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BOUNDS AND ESTIMATES ON THE EFFECTIVE PROPERTIES FOR NONLINEAR COMPOSITES

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Abstract. In this paper we derive lower bounds and upper bounds on the effective properties for nonlinear heterogeneous systems. The key result to obtain these bounds is to derive a variational principle, which generalizes the variational principle by P. Ponte Castaneda from 1992. In general, when the Ponte Castaneda variational principle is used one only gets either a lower or an upper bound depending on the growth conditions. In this paper we overcome this problem by using our new variational principle together with the bounds presented by Lukkassen, Persson and Wall in 1995. Moreover, we also present some examples where the bounds are so tight that they may be used as a good estimate of the effective behavior.

Keywords: homogenization, effective properties, variational methods, nonlinear composites

MSC 2000: 35B27, 73B27

1. Introduction

A composite material consists of two ore more distinct materials which are intimately mixed. Even if the composite material is strongly heterogeneous on a local level it will behave as a homogeneous medium when the size of the typical heterogeneity becomes small compared to the specimen under consideration. To find the effective properties which describe the composite by using the knowledge the physical properties of the constituents is an extremely difficult task. The field of mathematics that rigorously defines the notion of effective properties is called homogenization. Mathematical models of different physical phenomena in composite materials often involve functions of the form $g(\frac{x}{\varepsilon}, \xi)$ where ε is a small parameter. The function

$g(x,\xi)$ is assumed to be periodic, almost periodic, or to be a realization of a stationary random field in the first variable. In this paper we will consider the periodic case and we will denote the cell of periodicity by Y. This means that $g(\frac{x}{\varepsilon},\xi)$ is a rapidly oscillating function for small values of ε. Many problems in homogenization are concerned with the asymptotic behavior as ε tends to 0 of integral functionals of the form

$$\int_\Omega g\big(\frac{x}{\varepsilon}, Du(x)\big)\,\mathrm{d}x, \quad \Omega\subset\mathbb{R}^N,$$

where u belongs to some subset of the space $H^{1,p}(\Omega)$ and represents the state of the material. Functionals of this form appear naturally when the physical problem is described by some minimum energy principle of the form

$$E_\varepsilon = \min\left\{\int_\Omega g\Big(\frac{x}{\varepsilon}, Du(x)\Big)\,\mathrm{d}x + \int_\Omega fu\,\mathrm{d}x\right\}.$$

It is well known that if $g(x,\cdot)$ satisfies suitable growth conditions, then the energy $E_\varepsilon\to E_{\text{hom}}$ as $\varepsilon\to 0$, where

$$E_{\text{hom}} = \min\left\{\int_\Omega g_{\text{hom}}(Du(x))\,\mathrm{d}x + \int_\Omega fu\,\mathrm{d}x\right\}$$

with

$$g_{\text{hom}}(\xi) = \min_{u\in H^{1,p}_{\text{per}}(Y)}\int_Y g(x,\xi+Du)\,\mathrm{d}x. \tag{1}$$

Note that g_{hom} does not depend on x, i.e., g_{hom} describes a homogeneous material. Moreover, g_{hom} is given by a minimum problem where the underlying domain is the periodic cell Y. This means that E_{hom} approximates the actual energy E_ε for small values of ε. We say that the effective properties or homogenized properties are defined by g_{hom}. The limit process $E_\varepsilon\to E_{\text{hom}}$ is usually studied by the so called Γ-convergence. For more information concerning Γ-convergence the reader is referred e.g. to [4] and [6].

When the distribution of the different materials is known it is possible to use some numerical method to compute $g_{\text{hom}}(\xi)$ by the formula (1). Another approach is to find lower and upper bounds of $g_{\text{hom}}(\xi)$ for different classes of material distributions. In this paper we will be concerned with the latter approach. The study of estimating the effective behavior of heterogeneous systems is a classical problem that has attracted the attention of numerous investigators in many fields of applications. It is well known that the effective properties of a composite material are strongly dependent on the microstructure and that they are not given by any simple weighted average of the properties of the constituent materials. Most of the efforts so far have been concentrated on the effective behavior of linear systems. For linear problems very much is known about bounds for different problems and classes of materials. There are also many results concerning microstructures for which the bounds are attained. Many of the results in this direction can be found in the book [6]. For nonlinear material behavior much less is known.

Many of the results in this direction are based on an extension of the Hashin-Shtrikman variational principle to nonlinear problems, see [10] and [11], or on the variational principle by Ponte Castaneda which makes it possible to obtain nonlinear bounds from any bounds known for the corresponding linear problem, see e.g. [2] and [3]. For the p-Poisson equation some bounds were presented in [9]. These bounds have played a central role in the recently developed optimal structures obtained by using reiterated homogenization, i.e. introducing g in the form $g(\frac{x}{\varepsilon}, \dots, \frac{x}{\varepsilon^m}, \xi)$, see [1], [7] and [13].

In this paper we will derive new bounds and present some examples where these bounds are very tight. The main idea in the proofs is to combine the results in [9] with a generalized form of the Ponte Castaneda variational principle. We remark that in general when the Ponte Castaneda variational principle is used one only gets either a lower bound or an upper bound, while the new upper and lower bounds are valid at the same time for some ξ.

This paper is organized in the following way: In order not to disturb our discussion later on we present some necessary preliminaries in Section 2. The announced variational principles are derived in Section 3. The new bounds are presented and proved in Section 4. Finally, Section 5 is reserved for some concrete applications and examples.

