star-shaped with respect to some ball $B(u_0, \varrho)$ ($\varrho > 0$). When m is in contact without friction with the boundary of C, the reaction force R is normal to the boundary, i.e.

$$R \in -N_C(x).$$

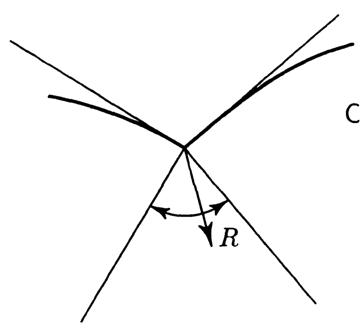

Fig. 1. Constrained equilibrium of a material point

Therefore, if $f_0(x)$ is an external force acting on m, it is necessary and sufficient for the equilibrium to occur that

$$x \in C \quad \text{and} \quad f_0 = -R \in N_C(x),$$

or also

$$x \in C \quad \text{and} \quad \langle -f_0(x), v\rangle \geqslant 0, \; \forall v \in T_C(x).$$

We suppose that

$$f_0(x) := m \cdot g + f(x),$$

where $m \cdot g$ is the gravity force and $f(x) = -A(x)$ where A is a continuous function of x. For instance, if f is a force which derives from a potential, i.e. there exists a differentiable function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = -\operatorname{grad}\Phi(x),$$

then $A(x) = \operatorname{grad}\Phi(x)$. Therefore, we have

$$\langle A(x) - mg, v\rangle \geqslant 0, \ \forall v \in T_C(x)$$

(here $\langle x, y\rangle := x_i y_i$). This simple example shows that Problem P is nothing else that a general expression of the classical principle of virtual work [31, 32].

Proposition 4.1. *Let C be a nonempty subset of $\mathbb{R}^3$. The set-valued function $x \to \partial d_C(x)$ is pseudomonotone.*

Proof. By [10; Proposition 2.1.2] the set $\partial d_C(x)$ is nonempty convex and compact. By [10; Proposition 2.1.5], the function $x \to \partial d_C(x)$ is upper semicontinuous. It remains to prove that if $u_n \to u$ and if $z_n \in \partial d_C(u_n)$ then for each $v \in \mathbb{R}^3$ there exists $z(v) \in \partial d_C(u)$ such that $\liminf\langle z_n, u_n - v\rangle \geqslant \langle z(v), u - v\rangle$. Let $v \in \mathbb{R}^3$ be given. We have

$$\liminf\langle z_n, u_n - v\rangle = -\limsup\langle z_n, v - u_n\rangle \geqslant -\limsup d_c^0(u_n, v - u_n).$$

The map $d_C^0(x, y)$ is upper semicontinuous as a function of (x, y) [10, Proposition 2.1.1], and thus

$$\liminf\langle z_n, u_n - v\rangle \geqslant -d_C^0(u, v - u). \tag{4.1}$$

For each $y \in \mathbb{R}^3$ there exists $z(y) \in \partial d_C(u)$ such that [10, Proposition 2.1.2]

$$d_C^0(u, y) = \langle z, y\rangle.$$

Therefore, there exists $z(v) \in \partial d_C(u)$ such that

$$\liminf\langle z_n, u_n - v\rangle \geqslant \langle z(v), u - v\rangle.$$

□

Since A is continuous, it is clear that $A + \lambda \partial d_C$ is pseudomonotone for each $\lambda \geqslant 0$. Therefore, we are in position to use the basic results presented in Section 3. For instance, if C is closed and star-shaped with respect to $B(0, \varrho)$ ($\varrho > 0$) and if $A(x) = A \cdot x$ where A is a positive semidefinite matrix, then by using Proposition 2.2 together with Corollary 2.1, it is easy to see that

$$\langle g, v\rangle < 0, \ \forall v \in C_\infty \cap \operatorname{Ker}(A + A^{\mathsf{T}}) \setminus \{0\}$$

is a sufficient condition for the existence of an equilibrium.

Further applications in the field of adhesive grasping problems in robotics can be found in [16].

