Luka Neralić; Sanjo Zlobec

LFS functions in multi-objective programming

Applications of Mathematics, Vol. 41 (1996), No. 5, 347–366

Persistent URL: http://dml.cz/dmlcz/134331

41 (1996) APPLICATIONS OF MATHEMATICS No. 4, 347–366

LFS FUNCTIONS IN MULTI-OBJECTIVE PROGRAMMING[1]

LUKA NERALIĆ, Zagreb, SANJO ZLOBEC, Montreal

(Received October 10, 1995)

Summary. We find conditions, in multi-objective convex programming with nonsmooth functions, when the sets of efficient (Pareto) and properly efficient solutions coincide. This occurs, in particular, when all functions have locally flat surfaces (LFS). In the absence of the LFS property the two sets are generally different and the characterizations of efficient solutions assume an asymptotic form for problems with three or more variables. The results are applied to a problem in highway construction, where the quantity of dirt to be removed and the uniform smoothness of the shape of a terrain are optimized simultaneously.

Keywords: multi-objective program, efficient (Pareto) solution, properly efficient solution, LFS function, convex program, l_1 norm, l_∞ norm, simultaneous optimization

AMS classification: 90C29, 41A28

1. INTRODUCTION

Differentiable convex functions with "locally flat surfaces" (abbreviation: "LFS functions") have been recently introduced in [18, 20]. The notion has been extended to nonsmooth functions in [15]. An important feature of mathematical programs with these functions is that optimality of a feasible point is characterized by the Karush-Kuhn-Tucker conditions.

In this paper we study multi-objective convex program with nonsmooth LFS objective functions. We find conditions on the constraints that guarantee that all efficient (Pareto) solutions are properly efficient. This is the case, in particular, when all functions in the program are LFS. The equality of the two solution sets means that every Pareto solution enjoys additional uniform "stability" properties.

[1] Research partly supported by the Research Council of Croatia and by a grant from NSERC of Canada. Presented in part as the National Contribution of Croatia at the 14th Triennial IFORS Conference, Vancouver, B.C., Canada, July 8–12, 1996.

In the absence of LFS constraints, the two sets are generally different. Moreover, the characterizations of Pareto optima then assume an asymptotic form (involving the closure of a set) for programs in three or more variables. The results are readily applicable to bi-objective approximation problems involving l_1 and l_∞ norms. We illustrate the theory on a highway construction problem.

The paper is organized as follows: In Section 2 we recall the notion of LFS functions. The well known classic Charnes-Cooper observation is modified for convex functions in Section 3. While the classic version is used here to prove the equivalence of efficient and properly efficient points for LFS functions, its modified version is used to relax the statement on the signs of the weights of the objective functions; see Section 4. The general case and the asymptotic form of the optimality conditions are studied in Section 5 and the application is given in Section 6.

2. LFS FUNCTIONS

Let us introduce and recall some basic facts about LFS functions. For the sake of simplicity we assume that all functions in the sequel are defined on the entire Euclidean space $\mathbb{R}^n$. An LFS function $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is defined locally, at an arbitrary but fixed $x^* \in \mathbb{R}^n$, using the following three cones with a vertex 0:

$$\begin{aligned}
D_f^{\leqslant}(x^*) &= \{d \in \mathbb{R}^n\colon f(x^* + \alpha d) \leqslant f(x^*), 0 < \alpha < \alpha^* \text{ for some } \alpha^* > 0\}\\
D_f^{<}(x^*) &= \{d \in \mathbb{R}^n\colon f(x^* + \alpha d) < f(x^*), 0 < \alpha < \alpha^* \text{ for some } \alpha^* > 0\}\\
D_f^{=}(x^*) &= \{d \in \mathbb{R}^n\colon f(x^* + \alpha d) = f(x^*), 0 < \alpha < \alpha^* \text{ for some } \alpha^* > 0\}.
\end{aligned}$$

These are the "cones of nonincrease, decrease, and constancy of f at x^*", respectively. For their properties see [2].

2.1 Definition. [14, 15, 20] A convex function $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ is said to have a locally flat surface at $x^* \in \mathbb{R}^n$ if

$$D_f^{\leqslant}(x^*) \quad \text{is } \textit{polyhedral} \text{ when } D_f^{<}(x^*) \neq \emptyset$$

or if

$$D_f^{=}(x^*) = \{d\colon f'(x^*; d) = 0\} \quad \text{and } \textit{polyhedral} \text{ when } D_f^{<}(x^*) = \emptyset.$$

Here $f'(x^*; d)$ denotes the directional derivative of f at x^* in the direction d. If f is a differentiable convex function then the above definition reduces to

$$N(\nabla f(x^*)) = D_f^{=}(x^*).$$

Here $N(\nabla f(x^*))$ denotes the null-space of the gradient. Note that LFS is a local property. However, some important classes of functions, e.g., linear functions, l_1 norm, l_∞ norm, their linear combinations such as

$$f(x) = a^T x + \sum_{i \in J} |x_i|$$

where $J \subset \{1, \ldots, n\}$, and the distance function

$$f(x) = \min\{\|(x - x^*) - z\| \colon z \in K\}$$

where K is a polyhedral cone, are LFS at every $x^* \in \mathbb{R}^n$. (Note that the Euclidean norm is not generally an LFS function.) For algebraic and geometric characterizations of LFS functions, and their importance in convex programming, see the recent papers [15, 18], (also [20].)