2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^N$ and let Y denote the unit cube in $\mathbb{R}^N$. We say that a function $\varphi : \mathbb{R}^N \to \overline{\mathbb{R}}$ is Y-periodic if $\varphi(x) = \varphi(x + e_i)$ for every $x \in \mathbb{R}^N$ and for every $i = 1, \dots, N$, where $\{e_1, \dots, e_n\}$ is the canonical basis in $\mathbb{R}^N$ and we call Y a cell of periodicity. We denote the Lebesgue measure of $Q \subset \mathbb{R}^n$ by $m(Q)$. The characteristic function of the set Q is denoted by χ_Q. Let $\mathcal{A}_j^- = \{A_{j,k_j}^-\}, k_j = 1, \dots, K_j$, be a partition of the unit cube with sides $\{e_1, \dots, e_{j-i}, e_{j+1}, \dots, e_N\}$ and let $E_{j,k_j}^- = A_{j,k_j}^- \times (0,1)$. Let $\{A_k^-\}$ be a partition of Y where A_k^- is of the form

$$A_k^- = \Omega_i \cap E_{1,k_1}^- \cap \dots \cap E_{N,k_N}^-.$$

By I we mean the index set such that $Y = \cup_{k\in I} A_k^-$ and

$$I^i_{j,k_j} = \{k \in I\colon\ A_k^- \subset \Omega_i,\ A_k^- \subset E^-_{j,k_j}\}.$$

In the corresponding way we define $\mathcal{A}_j^+ = \{A^+_{j,k_j}\}, k_j = 1,\ldots,K_j$, to be a partition of the unit cube with sides $\{e_j\}$, $E^+_{j,k_j} = A^+_{j,k_j} \times (0,1)^{N-1}$ and $\{A_k^+\}$ is a partition of Y where A_k^+ are of the form

$$A_k^+ = \Omega_i \cap E^+_{1,k_1} \cap \ldots \cap E^+_{N,k_N}.$$

By J we mean the index set such that $Y = \cup_{k\in J} A_k^+$ and we denote

$$J^i_{j,k_j} = \{k \in I\colon\ A_k^+ \subset \Omega_i,\ A_k^+ \subset E^+_{j,k_j}\}.$$

Moreover, let g denote a function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ and assume that g satisfies the following conditions:

(i) The function $x \mapsto g(x,s)$ is measurable and Y-periodic for every s.

(ii) The function $s \mapsto g(x,s)$ is convex for every x.

(iii) $g(x,0) = 0$ for every x.

(iv) $c_1|s|^p \leqslant g(x,s) \leqslant c_0 + c_2|s|^p$ for every x and s where $c_0, c_1, c_2 > 0$ and $1 < p < \infty$.

In [9] some new upper and lower bounds for the p-Poisson equation were presented, i.e. when $g(x,s) = a(x)\,|s|^p$. Below we review some of these results. The main result was that g_{hom} defined by [1] satisfies the following bounds:

$$q_j^-(a,k,p) \leqslant g_{\text{hom}}(ke_j) \leqslant q_j^+(a,k,p), \tag{2}$$

where

$$q_j^-(a,k,p) = |k|^p \int_Y \left(\int_0^1 a^{\frac{1}{1-p}}\,\mathrm{d}x_j\right)^{1-p} \mathrm{d}x,$$
$$q_j^+(a,k,p) = |k|^p \left(\int_Y \left(\int_0^1 \ldots \int_0^1 a\,\mathrm{d}x_1 \ldots \mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1} \ldots \mathrm{d}x_n\right)^{\frac{1}{1-p}} \mathrm{d}x\right)^{1-p}.$$

The result reads that for the lower bound we first calculate the p-harmonic mean of a along each line in the e_j direction to obtain a function which only depends on the variables $x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N$ and then we compute the arithmetic mean of this function. The upper bound is given by first calculating the arithmetic mean of a for each fixed x_j to obtain a function which only depends on the variable x_j and then computing the p-harmonic mean of this function.

In the isotropic and linear case, $p = 2$, this implies that we have an estimate of $g_{\mathrm{hom}}(\xi)$ for arbitrary ξ, namely

$$|\xi|^2 q_j^-(a,1,2) \leqslant g_{\mathrm{hom}}(\xi) \leqslant |\xi|^2 q_j^+(a,1,2)$$

for $j = 1,\dots,N$. It was also shown that it is possible to obtain approximations Q_j^- and Q_j^+ of the bounds q_j^- and q_j^+ respectively, where the p-harmonic mean and the arithmetic mean are calculated only finitely many times. Indeed,

$$Q_j^-(a,k,p,\mathcal{A}_j^-) \leqslant q_j^-(a,k,p) \leqslant g_{\mathrm{hom}}(ke_j) \leqslant q_j^+(a,k,p) \leqslant Q_j^+(a,k,p,\mathcal{A}_j^+),$$

where the approximative bounds have the form

$$Q_j^-(a,k,p,\mathcal{A}_j^-) = |k|^p \int_Y a^- \,\mathrm{d}x, \tag{3}$$

$$Q_j^+(a,k,p,\mathcal{A}_j^+) = |k|^p \left(\int_Y (a^+)^{\frac{1}{1-p}} \,\mathrm{d}x\right)^{1-p} \tag{4}$$

with

$$\begin{aligned}
a^-(x) &= \sum_{k_j=1}^{K_j} \chi_{E_{j,k_j}^-}(x) \left(\frac{1}{m(E_{j,k_j}^-)} \int_{E_{j,k_j}^-} a^{\frac{1}{1-p}} \,\mathrm{d}x\right)^{1-p}, \\
a^+(x) &= \sum_{k_j=1}^{K_j} \chi_{E_{j,k_j}^+}(x) \frac{1}{m(E_{j,k_j}^+)} \int_{E_{j,k_j}^+} a \,\mathrm{d}x.
\end{aligned}$$

It was also shown that if $\mathcal{B}_j^-$ and $\mathcal{B}_j^+$ are refinements of $\mathcal{A}_j^-$ and $\mathcal{A}_j^+$ respectively, then

$$Q_j^-(a,k,p,\mathcal{A}_j^-) \leqslant Q_j^-(a,k,p,\mathcal{B}_j^-) \leqslant Q_j^+(a,k,p,\mathcal{B}_j^+) \leqslant Q_j^+(a,k,p,\mathcal{A}_j^+).$$

This means that the approximation of the bounds will be better and better for finer and finer partition. In the special case $p = 2$ a more general result was proved in [8], namely

$$\sum_{j=1}^N \xi_j^2 q_j^-(a,1,2) \leqslant g_{\mathrm{hom}}(\xi) \leqslant \sum_{j=1}^N \xi_j^2 q_j^+(a,1,2). \tag{5}$$

More results concerning bounds of this type can be found in [7]. Especially, there are some very interesting results concerning optimal micro structures in the sense that the bounds are attained.