5. On a laminated plate problem

We consider a laminated plate consisting of two isotropic and homogeneous laminae and the binding material between them. We suppose that each lamina $\Omega^{(\alpha)} \subset \mathbb{R}^2$ ($\alpha = 1, 2$) has a constant thickness $h^{(\alpha)}$. Each lamina is identified with a bounded open and connected subset of $\mathbb{R}^2$ and its boundary $\Gamma^{(\alpha)}$ is assumed appropriately smooth. Let also the interlaminar binding material occupy a measurable subset Ω' such that $\mu(\Omega') > 0$ (here μ denotes the 2-dimensional Lebesgue measure), $\Omega' \subset \Omega^{(1)} \cap \Omega^{(2)}$, and $\bar{\Omega}' \cap \Gamma^{(1)} = \emptyset$, $\bar{\Omega}' \cap \Gamma^{(2)} = \emptyset$. The system is referred to a fixed right-handed cartesian coordinate system $0x_1x_2x_3$ and the middle plane of each lamina coincides with the $0x_1x_2$ plane. By $\zeta^{(\alpha)}(x)$ we denote the vertical deflection of the point $x = (x_1, x_2, x_3)$ of the α-th lamina, and by $u^{(\alpha)} = (u_1^{(\alpha)}, u_2^{(\alpha)})$ the horizontal displacement of the α-th lamina.

The theory of Von Kármán plates gives rise to the following system of partial differential equations:

$$k^{(\alpha)}\Delta^2\zeta^{(\alpha)} - h^{(\alpha)}(\sigma_{ij}^{(\alpha)}\zeta_{,j}^{(\alpha)})_{,i} = f^{(\alpha)} \ \text{ in } \Omega^{(\alpha)} \tag{5.1}$$

$$\sigma_{ij,j}^{(\alpha)} = 0 \ \text{ in } \Omega^{(\alpha)} \tag{5.2}$$

$$\sigma_{ij}^{(\alpha)} = C_{ijkl}^{(\alpha)}(\varepsilon_{ij}^{(\alpha)} + \tfrac{1}{2}\zeta_{,k}^{(\alpha)}\zeta_{,l}^{(\alpha)}) \ \text{ in } \Omega^{(\alpha)}. \tag{5.3}$$

Here, $i, j, k, l = 1, 2$, Δ^2 is the biharmonic operator, $k^{(\alpha)}$ is the bending rigidity of the α-th plate, $f^{(\alpha)} = (0, 0, F^{(\alpha)})$ is the distributed vertical load acting on the α-th lamina. The tensors $\sigma^{(\alpha)} = \{\sigma_{ij}^{(\alpha)}\}$ and $\varepsilon^{(\alpha)} = \{\varepsilon_{ij}^{(\alpha)}\}$ denote the stress and strain tensor respectively in the plane of the α-th lamina and $C^{(\alpha)} := \{C_{ijkl}^{(\alpha)}\}$ is the corresponding elasticity tensor, the components of which are assumed to be elements of $L^\infty(\Omega^{(\alpha)})$ and to satisfy the usual symmetry and ellipticity properties. We assume that $u^{(\alpha)}, v^{(\alpha)} \in [H^1(\Omega^{(\alpha)})]^2$ and that $\zeta^{(\alpha)}, z^{(\alpha)} \in H^2(\Omega^{(\alpha)})$.

From equation (5.2), multiplying by $z^{(\alpha)} \in H^2(\Omega^{(\alpha)})$, integrating, applying the Green-Gauss theorem and using classical computations in Von Kármán theory, we get the expressions

$$a_\alpha(\zeta^{(\alpha)}, z^{(\alpha)}) + h^{(\alpha)} R_\alpha(G_\alpha(\zeta^{(\alpha)}), P_\alpha(\zeta^{(\alpha)}, z^{(\alpha)})) = \int_{\Omega^{(\alpha)}} f^{(\alpha)} z^{(\alpha)} \, \mathrm{d}\Omega, \tag{5.4}$$

where a_α is the symmetric bilinear and semicoercive form

$$a(\zeta^{(\alpha)}, z^{(\alpha)}) = k^{(\alpha)} \int_{\Omega^{(\alpha)}} (1 - \nu^{(\alpha)}) \zeta^{(\alpha)}_{,ij} z^{(\alpha)}_{,ij} + \nu^{(\alpha)} \Delta\zeta^{(\alpha)} \Delta z^{(\alpha)} \, \mathrm{d}\Omega.$$