3. The Charnes-Cooper observation

Consider a multi-objective program

$$\text{(MP)} \qquad \text{Min}\{\Phi^k(x) \colon k \in Q\} \quad \text{s.t. } f^i(x) \leqslant 0, \quad i \in P$$

where the objective functions $\Phi^k \colon \mathbb{R}^n \mapsto \mathbb{R}$, $k \in Q$ and the constraints $f^i \colon \mathbb{R}^n \mapsto \mathbb{R}$, $i \in P$, are continuous functions, and the index sets P and Q are finite. Denote by

$$F = \{x \in \mathbb{R}^n \colon f^i(x) \leqslant 0, \quad i \in P\}$$

the feasible set of (MP) and take an arbitrary $x^* \in F$. The point is said to be an efficient (or Pareto) point if there is no other $x \in F$ such that

$$\Phi^k(x) \leqslant \Phi^k(x^*), \;\; k \in Q$$

with at least one strict inequality. Charnes and Cooper observed in [5] that $x^* \in F$ is efficient if, and only if, x^* is an optimal solution of the program

$$\begin{aligned} & \text{Min} \sum_{k \in Q} \Phi^k(x) \\ \text{(CCO)} \qquad & \text{s.t. } \Phi^k(x) \leqslant \Phi^k(x^*), \;\; k \in Q, \; x \in F. \end{aligned}$$

Their observation has been used to study and characterize efficient points as optimal solutions of mathematical programs, see, e.g., [2, 14, 16, 18]. In convex case their observation will now be sharpened to include non-negative weights.

First, we need more notation. Assume that the objectives and the constraints in (MP) are convex functions and that some $x^* \in F$ is fixed. For each $r \in Q$, we denote

$$F_r(x^*) = \{x \in F\colon \Phi^k(x) \leqslant \Phi^k(x^*),\ k \in Q \setminus \{r\}\}$$

and the index set

$$Q_r^=(x^*) = \{k \in Q \setminus \{r\}\colon x \in F_r(x^*) \Rightarrow \Phi^k(x) = \Phi^k(x^*)\}.$$

The pivotal set for the modification is

$$Q^=(x^*) = \bigcup_{r \in Q} Q_r^=(x^*).$$

We call this set "the minimal index set of active objectives". It can be calculated, essentially, in the same way as the minimal index sets of active constraints (see [2]). We will also need the index set

$$P^=(x^*) = \{i \in P\colon x \in F \cap F_0(x^*) \Rightarrow f^i(x) = 0\}$$

where

$$F_0(x^*) = \{x\colon \Phi^k(x) \leqslant \Phi^k(x^*), k \in Q\}.$$

Using the Lagrangian function, relative to x^*:

$$L(x, \lambda, u) = \sum_{k \in Q \setminus Q^=(x^*)} \lambda_k \Phi^k(x) + \sum_{i \in P \setminus P^=(x^*)} u_i f^i(x)$$

and the set

$$F^{\leqslant}(x^*) = \{x\colon f^i(x) \leqslant 0, i \in P^=(x^*)\} \cap \{x\colon \Phi^k(x) = \Phi^k(x^*), k \in Q^=(x^*)\}.$$

the following result was essentially proved in [16]. (It is a slight modification of [16, Theorem 2.4] obtained after using Remark (i) from the end of Section 2 in [16]. Its differentiable version was given in [19].)

3.1. Theorem ([16]). *Consider the convex multi-objective program (MP). A point $x^* \in F^{\leqslant}(x^*)$ is an efficient point if, and only if, either $Q \setminus Q^=(x^*) = \emptyset$, or there exists $\lambda^* = (\lambda_k^*) \geqslant 0$, $k \in Q \setminus Q^=(x^*)$, $\lambda^* \neq 0$ and $u^* = (u_i^*) \geqslant 0$, $i \in P \setminus P^=(x^*)$ such that*

$$L(x^*, \lambda^*, u) \leqslant L(x^*, \lambda^*, u^*) \leqslant L(x, \lambda^*, u^*)$$

for every $u \geqslant 0$ *and* $x \in F^{\leqslant}(x^*)$.

We are now ready to modify Charnes-Cooper observation.

3.2. Theorem. *Consider the convex multi-objective program* (MP). *A feasible point* x^* *is an efficient point if, and only if, either* $Q = Q^{=}(x^*)$ *or* x^* *is an optimal solution of the program*

$$\text{(MCCO)} \qquad \begin{gathered} \operatorname{Min} \sum_{k \in Q \setminus Q^{=}(x^*)} w_k \Phi^k(x) \\ s.t. \\ \Phi^k(x) = \Phi^k(x^*), \quad k \in Q^{=}(x^*) \\ x \in F \end{gathered}$$

for some $w_k \geqslant 0$, $k \in Q \setminus Q^{=}(x^*)$, *not all zero.*

Proof.. (Necessity:) If $x^* \in F$ is an efficient point, then

$$\sum_{k \in Q \setminus Q^{=}(x^*)} w_k \Phi^k(x^*) \leqslant \sum_{k \in Q \setminus Q^{=}(x^*)} w_k \Phi^k(x) + \sum_{i \in P \setminus P^{=}(x^*)} u_i f^i(x) \tag{3.1}$$

for some $w_k \geqslant 0$, $k \in Q \setminus Q^{=}(x^*)$, not all zero, and some $u_i \geqslant 0$, $i \in P \setminus P^{=}(x^*)$, and for every

$$x \in F^{\leqslant}(x^*) = \{x \colon f^i(x) \leqslant 0, i \in P^{=}(x^*)\} \cap \{x \colon \Phi^k(x) = \Phi^k(x^*), k \in Q^{=}(x^*)\}.$$

This follows from Theorem 3.1 after using the complementarity condition. Since

$$F \subset \{x \colon f^i(x) \leqslant 0, i \in P^{=}(x^*)\}$$

and $F \cap \{x \colon \Phi^k(x) = \Phi^k(x^*), k \in Q^{=}(x^*)\} \neq \emptyset$ (x^* being in both sets), it follows that (3.1) also holds for every x from the smaller set

$$F \cap \{x \colon \Phi^k(x) = \Phi^k(x^*), k \in Q^{=}(x^*)\}.$$

But, since $P^{=}(x^*) \subset P$, for every such x we have

$$\sum_{i \in P \setminus P^{=}(x^*)} u_i f^i(x) \leqslant 0$$

and (3.1) proves the necessity part.