We will use the bounds presented above in combination with an extension of the Ponte Castaneda variational principle to obtain more general nonlinear bounds for the homogenized operator g_{hom}.

3. Variational principles

In this section we derive some new variational principles which generalize the variational principles presented by Ponte Castaneda, see [3], in such a way that it is possible to use a nonlinear reference medium. We will present the variational principle using homogenization to define the effective properties while Castaneda presented the variational principle using the definition of effective properties by Hill.

Let us now define a function $f_r\colon\ \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ by

$$f_r(x,s) = \begin{cases} g(x, s^{\frac{1}{r}}), & s \geqslant 0, \\ \infty, & s < 0, \end{cases}$$

where $r \geqslant 1$. The function $s \mapsto f_r(x,s)$ is continuous for every x and $s > 0$ but not necessarily convex, $f(x,0) = 0$ for every x and $f(x,s) \geqslant 0$ for every x and s. The polar function, $f_r^*\colon\ \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$, corresponding to f_r is defined by

$$f_r^*(x,t) = \sup_{s\in\mathbb{R}}[st - f_r(x,s)] = \sup_{s\geqslant 0}\big[st - g(x, s^{\frac{1}{r}})\big]. \tag{6}$$

We note that f^* satisfies

$$f_r^*(x,t) \geqslant 0 \text{ and } f_r^*(x,0) = 0. \tag{7}$$

The bipolar function, $f_r^{**}\colon\ \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$, is defined by

$$f_r^{**}(x,s) = \sup_{t\in\mathbb{R}}[ts - f_r^*(x,t)]. \tag{8}$$

By (7) it follows that it is enough to take the supremum over $t \geqslant 0$ in (8) for $s \geqslant 0$, i.e.,

$$f_r^{**}(x,s) = \sup_{t\geqslant 0}[ts - f_r^*(x,t)] \text{ for } s \geqslant 0. \tag{9}$$

It turns out that this fact will be important further on. If r is chosen such that $s \mapsto f_r(x,s)$ is convex we have the equality

$$f_r^{**}(x,s) = f_r(x,s). \tag{10}$$

For a proof of this equality see e.g. [12], p. 91. Moreover, we define a nonnegative function F_r on $L^\alpha(\Omega)$, $1 \leqslant \alpha \leqslant \infty$, by

$$F_r(u) = \int_\Omega f_r(x, u(x))\,\mathrm{d}x.$$

The corresponding polar F_r^* on $L^{\alpha'}(\Omega)$, $\alpha' = \frac{\alpha}{\alpha-1}$, is then defined by

$$F_r^*(u^*) = \sup_{u\in L^\alpha(\Omega)} \biggl[\int_\Omega u(x)u^*(x)\,\mathrm{d}x - \int_\Omega f_r(x,u(x))\,\mathrm{d}x\biggr].$$

It can also be proved that for all $u^* \in L^{\alpha'}(\Omega)$ we have

$$F_r^*(u^*) = \int_\Omega f_r^*(x,u^*(x))\,\mathrm{d}x. \tag{11}$$

A proof of this statement can be found e.g. in [5], p. 271. The bipolar F_r^{**} on $L^\alpha\Omega$ is defined by

$$F_r^{**}(u) = \sup_{u^*\in L^{\alpha'}(\Omega)} \biggl[\int_\Omega u(x)u^*(x)\,\mathrm{d}x - \int_\Omega f_r^*(x,u^*(x))\,\mathrm{d}x\biggr]. \tag{12}$$

Moreover, $F_r^{**}(u)$ can be written as

$$F_r^{**}(u) = \int_\Omega f_r^{**}(x,u(x))\,\mathrm{d}x. \tag{13}$$

For a proof see [5], p. 273. Let us now study the minimization problem which appears in the definition of the homogenized energy density function g_{hom}, see (1):

$$g_{\mathrm{hom}}(\xi) = \inf_{u\in H^{1,p}_{\mathrm{per}}(Y)} \int_Y g(x,|\xi + Du(x)|)\,\mathrm{d}x,$$

where $H^{1,p}_{\mathrm{per}}(Y) = \{u \in H^{1,p}(Y)\colon\ u \text{ is } Y\text{-periodic}\}$. Note that the assumptions on g guarantee existence and uniqueness of the problem. We are now able to state a variational principle suitable for obtaining lower bounds.

Theorem 1. *If r is chosen such that $s \mapsto f_r(x,s)$ is convex, then*

$$\begin{aligned} g_{\mathrm{hom}}(\xi) &= \inf_{u\in H^{1,p}_{\mathrm{per}}(Y)} \int_Y g(x,|\xi + Du(x)|)\,\mathrm{d}x \\ &= \sup_{a\geqslant 0} \biggl[\inf_{u\in H^{1,r}_{\mathrm{per}}(Y)} \int_Y a(x)|\xi + Du(x)|^r\,\mathrm{d}x - \int_Y f_r^*(x,a(x))\,\mathrm{d}x\biggr], \end{aligned}$$

where $a \in L^\infty(Y)$.