Here $\nu^{(\alpha)} < \frac{1}{2}$ is the Poisson ratio of the α-th lamina, $P_\alpha \colon [H^2(\Omega^{(\alpha)})]^2 \to [L^2(\Omega^{(\alpha)})]^4$ is the completely continuous and quadratic function defined by

$$P_\alpha(\zeta, z) = \{\zeta^{(\alpha)}_{,i} z^{(\alpha)}_{,j}\},$$

$G_\alpha \colon H^2(\Omega^{(\alpha)}) \to [L^2(\Omega^{(\alpha)})]^4$ is the completely continuous and quadratic function defined by

$$G_\alpha(\zeta^{(\alpha)}) = \varepsilon^{(\alpha)}(u^{(\alpha)}) + \tfrac{1}{2} P_\alpha(\zeta^{(\alpha)}, \zeta^{(\alpha)}),$$

$R_\alpha(.,.) \colon [L^2(\Omega^{(\alpha)})]^4 \times [L^2(\Omega^{(\alpha)})]^4 \to \mathbb{R}$ is the continuous, symmetric and coercive bilinear form defined by

$$R_\alpha(M, N) := \int_{\Omega^{(\alpha)}} C_{ijkl} M_{ij} N_{kl} \, \mathrm{d}\Omega,$$

where M and N are 2×2 tensors, and $u^{(\alpha)}(\zeta^{(\alpha)})$ is the plane displacement corresponding to the vertical deflection and merely determined as the solution of the variational equality

$$R_\alpha(\varepsilon^{(\alpha)} + \tfrac{1}{2} P_\alpha(\zeta^{(\alpha)}, \zeta^{(\alpha)}), \varepsilon(v^{(\alpha)} - u^{(\alpha)})) = 0, \ \forall v^{(\alpha)} \in [H^1(\Omega^{(\alpha)}]^2.$$

See [25–28] for more details.

Let us now define the operators $A_\alpha \colon H^2(\Omega^{(\alpha)}) \to H^2(\Omega^{(\alpha)})^*$ and $C_\alpha \colon H^2(\Omega^{(\alpha)}) \to H^2(\Omega^{(\alpha)})^*$ such that

$$a_\alpha(\zeta^{(\alpha)}, z^{(\alpha)}) = \langle A_\alpha \zeta^{(\alpha)}, z^{(\alpha)} \rangle_\alpha$$

and

$$h^{(\alpha)} R_\alpha(G_\alpha(\zeta^{(\alpha)}), P_\alpha(\zeta^{(\alpha)}, z^{(\alpha)})) = \langle C_\alpha \zeta^{(\alpha)}, z^{(\alpha)} \rangle_\alpha.$$

It is known that

(5.5) $A_\alpha \colon H^2(\Omega^{(\alpha)}) \to H^2(\Omega^{(\alpha)})^*$ is bounded symmetric linear and semicoercive;

(5.6) $\operatorname{Ker}(A_\alpha)$ is the space of polynomials of degree $\leqslant 1$;

(5.7) $C_\alpha\colon H^2(\Omega^{(\alpha)}) \to H^2(\Omega^{(\alpha)})^*$ is completely continuous and nonnegative, i.e.

$$\langle C_\alpha x^{(\alpha)}, x^{(\alpha)}\rangle_\alpha \geqslant 0,\ \forall x^{(\alpha)} \in H^2(\Omega^{(\alpha)});$$

(5.8) if $q^{(\alpha)} \in \operatorname{Ker}(A_\alpha)$ then

$$\langle C_\alpha x^{(\alpha)}, q^{(\alpha)}\rangle_\alpha = 0,\ \forall x^{(\alpha)} \in H^2(\Omega^{(\alpha)}).$$

Now we put $F^{(\alpha)} = G^{(\alpha)} + R^{(\alpha)}$, where $G^{(\alpha)} \in L^2(\Omega^{(\alpha)})$ is given, for instance the transversal load applied on the α-th lamina, and $(R^{(1)}, R^{(2)}) \in L^2(\Omega^{(1)}) \times L^2(\Omega^{(2)})$ is a known function of $(\zeta^{(1)}, \zeta^{(2)})$ introduced in order to formulate the stress in the interlaminar binding layer Ω' or to consider the presence of obstacles. We will assume a set-valued reaction-displacement law of the form

$$-R = -(R^{(1)}, R^{(2)}) \in N_C(\zeta), \tag{5.9}$$

where $\zeta := (\zeta^{(1)}, \zeta^{(2)})$ and C denotes a nonempty closed subset of the product space