(Sufficiency:) Suppose that x^* solves (MCCO). If x^* was not Pareto optimal, then there would exist $x' \in F$ such that

$$\begin{aligned} \Phi^k(x') &< \Phi^k(x^*), \quad k \in Q \setminus Q^=(x^*) \\ \Phi^k(x') &= \Phi^k(x^*), \quad k \in Q^=(x^*) \end{aligned}$$

by the definition of $Q^=(x^*)$. But this contradicts the assumption that x^* is optimal for (MCCO). □

This result is of a separate interest. In particular, it shows that, in some situations, one may reduce the number of objectives to be used for testing a point x^* for efficiency:

3.3. Example. Consider the program

$$\begin{aligned} &\text{Min}\left\{\Phi^1(x) = -x_1 + x_2,\ \Phi^2(x) = x_1^2 + x_2^2\right\} \\ &\text{s.t.} \\ &\qquad x_1^2 + x_2^2 \leqslant 2 \\ &\qquad -x_1 + x_2^2 \leqslant 1 \\ &\qquad x_1 - x_2 \leqslant 1 \\ &\qquad -x_1 \leqslant 0. \end{aligned}$$

Is $x^* = [0\ \ 1]^T$ Pareto optimal? We find that $Q^=(x^*) = \{1\}$ and x^* is Pareto optimal if, and only if, x^* solves

$$\begin{aligned} &\text{Min } \Phi^2(x) \\ &\text{s.t. } \Phi^1(x) = 1 \\ &\qquad x \in F. \end{aligned}$$

Since this is not the case, x^* is not Pareto optimal. How about $y^* = (1/2)[1\ \ 1]^T$? Now $Q^=(y^*) = \{1\}$ and again we find that y^* is not an optimal solution of the above program. But for $z^* = [0\ \ 0]^T$ we find $Q^=(z^*) = \{2\}$. It follows that z^* is Pareto optimal.

We recall that optimality for multi-objective programs can be defined in other ways. For instance, $x^* \in F$ is a "weak Pareto optimum" if there is no $x \in F$ such that

$$\Phi^k(x) < \Phi^k(x^*), \quad k \in Q.$$

An immediate implication of the modified CCO for weak Pareto optima follows.

3.4. Corollary. *Consider the convex multi-objective program* (MP). *If* $Q^{=}(x^*) = \emptyset$ *at some feasible* x^*, *then* x^* *is a weak Pareto optimum if, and only if,* x^* *is a Pareto optimum.*

3.5. Example. Consider the program

$$\begin{aligned} &\text{Min}\left\{\Phi^1(x) = x_1, \Phi^2(x) = x_2\right\} \\ &\text{s.t.} \\ &\qquad x_1 \geqslant 0, \quad x_2 \geqslant 0. \end{aligned}$$

Here $x^* = [0 \;\; 1]^T$ is a weak Pareto optimum. But $Q^{=}(x^*) \neq \emptyset$ and x^* is not Pareto optimal.

We will use the modified observation in the context of LFS functions at the end of the next section.

4. The equivalence of efficient and properly efficient points

Consider again the program (MP) and its arbitrary efficient point x^*. If x^* enjoyes the following additional property:

"There exists a number $\beta > 0$ such that, for every $k \in Q$ and every $x \in F$ satisfying $\Phi^k(x) < \Phi^k(x^*)$ there exists, in the index set

$$Q' = \{s \in Q \setminus \{k\} : \Phi^s(x) > \Phi^s(x^*)\}$$

at least one $s \in Q'$ such that

$$\frac{\Phi^k(x^*) - \Phi^k(x)}{\Phi^s(x) - \Phi^s(x^*)} \leqslant \beta\text{"}$$

then x^* is said to be a properly efficient (or strong Pareto) point. These points are of fundamental importance in welfare economics; see, e.g., [5, 8, 9]. They have also been used in other areas, e.g., in engineering; see [13, 17].

It is well known that for a convex (MP) the sets of efficient and properly efficient points are different (the former being the closure of the later). The two sets may be different even in the unconstrained cases or when the objectives are linear. This is demonstrated by the following examples:

4.1. Example. Consider the two objectives

$$\Phi^1(x) = \max\{0, x^2 \operatorname{sign} x\} \;\text{ and }\; \Phi^2(x) = (x-1)^2.$$

Here $x^* = 0$ is an efficient point that is not properly efficient.

4.2. Example. Consider the linear objectives

$$\Phi^1(x) = x_1 \quad \text{and} \quad \Phi^2(x) = x_2$$

and the feasible set determined by

$$x_1^2 + x_2^2 \leqslant 1.$$

Here the point $x^* = (-1 \quad 0)^T$ is efficient, but not properly efficient.

Our main objective is to find a simple condition when the two sets coincide. We assume that the objectives are LFS at some arbitrary fixed feasible point x^*. The constraints can be split into those that are LFS at x^* and those that are not. Denote the index set of LFS constraints by $\mathscr{L}$ and the non-LFS constraints by R. Now $P = \mathscr{L} \cup R$ and $\mathscr{L} \cap R = \emptyset$ and (MP) can be rewritten as follows:

$$(\mathrm{MP})' \qquad \begin{aligned} &\operatorname{Min}\{\Phi^k(x) \colon k \in Q\} \\ &\text{s.t.} \\ &f^i(x) \leqslant 0, \quad i \in \mathscr{L} \\ &f^j(x) \leqslant 0, \quad j \in R. \end{aligned}$$

Denote the active constraints from the sets $\mathscr{L}$, R, and P at x^* by

$$\begin{aligned} \mathscr{L}(x^*) &= \{i \in \mathscr{L} \colon f^i(x^*) = 0\} \\ R(x^*) &= \{j \in R \colon f^j(x^*) = 0\} \\ P(x^*) &= \mathscr{L}(x^*) \cup R(x^*) \end{aligned}$$

respectively, and those from R that are active on the entire subset

.

$$F_* = \{x \colon \Phi^k(x) \leqslant \Phi^k(x^*), \ k \in Q\} \cap F$$

by $R_*^=$, i.e.,

$$R_*^= = \{j \in R \colon x \in F_* \Rightarrow f^j(x) = 0\}.$$

The presence of the index set $R_*^=$ sets the efficient and properly efficient points apart.