P r o o f. First we note that

$$\inf_{u\in H^{1,t}_{\mathrm{per}}(Y)} \int_Y g(x,|\xi + Du(x)|)\,\mathrm{d}x$$

is independent of the choice of $t \in [1, \infty)$, for a proof of this fact see e.g. [6], p. 423. From (10), (13), and (12) it follows that

$$
\begin{aligned}
&\inf_{u \in H^{1,p}_{\mathrm{per}}(Y)} \int_Y g(x, |\xi + Du(x)|)\,\mathrm{d}x \\
&\quad= \inf_{u \in H^{1,r}_{\mathrm{per}}(Y)} \int_Y f_r(x, |\xi + Du(x)|^r)\,\mathrm{d}x \\
&\quad= \inf_{u \in H^{1,r}_{\mathrm{per}}(Y)} \int_Y f_r^{**}(x, |\xi + Du(x)|^r)\,\mathrm{d}x \\
&\quad= \inf_{u \in H^{1,r}_{\mathrm{per}}(Y)} F_r^{**}(|\xi + Du(x)|^r) \\
&\quad= \inf_{u \in H^{1,r}_{\mathrm{per}}(Y)} \sup_{\substack{a \in L^\infty(Y) \\ a \geqslant 0}} \left[\int_Y a(x)|\xi + Du(x)|^r\,\mathrm{d}x - \int_Y f_r^*(x, a(x))\,\mathrm{d}x \right].
\end{aligned}
$$

The homogenized operator g_{hom} can thus be written as

$$
g_{\mathrm{hom}}(\xi) = \inf_{u \in H^{1,p}_{\mathrm{per}}(Y)} \sup_{\substack{a \in L^\infty(Y) \\ a \geqslant 0}} \left\{ \int_Y a(x)|\xi + Du(x)|^r\,\mathrm{d}x - \int_Y f_r^*(x, a(x))\,\mathrm{d}x \right\}.
$$

By using an appropriate version of the saddle point theorem, see [5], p. 175, it is possible to change the order of infimum and supremum obtaining

$$
g_{\mathrm{hom}}(\xi) = \sup_{\substack{a \in L^\infty(Y) \\ a \geqslant 0}} \left\{ \inf_{u \in H^{1,r}_{\mathrm{per}}(Y)} \int_Y a(x)|\xi + Du(x)|^r\,\mathrm{d}x - \int_Y f_r^*(x, a(x))\,\mathrm{d}x \right\},
$$

and the proof is complete. □

R e m a r k 1. Theorem 1 holds with an inequality for all r, since we always have that $f_r(x,t) \geqslant f_r^{**}(x,t)$.

We can now proceed in a similar way to obtain a variational principle suitable for obtaining upper bounds on $g_{\mathrm{hom}}(\xi)$. Let us define function $h_r\colon \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ by

$$
h_r(x,s) = \begin{cases} g(x, s^{\frac{1}{r}}), & s \geqslant 0, \\ -\infty, & s < 0, \end{cases}
$$

where $r \geqslant 1$. Let the superscript $*$ denote the corresponding concave polar. The variational principle can then be formulated as

Theorem 2. *If r is chosen such that $s \mapsto h_r(x,s)$ is concave, then*

$$g_{\text{hom}}(\xi) = \inf_{u \in H^{1,p}_{\text{per}}(Y)} \int_Y g(x, |\xi + Du(x)|)\,\mathrm{d}x$$
$$= \inf_{\substack{a \in L^\infty(Y) \\ a \geqslant 0}} \left[\inf_{u \in H^{1,r}_{\text{per}}(Y)} \int_Y a(x)|\xi + Du(x)|^r\,\mathrm{d}x - \int_Y h_r^*(x, a(x))\,\mathrm{d}x \right].$$

P r o o f. The proof follows by similar arguments as in the proof of Theorem 1. □

R e m a r k 2. Theorem 2 is valid with an inequality for all r, since we always have that $h_r(x,t) \leqslant h_r^{**}(x,t)$.

4. Bounds

In this section we will derive some bounds on the homogenized energy-density function g_{hom}. The main idea is to use the variational principles in Theorem 1 and Theorem 2 in combination with the known bounds for the p-Poisson equation, see (2) and (4), to obtain bounds for more general types of energy-density functions. We will study n-phase composites, i.e. the energy-density function $g(x,s)$ will be of the form

$$g(x,s) = \sum_{i=1}^{n} \chi_i(x) g_i(s), \tag{14}$$

where χ_i is the characteristic function of $\Omega_i = \{x \in Y \colon\ x \text{ is in the } i \text{ phase}\}$. First we give some results concerning lower bounds.

Theorem 3. *Let $\xi = \xi_j e_j$ and assume that the local energy density function g is such that the function $s \mapsto f_r(x,s)$ is convex. Then*

$$g_{\text{hom}}(\xi) \geqslant \sup_{a \in V} \left\{ q_j^-(a, \xi_j, r) - \sum_{i=1}^{n} m(\Omega_i) v_i(a_i) \right\},$$

where

$$V = \left\{ a \in L^\infty \colon\ a(x) = \sum_{i=1}^{n} a_i \chi_i(x),\ 0 < k_1 \leqslant a_i \leqslant k_2 < \infty \right\}$$

and

$$v_i(a_i) = \sup_{s \geqslant 0} \left\{ s a_i - g_i(s^{\frac{1}{r}}) \right\}.$$

P r o o f. By taking the supremum over the smaller set V in the variational principle in Theorem 1 we obtain that

$$g_{\text{hom}}(\xi) \geqslant \sup_{a\in V}\Big\{\inf_{u\in H^{1,r}_{\text{per}}(Y)}\int_Y a(x)|\xi+Du(x)|^r\,\mathrm{d}x-\int_Y f_r^*(x,a(x))\,\mathrm{d}x\Big\}.$$

From (6) and (14) it follows that

$$g_{\text{hom}}(\xi) \geqslant \sup_{a\in V}\Big\{\inf_{u\in H^{1,r}_{\text{per}}(Y)}\int_Y a(x)|\xi+Du(x)|^r\,\mathrm{d}x-\sum_{i=1}^n m(\Omega_i)v_i(a_i)\Big\}.$$

Next, by applying the lower bound in (2) to the first term we get that

$$g_{\text{hom}}(\xi) \geqslant \sup_{a\in V}\Big\{q_j^-(a,\xi_j,r)-\sum_{i=1}^n m(\Omega_i)v_i(a_i)\Big\},$$

and the theorem is proved. □

If we restrict ourselves to the case when $r=2$ is a possible choice we can even obtain a lower bound in a general direction ξ.