$$X := H^1(\Omega^{(1)}) \times H^1(\Omega^{(2)})$$

for which the corresponding duality product is defined by

$$\langle .,.\rangle = \langle .,.\rangle_1 + \langle .,.\rangle_2$$

and the corresponding norm by

$$\|.\|^2 = \|.\|^2_{H^1(\Omega^{(1)})} + \|.\|^2_{H^1(\Omega^{(2)})}.$$

Formula (5.9) implies that

$$\langle R, h\rangle \geqslant 0,\ \forall h \in T_C(\zeta). \tag{5.10}$$

Let $G \in X^*$ be defined by

$$\langle G, z\rangle = \sum_{\alpha=1}^{2} \int_{\Omega^{(\alpha)}} G^{(\alpha)} z^{(\alpha)}\, \mathrm{d}\Omega,$$

and let us define the operators $A\colon X \to X^*$ and $C\colon X \to X^*$ by the formulae

$$\langle A\zeta, z\rangle = \langle A_1\zeta^{(1)}, v^{(1)}\rangle + \langle A_2\zeta^{(2)}, v^{(2)}\rangle,$$

and

$$\langle T\zeta, z\rangle = \langle C_1\zeta^{(1)}, v^{(1)}\rangle + \langle C_2\zeta^{(2)}, v^{(2)}\rangle.$$

From (5.4) and (5.10) we obtain the problem

(5.11) Find $\zeta \in C$ such that

$$\langle A\zeta + T\zeta - G, z\rangle \ \forall z \in T_C(\zeta).$$

If $C := \bigcup_{j=1}^{N} C_j$ where each set C_j is assumed to be nonempty convex and closed and if $\operatorname{int}\left\{\bigcap_{j=1}^{N} C_j\right\} \cap \operatorname{Ker}(A) \neq \emptyset$ then C is star-shaped with respect to a ball $B(u_0, \varrho)$ with $u_0 \in \operatorname{Ker} A$. Thus assumptions ($\mathrm{H}_1'$) and (iii) of Theorem 3.2 are satisfied. By properties (5.7) and (5.8), it is clear that T is bounded pseudomonotone and satisfies

$$\langle T\zeta, \zeta - u_0\rangle \geqslant 0, \ \forall \zeta \in X.$$

By properties (5.5) and (5.6), it is clear that assumptions (i) and (ii) are also satisfied, and thus we get the solvability of problem (5.11) for each $G \in L^2(\Omega^{(1)}) \times L^2(\Omega^{(2)})$ satisfying the inequality

$$\sum_{\alpha=1}^{2} \int_{\Omega^{(\alpha)}} G^{(\alpha)} q^{(\alpha)} \, \mathrm{d}\Omega < 0, \ \forall q \in C_\infty \cap \operatorname{Ker}(A) \setminus \{0\}. \tag{5.12}$$

6. On the generalized Signorini-like problem in elasticity

Let Ω be a body identified with a bounded open connected subset of $\mathbb{R}^3$ referred to a coordinate system $\{0, x_1, x_2, x_3\}$. We assume that the surface of the body is regular (i.e. Γ is a hypersurface of class C^m ($m \geqslant 1$) and Ω is located on one side of Γ). It is assumed that Ω is subjected to a density force F. Surface tractions t are applied to a portion Σ of Γ. The body Ω is assumed to be fixed along an open subset Γ_U of Γ (possibly empty). We suppose that $\Gamma_U \cap \Sigma = \emptyset$.

Let $\sigma = \{\sigma_{ij}\}$ be the stress tensor and let $n = \{n_i\}$ be the outward unit normal vector on Γ. We denote by $S = \{S_i\}$ the stress vector on Γ, i.e. $S_i = \sigma_{ij} \cdot n_j$.

Let u denote the displacement field of the body. We consider the case of infinitesimal deformations of the body and suppose that the body is characterized by a Cauchy elastic law, i.e. $\sigma_{ij} = C_{ijkl} \cdot \varepsilon_{ij}$ where $\varepsilon = \{\varepsilon_{ij}(u)\}$ is the strain tensor and

$C = \{C_{ijkl}(x)\}$ is the linear-elasticity tensor. The elasticity tensor $C \in [L^\infty(\Omega)]^{81}$ is supposed to satisfy the classical symmetry properties

$$C_{ijkl}(x) = C_{klij}(x) = C_{jikl}(x),$$

and the ellipticity property

$$C_{ijkl}(x)\zeta_{ij}\zeta_{kl} \geqslant c \cdot \xi_{ij} \cdot \xi_{ij} \ (c > 0)$$

for a.e. $x \in \Omega$, and for all 3×3 symmetric matrices ζ.