We will apply the classic Charnes-Cooper observation to (MP). The following result, recently prooved in [15, Theorem 3.1], will be used in the proof. In addition to various cones, the result also uses the minimal index set of active constraints

$$R^= = \{j \in R \colon x \in F \Rightarrow f^j(x) = 0\}.$$

This set is defined for the single objective convex program

$$\text{(SP)} \qquad \begin{gathered} \operatorname{Min} f(x) \\ \text{s.t.} \\ f^i(x) \leqslant 0, \quad i \in \mathscr{L} \\ f^j(x) \leqslant 0, \quad j \in R. \end{gathered}$$

4.3. Theorem ([15]). *A feasible point x^* of the convex program* (SP) *is optimal if, and only if*

$$D_f^{<}(x^*) \bigcap \Big\{ \bigcap_{j \in R(x^*) \setminus R^=} D_{f^j}^{<}(x^*) \Big\} \bigcap \operatorname{conv} \Big\{ \bigcap_{j \in R^=} D_{f^j}^{=}(x^*) \Big\} \bigcap \Big\{ \bigcap_{i \in \mathscr{L}(x^*)} D_{f^i}^{\leqslant}(x^*) \Big\} = \emptyset.$$

In the absence of the set $R_*^=$ we have the following result:

4.4. Theorem. *Consider the multi-objective convex program* (MP) *and a feasible point x^*. Assume that all objectives are LFS at x^* and that $R_*^= = \emptyset$. Then the following three statements are equivalent:*

(i) *x^* is efficient.*

(ii) *x^* is properly efficient.*

(iii) *The system*

$$\sum_{k \in Q} w_k h^k + \sum_{i \in P(x^*)} \lambda_i h^i = 0$$
$$w_k > 0, \quad k \in Q, \quad \lambda_i \geqslant 0, \quad i \in P(x^*)$$

is consistent for some subgradients

$$h^k \in \partial \Phi^k(x^*), \quad k \in Q, \quad h^i \in \partial f^i(x^*), \quad i \in P(x^*).$$

Proof. Since (ii)$\Rightarrow$(i) is obvious, it is enough to prove the implications

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}.$$

First, the classic Charnes-Cooper observation, Theorem 4.3, and the fact that $R_*^= = \emptyset$, tell us that the point x^* is efficient if, and only if, there exist some subgradients

$$h^0 \in \partial\Big(\sum_{k \in Q} \Phi^k(x^*) \Big), \; h^k \in \partial \Phi^k(x^*), \; k \in Q, \; h^i \in \partial f^i(x^*), \; i \in P(x^*)$$

such that the system

$$
(4.1) \qquad \begin{gathered} h^0 + \sum_{k\in Q} \lambda_k h^k + \sum_{i\in P(x^*)} \lambda_i h^i = 0 \\ \lambda_k \geqslant 0, \quad k \in Q, \quad \lambda_i \geqslant 0, \quad i \in P(x^*) \end{gathered}
$$

is consistent. Using the familiar properties of the subgradients, we note that

$$
\partial\Big(\sum_{k\in Q} \Phi^k(x^*)\Big) = \sum_{k\in Q} \partial\Phi^k(x^*).
$$

Hence h^0 can be decomposed as

$$
h^0 = \sum_{k\in Q} H^k
$$

for some $H^k \in \partial\Phi^k(x^*)$, $k \in Q$. The system (4.1) can now be written as

$$
\begin{gathered} \sum_{k\in Q} (H^k + \lambda_k h^k) + \sum_{i\in P(x^*)} \lambda_i h^i = 0 \\ \lambda_k \geqslant 0, \quad k \in Q, \quad \lambda_i \geqslant 0, \quad i \in P(x^*). \end{gathered}
$$

By convexity of the subdifferentials $\partial\Phi^k(x^*)$, $k \in Q$, it follows that

$$
H^k + \lambda_k h^k = w^k (h')^k
$$

for some $w^k > 0$ and some $(h')^k \in \partial\Phi^k(x^*)$. Hence (4.1) implies the statement (iii).

If (iii) holds, then x^* is an optimal solution of the program

$$
\begin{aligned} &\text{Min} \sum_{k\in Q} w_k \Phi^k(x) \\ &\text{s.t.} \quad f^i(x) \leqslant 0, \quad i \in P \end{aligned}
$$

again by Theorem 4.3. Since the weights $w_k \in Q$ in the objective function are all positive, the point x^* is properly efficient. This establishes (iii)$\Rightarrow$(ii) and the proof is complete. □

The situation described in Example 4.3 shows that the first two statements of the theorem are not equivalent if $R_*^= \neq \emptyset$. (In that example $R = R_*^= = \{1\}$.)

The next example shows that the presence of this set generally destroys the equivalence between the first and the third statement.

4.5. Example. Consider the bi-objective program

$$\begin{aligned} &\operatorname{Min}\{\Phi^1 = x_3, \Phi^2 = 0\} \\ &\quad \text{s.t.} \\ &f^1(x) = x_1 \leqslant 0 \\ &f^2(x) = \operatorname{dist}(x - x^*, C) \leqslant 0 \end{aligned}$$

where C is the "ice-cream" cone

$$C = \{x \in \mathbb{R}^3 : 2x_1x_2 \geqslant x_3^2,\ x_1 \geqslant 0,\ x_2 \geqslant 0\}$$

and $x^* = 0 \in \mathbb{R}^3$. Note that the objectives and the first constraint are LFS and that f^2 is convex but not LFS at $x^* = 0$. Here $\mathscr{L} = \{1\}$, $R = \{2\}$. We calculate

$$\begin{aligned} F_* &= C \cap \{x = (x_i) \in \mathbb{R}^3 : x_1 \leqslant 0, x_3 \leqslant 0\} \\ &= \{[0 \ \ x_2 \ \ 0]^T : x_2 \geqslant 0\} \end{aligned}$$

and find that $R_*^= = \{2\}$. The origin is an efficient (and also a properly efficient) point but the system in the statement (iii) is

$$w_1[0 \ \ 0 \ \ 1] + w_2[0 \ \ 0 \ \ 0] + \lambda_1[1 \ \ 0 \ \ 0] = [0 \ \ 0 \ \ 0]$$

for some $w_1 > 0, w_2 > 0$ and $\lambda_1 \geqslant 0$; clearly inconsistent.