Theorem 4. *Assume that the local energy density function g is such that the function $s\mapsto f_2(x,s)$ is convex. Then*

$$g_{\text{hom}}(\xi) \geqslant \sup_{a\in V}\Big\{\sum_{j=1}^N \xi_j^2 q_j^-(a,1,2)-\sum_{i=1}^n m(\Omega_i)v_i(a_i)\Big\},$$

where

$$V=\Big\{a\in L^\infty\colon\ a(x)=\sum_{i=1}^n a_i\chi_i(x),\ 0<k_1\leqslant a_i\leqslant k_2<\infty\Big\}$$

and

$$v_i(a_i)=\sup_{s\geqslant 0}\Big\{sa_i-g_i(s^{\frac12})\Big\}.$$

P r o o f. By using the variational principle in Theorem 1 with $r=2$ in combination with the lower bound in (4) we obtain that

$$g_{\text{hom}}(\xi) \geqslant \sup_{\substack{a\in L^\infty(Y)\\ a\geqslant 0}}\Big\{\sum_{j=1}^N \xi_j^2 q_j^-(a,1,2)-\int_Y f_2^*(x,a(x))\,\mathrm{d}x\Big\}.$$

From (6) and (14) it follows that

$$g_{\text{hom}}(\xi) \geqslant \sup_{\substack{a\in L^\infty(Y)\\ a\geqslant 0}}\Big\{\sum_{j=1}^N \xi_j^2 q_j^-(a,1,2)-\sum_{i=1}^n m(\Omega_i)v_i(a_i)\Big\}.$$

The theorem follows by taking the supremum over all a in the smaller set V. □

R e m a r k 3. For the special case of powerlaw materials, i.e., when

$$g(x,s)=\sum_{i=1}^{n}\chi_i(x)\lambda_i s^p=\lambda(x)s^p,$$

we can choose $r=p$ in Theorem 3 and find that

$$v_i(a_i)=\sup_{s\geqslant 0}\{a_i s-\lambda_i s\}=\begin{cases}0 & \text{if } a_i\leqslant\lambda_i,\\ \infty & \text{if } a_i>\lambda_i.\end{cases}$$

This means that the optimal choice of $a\in V$ is

$$a(x)=\sum_{i=1}^{n}\chi_i(x)\lambda_i,$$

and we get that

$$g_{\mathrm{hom}}(\xi)\geqslant q_j^-(\lambda,\xi_j,p),$$

i.e. we recover the bound in (2).

We can now, by using the same ideas, derive upper bounds corresponding to Theorem 4 and Theorem 3.

Theorem 5. *Let $\xi=\xi_j e_j$ and assume that the local energy density function g is such that the function $s\mapsto h_r(x,s)$ is concave. Then*

$$g_{\mathrm{hom}}(\xi)\leqslant\inf_{a\in V}\left\{q_j^+(a,\xi_j,r)-\sum_{i=1}^{n}m(\Omega_i)v_i(a_i)\right\},$$

where

$$V=\left\{a\in L^\infty\colon\ a(x)=\sum_{i=1}^{n}a_i\chi_i(x),\ 0<k_1\leqslant a_i\leqslant k_2<\infty\right\}$$

and

$$v_i(a_i)=\inf_{s\geqslant 0}\left\{sa_i-g_i(s^{\frac{1}{r}})\right\}.$$

Theorem 6. *Assume that the local energy density function g is such that the function $s\mapsto h_2(x,s)$ is concave. Then*

$$g_{\mathrm{hom}}(\xi)\leqslant\inf_{a\in V}\left\{\sum_{j=1}^{N}\xi_j^2 q_j^+(a,1,2)-\sum_{i=1}^{n}m(\Omega_i)v_i(a_i)\right\},$$

where

$$V=\left\{a\in L^{\infty}\colon\ a(x)=\sum_{i=1}^{n}a_i\chi_i(x),\ 0<k_1\leqslant a_i\leqslant k_2<\infty\right\}$$

and

$$v_i(a_i)=\inf_{s\geqslant 0}\left\{sa_i-g_i(s^{\frac{1}{2}})\right\}.$$

R e m a r k 4. For powerlaw materials we find that

$$g_{\text{hom}}(\xi)\leqslant q_j^{+}(\lambda,\xi_j,p),$$

i.e. we recover the bound in 2, cf. Remark 3.

R e m a r k 5. The computations of the bounds in Theorems 3–6 involve an optimization problem with $2n$ variables which in general has to be done numerically.

R e m a r k 6. By using the approximative bounds in (3) and (4) we can obtain bounds corresponding to those in Theorems 3–6.

To obtain the bounds, a has so far been chosen to be constant in each phase. We will now study the question how a should be chosen to recover the bounds in (2) for the special case of powerlaw materials. To do this we need the following lemma of independent interest:

Lemma 7. *Let $w\in L^p(E)$, $p>1$ and assume that $0<k_1\leqslant a(x)\leqslant k_2<\infty$. Then*

$$\inf_{w\in W}\frac{1}{|E|}\int_E a(k+w)^p\,\mathrm{d}x=k^p\left(\frac{1}{|E|}\int_E a^{\frac{1}{1-p}}\,\mathrm{d}x\right)^{1-p},$$

where

$$W=\left\{w\in L^p(E)\colon\ w\geqslant -k,\ \int_E w\,\mathrm{d}x=0\right\},$$

and the infimum is attained for

$$w(x)=k|E|\left(\int_E a^{\frac{1}{1-p}}\,\mathrm{d}x\right)^{-1}a^{\frac{1}{1-p}}(x)-k.$$