We suppose that a nonempty measurable part Ω' ($\mu(\Omega') > 0$) of the body Ω is constrained to lie inside a closed convex box $Q \subset \mathbb{R}^3$. We assume that $\Omega' \subset Q$. The displacement field u satisfies the following system of equations:

$$-\partial\sigma_{ij}/\partial x_j = F_i + R_i \ \text{ in } \Omega, \tag{6.1}$$

$$S_i = t_i \ \text{ on } \Sigma, \tag{6.2}$$

$$x + u(x) \in Q, \ \forall x \in \Omega', \tag{6.3}$$

$$u = 0 \ \text{ on } \Gamma_U, \tag{6.4}$$

where the reaction force R is introduced in order to describe the action of the constraints on the body. We assume also that $\Gamma = \Sigma \cup \Gamma_U \cup E$, where $\mu(E) = 0$ (μ denotes the 3-dimensional Lebesgue measure). Moreover, we put

$$F_i(x, u(x)) = f_i(x) - G_i(x, u),$$

where f_i denotes a body force density which does not depend of the displacement. The displacement of a particle $x \in \Omega$ after deformation becomes $x + u(x)$ so that (6.3) ensures that part Ω' of the body lies in Q.

Let X be the Hilbert space defined by

$$X := \{v \in [H^1(\Omega)]^3 : v = 0 \ \text{ on } \Gamma_U\}.$$

(The boundary condition $v = 0$ on Γ_U is considered in the usual trace sense.) In order to formulate a frictionless contact between the body and the box, we introduce the abstract multivalued law

$$-R \in N_C(u),$$

where C is the closed convex subset of X defined by

$$C := \{v \in X : x + v(x) \in Q, \ \text{a.e. on } \Omega'\}.$$

Then, as a weak formulation of system (6.1)–(6.4) we consider the variational inequality

$$u \in C\colon a(u, v-u) + \langle Bu, v-u\rangle \geqslant \langle f, v-u\rangle, \ \forall v \in C, \tag{6.5}$$

where $a(u,v)$ is the bilinear continuous symmetric form

$$a(u,v) = \int_\Omega C_{ijhk}\varepsilon_{ij}(u)\varepsilon_{hk}(v)\,\mathrm{d}\Omega,$$

$B\colon X \to X^*$ is the nonlinear operator defined by

$$\langle Bu, v\rangle = \int_\Omega G_i(x, u_i)v_i\,\mathrm{d}\Omega$$

and $\langle v, f\rangle$ is the linear continuous form

$$\langle v, f\rangle = \int_\Omega f_i v_i\,\mathrm{d}\Omega + \int_\Sigma t_i v_i\,\mathrm{d}s$$

with $f \in [L^2(\Omega)]^3$ and $t \in [L^2(\Sigma)]^3$.

Formulation (6.5) is obtained by using classical distributional tools. We refer the reader to [21] and [33] for more details.

Let $A\colon X \to X^*$ be the bounded linear and symmetric operator defined by

$$\langle Au, v\rangle = a(u,v), \ \forall u, v \in X.$$

It is known that if $\mu(\Gamma_U) > 0$ then A is coercive. However, if $\Gamma_U = \emptyset$ then A is only semicoercive and

$$\operatorname{Ker} A = \{v \in X : v(x) = a \wedge x + b, \ a, b \in \mathbb{R}^3\},$$

where $\wedge$ denotes the vector product.

Case 1. Suppose that G_i obeys a potential law, i.e. there exists a functional

$$\Phi_i\colon \Omega \times \mathbb{R} \to \mathbb{R} \ \text{ such that } \ \Phi_i(x,.) \in C^1(\Omega, \mathbb{R}) \ \text{ for all } \ x \in \Omega \ \text{ and}$$

$$G_i(x,.) = -\nabla\Phi_i(x,.).$$

If $\Phi_i(x,.)$ is convex for all $x \in \Omega$, then the corresponding operator B is monotone and hemicontinuous [34]. By using Theorem 3.1, Theorem 3.2 and Remark 3.2 (ii) together with Proposition A.1, we get the following result.