An important special case follows.

4.6. Corollary. *Consider the multi-objective convex program* (MP) *and its feasible point x^*. Assume that all objective functions and the constraints are LFS at x^*. Then the following statements are equivalent:*

(i) *x^* is efficient.*

(ii) *x^* is properly efficient.*

(iii) *The system*

$$0 \in \sum_{k\in Q} w_k \partial\Phi^k(x^*) + \sum_{i\in P(x^*)} \lambda_i \partial f^i(x^*)$$
$$w_k > 0, \quad k \in Q, \quad \lambda_i \geqslant 0, \quad i \in P(x^*)$$

is consistent.

Proof. Since all functions are LFS at x^*, we have $R = \emptyset$, and hence $R_*^= = \emptyset$. □

The above Corollary 4.6 is applicable to a wide class of multi-objective programs:

4.7. Example. Consider the multi-objective program

$$\begin{aligned} \text{Min} & \left\{|x_1| + |x_2|,\ x_3,\ \operatorname*{Max}_{i=1,2} |x_i|\right\} \\ \text{s.t.} \quad & |x_1| + |x_2| + |x_3| \leqslant 10 \\ & x_1 - |x_2| + 2x_3 \leqslant 20 \\ & \exp(-x_3) \leqslant 1. \end{aligned}$$

Since all functions are LFS at every point here, the efficient and properly efficient points coincide.

Remarks. (i) Since the set $R_*^=$ is defined relative to F_* (and not F), this set is generally nonempty in the presence of Slater's condition.

(ii) If all functions in (MP) are linear, then they are LFS, and the above results are readily applicable. This is well known.

(iii) In view of the modified Charnes-Cooper observation (Theorem 3.2) we note that, for LFS objectives, an efficient point may induce a zero weight for an objective Φ^k, $k \in Q \setminus Q^=(x^*)$. However, in that case there also exists a positive weight for the same objective (by Corollary 4.6).

5. The general case

If the objectives are LFS, but not all the constraints, then the sets of efficient and properly efficient points do not generally coincide. It is curious that in this case the characterization of efficient points assumes an asymptotic form (involving the closure of a cone). In what follows we denote the polar of a set M by M^+, i.e.,

$$M^+ = \{u \colon u^T x \geqslant 0, \quad \text{for every } x \in M\}.$$

Recall that M^+ is a closed convex cone for any M.

5.1. Theorem. *Consider the multi-objective convex program (MP) and its feasible point x^*. Assume that the objectives Φ^k, $k \in Q$ are LFS at x^*. Then x^* is an efficient point if, and only if, the system*

$$\sum_{k \in Q} (h^k)^T + \sum_{j \in R(x^*) \setminus R_*^=} \lambda_j (h^j)^T \in$$

$$\operatorname{cl}\left\{ \Big(\bigcap_{j \in R_*^=} D_{f^j}^=(x^*) \Big)^+ + \sum_{k \in Q} \{u_k \partial \Phi^k(x^*) \colon u_k \leqslant 0\} + \sum_{i \in \mathscr{L}(x^*)} \{v_i \partial f^i(x^*) \colon v_i \leqslant 0\} \right\}$$

$$\lambda_j \geqslant 0, \quad j \in R(x^*) \setminus R_*^=$$

is consistent for some

$$h^k \in \partial\Phi^k(x^*), \quad k \in Q \quad and \quad h^j \in \partial f^j(x^*), \quad j \in R(x^*) \setminus R_*^=.$$

P r o o f. Apply Theorem 4.3 to (CCO) and use the fact that

$$\left\{D_{\Phi^i}^{\leqslant}(x^*)\right\}^+ = \{u_k \partial\Phi^k(x^*) \colon u_k \leqslant 0\}, \quad k \in Q$$

and

$$\left\{D_{f^i}^{\leqslant}(x^*)\right\}^+ = \{v_i \partial f^i(x^*) \colon v_i \leqslant 0\}, \quad i \in P(x^*);$$

these functions being LFS at x^*. □

The closure requirement in the above theorem can be omitted for programs in one or two variables, but not, generally, for programs in three or more variables. This follows from the fact that the sum of two closed convex cones is not generally closed in $\mathbb{R}^r$, when $r \geqslant 3$. The claim will be demonstrated by an example.

5.2. E x a m p l e. Consider the single-objective program with an LFS objective:

$$\begin{aligned} &\operatorname{Min} \Phi^1(x) = x_3 \\ &\text{s.t.} \qquad f^1(x) = x_1 \leqslant 0 \\ &\qquad\qquad f^2(x) = g(x) \leqslant 0 \end{aligned}$$

where $g\colon \mathbb{R}^3 \to \mathbb{R}$ is defined as follows:

$$g(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } 2x_1x_2 \geqslant x_3^2, x_1 \geqslant 0, x_2 \geqslant 0, \\ x_1^2 + x_2^2 + x_3^2, & \text{if } 2x_1x_2 \geqslant x_3^2, x_1 \leqslant 0, x_2 \leqslant 0, \\ \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - (x_1 + x_2)\{[\frac{1}{2}(x_1 - x_2)]^2 + \frac{1}{2}x_3^2\}^{1/2}, & \\ & \text{otherwise.} \end{cases}$$