P r o o f. By the reversed Hölder inequality and the zero average constraint we have

$$\begin{aligned}(15)\qquad \frac{1}{|E|}\int_E a(k+w)^p\,\mathrm{d}x&\geqslant\frac{1}{|E|}\left(\int_E a^{\frac{1}{1-p}}\,\mathrm{d}x\right)^{1-p}\left(\int_E(k+w)\,\mathrm{d}x\right)^{p}\\&=k^p\left(\frac{1}{|E|}\int_E a^{\frac{1}{1-p}}\,\mathrm{d}x\right)^{1-p}.\end{aligned}$$

We have equality in (15) if

$$ca^{\frac{1}{1-p}} = k + w.$$

Moreover, in order to satisfy the zero average constraint, the constant c must be given by

$$c = k|E| \left(\int_E a^{\frac{1}{1-p}} \, \mathrm{d}x \right)^{-1},$$

and the proof of the lemma is complete. $\square$

Theorem 8. *If* $g(x,s) = \sum_{i=1}^{n} \chi_i(x)\lambda_i s^p$, $\xi = \xi_j e_j$, $1 \leqslant j \leqslant N$, *and* $p \geqslant 2$. *Then*

$$\begin{aligned} g_{\mathrm{hom}}(\xi) &\geqslant \sup_{a \in V_1} \left\{ |\xi_j|^2 Q_j^-(a, 1, 2, \mathcal{A}_j^-) - \int_Y f_2^*(x, a) \, \mathrm{d}x \right\} \\ &= |\xi_j|^p Q_j^-(\lambda, 1, p, \mathcal{A}_j^-), \end{aligned}$$

where

$$V_1 = \left\{ a \in L^\infty \colon\, a(x) = \sum_{k \in I_k} a_k \chi_{A_k^-}(x),\ 0 < k_1 \leqslant a_k \leqslant k_2 < \infty \right\}.$$

P r o o f. Fix $1 \leqslant j \leqslant N$. Theorem 1 and the lower bound in (3) imply that

$$g_{\mathrm{hom}}(\xi) \geqslant \sup_{a \in V_1} \left\{ \xi_j^2 Q_j^-(a, 1, 2, \mathcal{A}_j^-) - \int_Y f_2^*(x, a) \, \mathrm{d}x \right\}.$$

We have to show that the right hand side is equal to $Q_j^-(\lambda, \xi_j, p, \mathcal{A}_j^-)$. An application of Lemma 7 with $k = 1$ and $p = 2$, yields

$$\begin{aligned} Q_j^-(a, 1, 2, \mathcal{A}_j^-) &= \sum_{k_j=1}^{K_j} \left(\frac{1}{m(E_{j,k_j}^-)} \int_{E_{j,k_j}^-} a^{-1} \, \mathrm{d}x \right)^{-1} m(E_{j,k_j}^-) \\ &= \sum_{k_j=1}^{K_j} \inf_{w \in W_1} \frac{1}{m(E_{j,k_j}^-)} \int_{E_{j,k_j}^-} a(1+w)^2 \, \mathrm{d}x \, m(E_{j,k_j}^-) \\ &= \inf_{w \in W_1} \sum_{k_j=1}^{K_j} \int_{E_{j,k_j}^-} a(1+w)^2 \, \mathrm{d}x \\ &= \inf_{w \in W_1} \sum_{k_j=1}^{K_j} \sum_{i=1}^{n} \sum_{k \in I_{j,k_j}^i} a_k (1+w_k)^2 m(A_k^-), \end{aligned} \tag{16}$$

where

$$W_1 = \Big\{ w \in L^2(Y)\colon\ w = \sum_{k\in I} w_k \chi_{A_k^-},\ \int_{E^-_{j,k_j}} w \,\mathrm{d}x = 0,\, w \geqslant 1 \Big\}.$$

We also have

$$\int_Y f_2^*(x,a)\,\mathrm{d}x = \int_Y \sup_{s\geqslant 0} \Big\{ s \sum_{k=I} a_k \chi_{A_k^-} - \sum_{i=1}^{n} \chi_{\Omega_i} g_i(s^{\frac12}) \Big\}\,\mathrm{d}x \tag{17}$$

$$= \int_Y \sum_{k_j=1}^{K_j} \sum_{i=1}^{n} \sum_{k\in I^i_{j,k_j}} \chi_{A_k^-} \sup_{s\geqslant 0} \big\{ s a_k - g_i(s^{\frac12}) \big\}\,\mathrm{d}x$$

$$= \sum_{k_j=1}^{K_j} \sum_{i=1}^{n} \sum_{k\in I^i_{j,k_j}} v_i(a_k) m(A_k^-).$$

By (16), (17) and a saddle point theorem, see [5], we obtain

$$\sup_{a\in V_1} \Big\{ \xi_j^2 Q_j^-(a,1,2,\mathcal{A}_j^-) - \int_Y f_2^*(x,a)\,\mathrm{d}x \Big\}$$

$$= \sup_{a\in V_1} \Big\{ \inf_{w\in W_1} \sum_{k_j=1}^{K_j} \sum_{i=1}^{n} \sum_{k\in I^i_{j,k_j}} a_k \xi_j^2 (1+w_k)^2\, m(A_k^-)$$

$$- \sum_{k_j=1}^{K_j} \sum_{i=1}^{n} \sum_{k\in I^i_{j,k_j}} v_i(a_k) m(A_k^-) \Big\}$$

$$= \inf_{w\in W_1} \Big\{ \sum_{k_j=1}^{K_j} \sum_{i=1}^{n} \sum_{k\in I^i_{j,k_j}} \sup_{t\geqslant 0} [t\xi_j^2 (1+w_k)^2 - v_i(t)] m(A_k^-) \Big\}.$$