Proposition 6.1. i) *if* $\mu(\Gamma_U) > 0$ *then problem* (6.5) *has at least one solution for each* $f \in X^*$. ii) *if* $\Gamma_U = \emptyset$ *and* $0 \in \mathrm{Ker}(B)$ *then problem* (6.5) *has at least one solution for each* $f \in X^*$ *satisfying the inequality*

$$\langle f, e \rangle < 0, \ \forall e \in \{v \in X : v(x) = a \wedge x + b \in Q_\infty \ \text{a.e. on} \ \ \Omega', \ a, b \in \mathbb{R}^3\}.$$

Case 2. Suppose that G_i is a Carathéodory function satisfying assumptions $(\mathrm{F}_1^{(i)})$ and $(\mathrm{F}_2^{(i)})$. Then as a consequence of Theorem 3.3 and Theorem 3.4, we get the following result.

Proposition 6.2. *if* $\mu(\Gamma_U) > 0$ *then problem* (6.5) *has at least one solution for each* $f \in X^*$, *and ii) if* $\Gamma_U = \emptyset$, *then problem* (6.5) *has at least one solution for each* $f \in X^*$ *satisfying the inequality*

$$\sum_{\alpha=1}^{2} \int_{\Omega^{+(\alpha)}(e)} (f^{(\alpha)})_\infty(x) \cdot e^{(\alpha)} \, \mathrm{d}x + \int_{\Omega^{-(\alpha)}(e)} (f^{(\alpha)})^\infty(x) \cdot e^{(\alpha)} \, \mathrm{d}x > \langle g, e \rangle$$

$$\forall e \in \{v \in X : v(x) = a \wedge x + b \in Q_\infty \ \ \text{a.e. on} \ \ \Omega', \ a, b \in \mathbb{R}^3\}.$$

Remark 5.1. i) Using our approach, we have obtained new existence results for hemivariational inequalities and variational inequalities. ii) we have proved the solvability of semicoercive problems as long as a compatibility condition on the right hand term f is satisfied. Such compatibility conditions can be seen as generalized Lazer-Landesman conditions [19].

Annex: On the box-set

Let $\Omega^{(\alpha)}$ $(\alpha = 1, \ldots, N)$ be a nonempty open subset of $\mathbb{R}^n$ $(n \in \mathbb{N}, n \geqslant 1)$ and let Ω' be a measurable subset such that $\mu(\Omega') > 0$ and $\Omega' \subset \Omega^{(\alpha)}$, $\forall \alpha = 1, \ldots, N$. Let Q be a nonempty subset of $\mathbb{R}^n$. We assume that $\Omega' \subset Q$.

The following proposition holds true:

Proposition A.1. *Let* X *be a real Banach space such that* X *is continuously embedded in* $L^p(\Omega^{(1)}) \times \ldots \times L^p(\Omega^{(N)})$ $(p \geqslant 1)$ *and let* C *be defined by*

$$C := \{u \in X : x + u(x) \in Q, \ \ \text{a.e. on} \ \ \Omega'\}.$$

then

i) $O \in C$;

ii) $C_\infty \subset \{v \in X \mid v(x) \in Q_\infty,\ \text{a.e. on } \Omega'\}$;
iii) *if Q is closed convex then C is closed convex and*

$$C_\infty = \{v \in X \mid v(x) \in Q_\infty,\ \text{a.e. on } \Omega'\}.$$

Proof. i) Trivial. ii) Let $v \in C_\infty$. There exist $\{t_n \mid n \in \mathbb{N}\}$ and $\{v_n \mid n \in \mathbb{N}\}$ such that $v_n \to v, t_n \to \infty$ and $t_n \cdot v_n \in C$, that is

$$x + t_n \cdot v_n(x) \in Q, \text{ a.e. on } \Omega'. \tag{A.1}$$

Since $X \hookrightarrow L^p(\Omega^{(1)}) \times \ldots \times L^p(\Omega^{(N)})$ continuously, there exists a subsequence (again denoted by v_n) such that $v_n^{(\alpha)}(x) \to v^{(\alpha)}(x)$ a.e. on $\Omega^{(\alpha)}$. This together with (A.1) implies that $v(x) \in Q_\infty$ for a.e. $x \in \Omega'$. ii) It is clear that C is closed and convex. Therefore C_∞ can also be written as

$$\bigcap_{\mu>0} (C - z)/\mu,$$

where z is any element of C. Let $v \in X$ be such that $v(x) \in Q_\infty$ a.e. on Ω'. Then we obtain

$$x + \lambda v(x) \in Q \text{ a.e. on } \Omega',\ \forall \lambda > 0.$$

Taking a sequence $\lambda_n \to +\infty$, we find that $x + \lambda_n v \in C$ and thus $v \in C_\infty$. □

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