The function g was introduced in [18]. It is convex and differentiable at $x^* = 0 \in \mathbb{R}^3$. Its cone of directions of constancy at the origin is the ice-cream cone C from Example 4.5. Here the feasible set is

$$F = \{[0 \;\; x_2 \;\; 0]^T \colon x_2 \geqslant 0\}$$

and the optimal solution is the origin $x^* = 0$. We also find that

$$Q = \{1\}, \quad R(x^*) = R_*^= = \{2\}, \quad \mathscr{L}(x^*) = \{1\}.$$

The system in Theorem 5.1 becomes

$$[0 \;\; 0 \;\; 1] \in \mathrm{cl}\{C + [v_1 \;\; 0 \;\; u_1] \colon u_1 \leqslant 0, \; v_1 \leqslant 0\}.$$

Using the sequences

$$c^k = [k \;\; 1/2k \;\; 1], \; v_1^k = [-k \;\; 0 \;\; 0], \quad \text{and} \quad u = [0 \;\; 0 \;\; 0], \; k = 1, 2, \ldots$$

the closure condition is satisfied, since

$$[0 \;\; 0 \;\; 1] = \lim_{k \to \infty} [0 \;\; 1/2k \;\; 1].$$

Without the closure, the system is

$$[0 \;\; 0 \;\; 1] = [c_1 + v_1, \;\; c_2, \;\; c_3 + u_1]$$

for some $2c_1c_2 \geqslant c_3^2$, $c_1 \geqslant 0$, $c_3 \geqslant 0$, $u_1 \leqslant 0$, $v_1 \leqslant 0$. Its consistency would imply $c_2 = 0$ and further $c_3 = 0$ and $u_1 = 1$. The later contradicts $u_1 \leqslant 0$. This confirms that, generally, the closure condition cannot be omitted in Theorem 5.1.

If the cones

$$(5.1) \qquad D^{=}_{f^j}(x^*), \quad j \in R^{=}_*$$

are polyhedral, then the characterization is significantly simplified:

5.3. Corollary. *Consider the multi-objective convex program* (MP) *and its feasible point x^*. Assume that the objectives Φ^k, $k \in Q$ are* LFS *at x^* and that the cones* (5.1) *are polyhedral. Then x^* is an efficient point if, and only if, the system*

$$\sum_{k \in Q} w_k (h^k)^T + \sum_{i \in P(x^*) \setminus R^{=}_*} \lambda_i (h^i)^T \in \left\{ \bigcap_{j \in R^{=}_*} D^{=}_{f^j}(x^*) \right\}^+$$
$$w_k > 0, \; k \in Q; \; \lambda_i \geqslant 0, \; i \in P(x^*) \setminus R^{=}_*$$

is consistent for some

$$h^k \in \partial \Phi^k(x^*), \quad k \in Q \quad \text{and} \quad h^i \in \partial f^i(x^*), \quad i \in P(x^*) \setminus R^{=}_*.$$

5.4. Example. Consider the program

$$\begin{aligned} &\mathrm{Min}\{|x_1|, \; |x_2|\} \\ &\text{s.t.} \qquad x_1^2 + x_2^2 \leqslant 1. \end{aligned}$$

At $x_1^* = -1,\ x_2^* = 0$, we find that

$$F_* = \{x = (x_i) \in \mathbb{R}^2 \colon |x_1| \leqslant 1,\ |x_2| \leqslant 0,\ x_1^2 + x_2^2 \leqslant 1\}$$

and hence $R = \{1\}$, but $R_*^= = \emptyset$. The cone (5.1) is $\mathbb{R}^2$, a polyhedral set. Therefore x^* is an efficient point if, and only if, the system

$$w_1[-1\ 0] + w_2[0\quad \varrho] + \lambda[-2\quad 0] = [0\quad 0]$$

is consistent for some $w_1 > 0$, $w_2 > 0$, $\lambda \geqslant 0$ and some ϱ from the interval $[-1,\ 1]$. But this system is inconsistent. Hence x^* cannot be an efficient point.

For the sake of comparison we recall the program from Example 4.2.

5.5. Example. Consider

$$\begin{aligned} &\operatorname{Min}\{x_1,\ x_2\} \\ &\text{s.t.} \qquad x_1^2 + x_2^2 \leqslant 1. \end{aligned}$$

At $x_1^* = -1$, $x_2^* = 0$, this time we have

$$\begin{aligned} F_* &= \{x = (x_i) \in \mathbb{R}^2 \colon x_1 \leqslant -1,\ x_2 \leqslant 0,\ x_1^2 + x_2^2 \leqslant 1\} \\ &= \{x^*\}. \end{aligned}$$

Hence

$$R = R_*^= = \{1\}.$$

The system from Corollary 5.3, is

$$w_1[-1\quad 0] + w_2[0\quad \varrho] \in \{[0\quad 0]\}^+$$

for some $w_1 > 0$, $w_2 > 0$ and some $-1 \leqslant \varrho \leqslant 1$; clearly satisfied. Therefore x^* is efficient.

6. Applications

The above results are readily applicable to simultaneous approximations involving l_1 and l_∞ norms. These norms have been used separately in many contexts from optimal control (e.g., [3]) to a highway construction (e.g., [6]). A possible advantage of using l_1 and l_∞ norms over the Euclidean norm is based on their observed robustness. Indeed, it has been reported, e.g., by Charnes [4] that "the linear programming models, following from least-absolute-value regressions have been notably robust and far less sensitive to approximate linear dependencies than "classical" unconstrained least squares regression. This is the experience of the last thirty-plus years." For a simultaneous approximation by l_1 and the Euclidean norm, see, e.g., [1, 11, 12].