From (10) and Lemma 7 it follows that

$$\sup_{a\in V_1} \Big\{ \xi_j^2 Q_j^-(a,1,2,\mathcal{A}_j^-) - \int_Y f_2^*(x,a)\,\mathrm{d}x \Big\}$$

$$= \sum_{k_j=1}^{K_j} \inf_{w\in W_1} \frac{1}{m(E^-_{j,k_j})} \int_{E^-_{j,k_j}} g(x, |\xi_j|\,|1+w|)\,\mathrm{d}x\, m(E^-_{j,k_j})$$

$$= |\xi_j|^p \sum_{k_j=1}^{K_j} \inf_{w\in W_1} \frac{1}{m(E^-_{j,k_j})} \int_{E^-_{j,k_j}} \lambda |1+w|^p \,\mathrm{d}x\, m(E^-_{j,k_j})$$

$$= |\xi_j|^p \sum_{k_j=1}^{K_j} \int_{E^-_{j,k_j}} \Big(\frac{1}{m(E^-_{j,k_j})} \int_{E^-_{j,k_j}} \lambda^{\frac{1}{1-p}}\,\mathrm{d}x \Big)^{1-p} \mathrm{d}x,$$

and the proof is complete. □

Theorem 9. *If* $g(x,s)=\sum\limits_{i=1}^{n}\chi_i(x)\lambda_i s^p$, $\xi=\xi_j e_j$, $1\leqslant j\leqslant N$ *and* $1<p\leqslant 2$, *then*

$$\begin{aligned} g_{\rm hom}(\xi) &\leqslant \sup_{a\in V_2}\Big\{|\xi_j|^2 Q_j^-(a,1,2,\mathcal{A}_j^+)-\int_Y h_2^*(x,a)\,\mathrm{d}x\Big\}\\ &=|\xi_j|^p Q_j^-(\lambda,1,p,\mathcal{A}_j^+),\end{aligned}$$

where

$$V_2=\Big\{a\in L^\infty\colon\ a(x)=\sum_{k\in J_k}a_k\chi_{A_k^+}(x),\ 0<k_1\leqslant a_k\leqslant k_2<\infty\Big\}.$$

P r o o f. The proof follows by similar arguments as in the proof of Theorem 8.

5. Examples

In this section we will use the results derived above to study some concrete examples where the bounds provide a very good information about the effective behavior of a composite with nonlinear material behavior. We will restrict ourselves to two-phase problems in $\mathbb{R}^2$, it is clear that the same arguments would hold for n-phase problems in $\mathbb{R}^N$.

E x a m p l e 1. Let the unit cell be a coated square in $\mathbb{R}^2$, where the square inclusion has side length $b=0.8$. The local energy density function g_i is of the form

$$g_i(s)=\frac{1}{2}\lambda_i s^2+\frac{1}{p}\kappa_i s^p,$$

where $p=4$, $i=1$ represents the matrix material and $i=2$ the inclusions. Let $\lambda_1=20$, $\lambda_2=10$, $\kappa_1=1$ and κ_2 be variable. Then Theorem 4 and an application of Lemma 7 implies that

$$\begin{aligned} g_{\rm hom}(\xi)\geqslant \inf_{t\geqslant -1}\Big\{&(1-b^2)g_1\Big(|\xi|\Big(\frac{b(1+t)^2+1}{1+b}\Big)^{\frac12}\Big)\\ &+b^2 g_2\Big(|\xi|\,\Big|1+\frac{b-1}{b}t\Big|\Big)\Big\},\end{aligned}$$

see curve 1 in Fig. 1 By using the wider class of a's used in Theorem 8 we obtain

$$\begin{aligned} g_{\rm hom}(\xi)\geqslant \inf_{t\geqslant -1}\Big\{&b(1-b)g_1\Big(|\xi|\Big|1+\frac{b}{b-1}t\Big|\Big)\\ &+b^2 g_2(|\xi|\,|1+t|)+(1-b)g_1(|\xi|)\Big\},\end{aligned}\tag{18}$$

see curve 2 in Fig. 1. Moreover, let $r = p = 4$ in Theorem 5. Then it follows by an application of Lemma 7 that

$$g_{\mathrm{hom}}(\xi) \leqslant \inf_{t \geqslant -1} \left\{ (1-b^2) g_1 \left(|\xi_j| \left(\frac{b}{1+b}(1+t)^r + \frac{1}{1+b}\left(1 + \frac{b}{b-1}t\right)^r \right)^{\frac{1}{r}} \right) \right. \\ \left. + b^2 g_2(|\xi|(1+t)) \right\}$$

for $j = 1, 2$, see curve 3 in Fig. 1. Similarly, by letting a belong to the wider class given in Theorem 9 we obtain

$$g_{\mathrm{hom}}(\xi) \leqslant \inf_{t \geqslant -1} \left\{ b(1-b) g_1(|\xi_j|\,|1+t|) + b^2 g_2(|\xi_j|\,|1+t|) \right. \\ \left. + (1-b) g_1 \left(|\xi_j| \left| 1 + \frac{b}{b-1}t \right| \right) \right\} \tag{19}$$

for $j = 1, 2$, see curve 4 in Fig. 1. Thus the problem of calculating the estimates for g_{hom} has been reduced to a minimization problem with respect to one single variable.

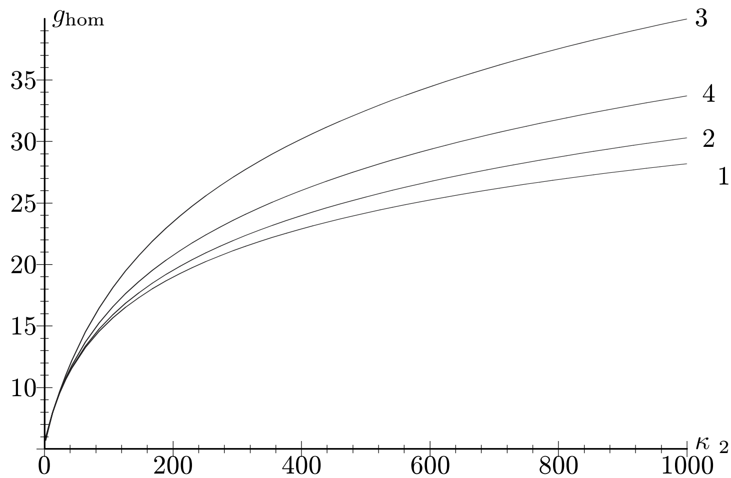

Figure 1. Estimates on g_{hom} for $\xi_j = 1$ corresponding to example 10.