6.1. Example. (Adopted from [6] to the bi-objective case.) Let $c = c(t)$ denote the shape of a terrain where a road with the shape $x = x(t)$ is to be built, $0 \leqslant t \leqslant T$. The usual assumption is that the cost is proportional to the amount of dirt to be added or removed. However, one can also require uniform "smootness" of the road relative to the shape of the terrain. Together with restrictions on the slopes and their rates of change, a model is

$$\begin{aligned}
\text{Min} & \left\{ \int_0^T |x(t) - c(t)|\,\mathrm{d}t,\ \operatorname*{Max}_{0\leqslant t\leqslant T} |x(t) - c(t)| \right\} \\
\text{s.t.} & \quad |\dot{x}(t)| \leqslant \beta_1 \\
& \quad |\ddot{x}(t)| \leqslant \beta_2,\quad 0 \leqslant t \leqslant T \\
& \quad x(0) = \alpha,\quad x(T) = \beta
\end{aligned}$$

where α, β, β_1, and β_2 are some prescribed constants. After a discretization at N unit intervals by t_i, $i = 0, 1, \ldots, N$ and, using the notation

$$c(t_i) = c_i,\ x(t_i) = x_i,\ \dot{x}(t_i) = x_{1,i},\ i = 0, 1, \ldots, N$$

the model becomes

$$\begin{aligned}
\text{Min} & \left\{ \sum_{i=1}^{N} |x_i - c_i|,\ \operatorname*{Max}_{i=1,2,\ldots,N} |x_i - c_i| \right\} \\
\text{s.t.} & \quad x_i - x_{i-1} = x_{1,i-1},\ i = 1, \ldots, N \\
& \quad -\beta_1 \leqslant x_{1,i} \leqslant \beta_1,\ i = 1, \ldots, N-1 \\
& \quad -\beta_2 \leqslant x_{1,i} - x_{1,i-1} \leqslant \beta_2,\ i = 1, \ldots, N-1 \\
& \quad x_0 = \alpha,\ x_N = \beta.
\end{aligned}$$

Since all functions are LFS we know that the efficient points are properly efficient and optimality can be checked by Corollary 4.6.

Let us illustrate how this works on a numerical example.

6.2. Numerical example. Consider the problem of building a road on the terrain of the shape $c = c(t)$ depicted in Figure 6.1 below. The shape of the road is $x = x(t)$ and it is required that $x(0) = 1$ and $x(6) = 1.5$. The slope should not exceed $\beta_1 = 0.1$ in magnitude. After a discretization by unit intervals, the problem becomes

$$\begin{aligned} &\operatorname{Min}\left\{\Phi^1 = \sum_{i=1}^{5} |x_i - c_i|,\ \Phi^2(x) = \underset{i=1,\ldots,5}{\operatorname{Max}} |x_i - c_i|\right\} \\ (6.1) \qquad &\text{s.t.} \qquad -0.1 \leqslant x_{i+1} - x_i \leqslant 0.1,\ i = 1,\ldots,5 \\ &x_0 = 1,\quad x_6 = 1.5. \end{aligned}$$

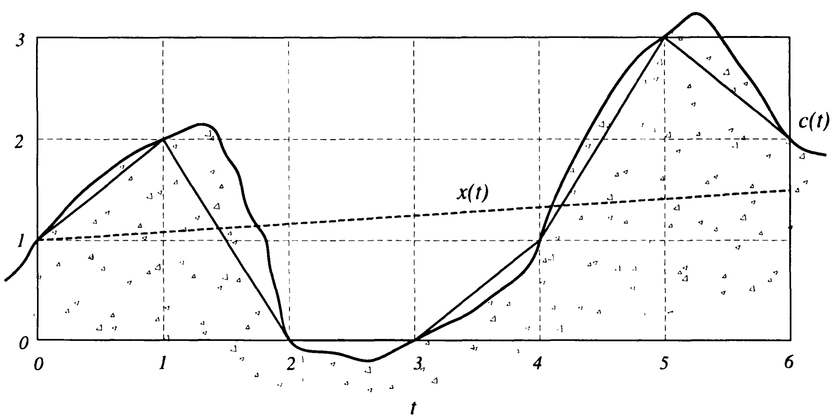

Fig. 6.1

The values of $c_i = c(t_i)$ are read from the figure. These are

t_i	0	1	2	3	4	5	6
c_i	1	2	0	0	1	3	2

Since the functions are LFS, every efficient solution can be obtained by solving the program

$$\operatorname{Min} \quad w_1\Phi^1(x) + w_2\Phi^2(x)$$

on the feasible set of (6.1) for some positive weights w_1 and w_2. Specify, e.g., $w_1 = w_2 = 1$. Applying the familiar substitutions

$$y_i^+ = \max(0, y_i) \quad \text{and} \quad y_i^- = \max(0, -y_i)$$

and

$$\varrho = \max |x_i - c_i|$$

to $y_i = x_i - c_i$, $i = 1, \ldots, 5$, we obtain a linear program of the form

$$\begin{aligned} &\operatorname{Min} d^T u \\ \text{s.t.} \qquad &Au \leqslant b \end{aligned}$$

where A is a 59×22 coefficient matrix. (Here 12 rows represent 6 two-sided constraints on $x_{i-1} - x_i$, $i = 0, \ldots, 5$. The initial conditions on x_0 and x_6 determine 4 next rows. Then 14 rows represent 7 equations $x_i - c_i = y_i^+ - y_i^-$, 7 rows represent $y_i^+ \geqslant 0$, 7 rows represent $y_i^- \geqslant 0$, 14 rows represent 7 two-sided inequalities $-\varrho \leqslant |x_i - c_i| \leqslant \varrho$, and, finally, 1 row represents $\varrho \geqslant 0$.) The size of the matrix is reduced by 20 rows after some economization. The elements of A are now -1, 0 and 1, the elements of d are all 0 and 1, while the elements of b range from -3 to 3. An optimal solution is

$$\begin{aligned} x^* &= (1.1\ 1.1\ 1.2\ 1.3\ 1.4)^T \\ \varrho^* &= 1.5 \\ y^{+*} &= (0\ 1.2\ 1.3\ 0.4\ 0.05)^T \\ y^{-*} &= (0.9\ 0\ 0\ 0\ 1.55)^T. \end{aligned}$$

Hence x^* is a desired optimal solution. (See Figure 6.1.) Its x^* is verified directly in the next example.