Let us now study the cell geometry in Example 1 for the two extreme cases: (i) stiff inclusions, (ii) soft inclusions. By stiff inclusions we mean that

$$g_2(s) = \begin{cases} 0, & s = 0, \\ \infty, & s \neq 0 \end{cases}$$

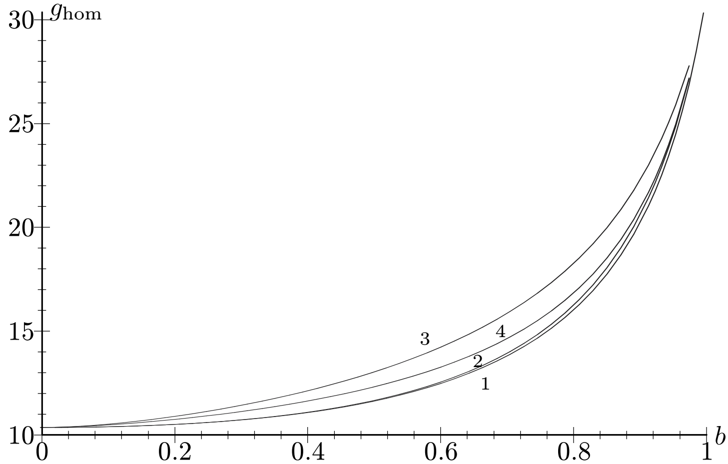

Figure 2. Estimates on g_{hom} for $\xi_j = 1$ and $\kappa_2 = 100$ corresponding to example 10.

and by soft inclusions we mean that $g_2(s) = 0$. For the general theory concerning problems with stiff and soft inclusions, the reader is referred to [6]. For stiff inclusions the lower bound and the upper bound corresponding to (18) and (19) take the forms

$$g_{\mathrm{hom}}(\xi) \leqslant (1-b)g_1\left(|\xi_j|\left|1+\frac{b}{1-b}\right|\right),$$
$$g_{\mathrm{hom}}(\xi) \geqslant b(1-b)g_1\left(|\xi|\frac{1}{1-b}\right) + (1-b)g_1(|\xi|),$$

respectively. For soft inclusions we obtain

$$g_{\mathrm{hom}}(\xi) \leqslant \inf_{t\geqslant -1}\left\{b(1-b)g_1\big(|\xi_j|(1+t)\big) + (1-b)g_1\left(|\xi_j|\left|1+\frac{b}{b-1}t\right|\right)\right\}, \tag{20}$$
$$g_{\mathrm{hom}}(\xi) \geqslant (1-b)g_1(|\xi|). \tag{21}$$

For stiff inclusions there is a similar formula, based on the lower Hashin-Shtrikman bound for linear problems, which is a lower bound among all isotropic nonlinear composites satisfying superquadratic growth in the matrix phase, but there is no corresponding upper bound. More information about this can be found in [2]. This lower bound has the form

$$g_{\mathrm{hom}}(\xi) \geqslant (1-b^2)g_1\left(\frac{\sqrt{(1+b^2)}}{1-b^2}|\xi|\right).$$

In [2] a corresponding estimate of g_{hom} for soft inclusions was also presented based on the Hashin-Shtrikman upper bound for linear composites. Note that this estimate is not a rigorous upper bound for nonlinear isotropic composites, it can nonetheless be interpreted as an estimate of an upper bound. The estimate is

$$g_{\mathrm{hom}}(\xi) \lesssim (1-b^2) g_1\left(|\xi|\sqrt{\left(\frac{1}{1+b^2}\right)}\right). \tag{22}$$

E x a m p l e 2. Let the unit cell be a coated square in $\mathbb{R}^2$, where the square inclusion has sidelength b. We also assume that the square inclusion is soft and that the local energy density function g_1 of the matrix material is of the form

$$g_i(s) = \frac{1}{2}s^2 + \frac{1}{p}10s^p,$$

where $p = 4$. Moreover, let $\xi_j = 1$. In Fig. 3 we have plotted the bounds in (22), (21) and (20) respectively for different values of b.

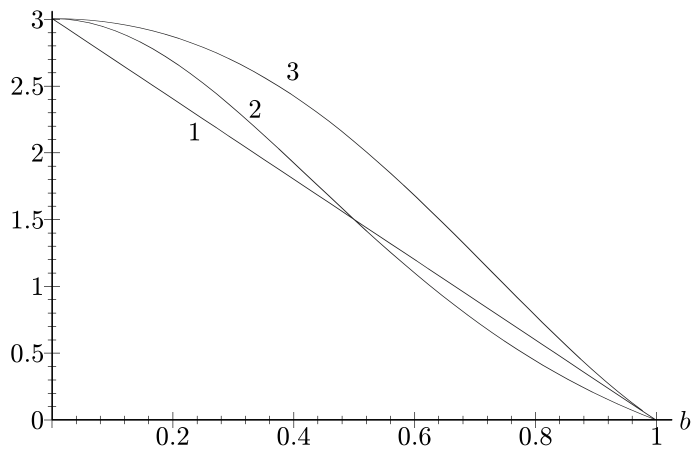

Figure 3. 1 – lower bound, 2 – upper estimate, 3 – upper bound for soft inclusions.

Note that curve 2 which corresponds to the estimate of the upper bound in (22) is below the lower bound in (21).

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