6.3. Example. Consider the program (6.1) and x^* from the preceding example. We wish to verify directly whether x^* is properly efficient.

First we find that

$$\partial \Phi^1(x^*) = \sum_{i=1}^{5} \partial |x_i^* - c_i| = (-1\ 1\ 1\ 1\ -1)^T$$

since $x_i^* \neq c_i$, $i = 1, \ldots, 5$. (Recall that c_i, $i = 1, \ldots, 5$ are the ordinates of the shape of the terrain. In particular, $c_5 = 3$.) Also, using the index set

$$I(x^*) = \left\{ k \colon |x_k - c_k| = \max_{i=1,\ldots,5} |x_i^* - c_i| \right\}$$

we find that

$$\begin{aligned} \partial \Phi^2(x^*) &= \text{conv hull}\{\text{sign}(x_i^* - c_i) e^i \colon i \in I(x^*)\} \\ &= \text{sign}(1.4 - 3) e^5 \\ &= (0\ 0\ 0\ 0\ -1)^T. \end{aligned}$$

Hence x^* is efficient, by Corollary 4.5, if, and only if, the system

$$w_1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + w_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$
$$+\lambda_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \lambda_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is consistent for some $w_1 > 0$, $w_2 > 0$, $\lambda_i \geqslant 0$, $i = 1, \ldots, 5$. The choice

$$w_1^* = w_2^* = 1, \ \lambda_1^* = \lambda_2^* = \lambda_5^* = 1, \ \lambda_3^* = 2, \ \lambda_4^* = 3$$

confirms the optimality.

Acknowledgement. The authors acknowledge a helpful discussion with Alejandro Balbas, Antonio Heras, and Moisei Mayergoiz. They thank the referee for his suggestions that resulted in a clearer presentation and for detecting a numerical error.

References

[1] *S. M. Allende, C. N. Bouza*: A parametric mathematical programming approach to the estimation of the coefficients of the linear regression model. Parametric Optimization and Related Topics III (J. Guddat et al., eds.). Akademie Verlag, Berlin, 1993, pp. 9–20.

[2] *A. Ben-Israel, A. Ben-Tal, S. Zlobec*: Optimality in Nonlinear Programming: A Feasible Directions Approach. Wiley Interscience, New York, 1981.

[3] *M. D. Canon, C. D. Cullum, E. Polak*: Theory of Optimal Control and Mathematical Programming. McGraw-Hill, New York, 1970.

[4] *A. Charnes*: Correspondence. 1989.

[5] *A. Charnes, W. W. Cooper*: Management Models and Industrial Applications of Linear Programming, Vol. I. Wiley, New York, 1961.

[6] *S. J. Citron*: Elements of Optimal Control. Holt, Rinehard and Winston, New York, 1969.

[7] *R. W. Farebrother*: The historical development of L_1 and L_∞ estimation procedures. Statistical Data Analysis Based on L_1-norm and Related Methods (Y. Dodge, ed.). North Holland, Amsterdam, 1987, pp. 37–63.

[8] *M. D. Intriligator*: Mathematical Optimization and Economic Theory. Prentice Hall, Englewood Cliffs, New Jersey, 1972.

[9] *S. Karlin*: Mathematical Methods in Theory of Games, Programming and Economics, Vol. I. Addison-Wesley, Reading, Massachussetts, 1959.

[10] *O. J. Karst*: Linear curve fitting using least deviations. Journal of the American Statistical Association *53* (1958), 118–132.
[11] *Lj. Martić*: Bicriterial programming in regression analysis. Proceedings of KOI '91 (Lj. Martić and L. Neralić, eds.). Faculty of Economics, Zagreb, 1991, pp. 37–45. (In Croatian.)
[12] *Lj. Martić*: A simple regression by l_1 and L_∞ criteria. Proceedings of KOI '92 (V. Bahovec, Lj. Martić and L. Neralić, eds.). Croatian Operational Research Society, Rovinj, 1992, pp. 17–32. (In Croatian.)
[13] *M. E. Salukvadze*: Vector-Valued Optimization Problems in Control Theory. Academic Press, New York, 1969.
[14] *F. Sharifi Mokhtarian*: Mathematical Programming with LFS Functions. M. Sc. Thesis, McGill University, Montreal, Quebec, 1992.
[15] *F. Sharifi Mokhtarian, S. Zlobec*: Mathematical Programming with LFS Functions. Utilitas Mathematica *45* (1994), 3–15.
[16] *M. van Rooyen, X. Zhou, S. Zlobec*: A saddle-point characterization of Pareto optima. Mathematical Programming *67* (1994), 77–88.
[17] *T. L. Vincent, W. J. Grantham*: Optimality in Parametric Systems. Wiley Interscience, New York, 1981.
[18] *X. Zhou, F. Sharifi Mokhtarian, S. Zlobec*: A simple constraint qualification in convex programming. Mathematical Programming *61* (1993), 385–397.
[19] *S. Zlobec*: Two characterizations of Pareto minima in convex multicriteria optimization. Aplikace Matematiky *29* (1984), 342–349.
[20] *S. Zlobec*: Characterizations of optimality in nonconvex programming. The Fourteenth Symposium on Mathematical Programming with Data Perturbations. The George Washington University, Washington, D. C., May 23, 1992 (tutorial lecture).

Authors' addresses: *Luka Neralić*, University of Zagreb, Faculty of Economics, Kennedyjev trg 6, 10000 Zagreb, Croatia; *Sanjo Zlobec*, McGill University, Department of Mathematics and Statistics, Burnside Hall, 805 Sherbrooke Street West, Montreal, Canada H3A 2K6.