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DECOMPOSITION OF AN UPDATED CORRELATION MATRIX VIA HYPERBOLIC TRANSFORMATIONS*

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Dedicated to Prof. Dr. Gerhard Opfer on the occasion of his 65th birthday.

Abstract. An algorithm for hyperbolic singular value decomposition of a given complex matrix based on hyperbolic Householder and Givens transformation matrices is described in detail. The main application of this algorithm is the decomposition of an updated correlation matrix.

Keywords: eigensystem of a correlation matrix, hyperbolic transformations, hyperbolic Householder transformation, hyperbolic Givens transformation, hyperbolic singular value decomposition

MSC 2000: 65F25, 65F30

1. Introduction

Let a sample data of some technical problem be collected in a matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$, where m represents the number of outputs and n the number of measurements on these outputs:

$$\mathbf{X} = (x_{ij}), \quad i = 1, \ldots, m; \quad j = 1, \ldots, n. \tag{1.1}$$

Usually $n \gg m$ and the matrix $\mathbf{X}$ is rectangular. We want to compute singular values of the matrix $\mathbf{X}$, i.e. to find an eigensystem of the correlation matrix $\mathbf{X}\mathbf{X}^{\mathrm{T}}$. A classical way how to perform this task is to use an algorithm for singular value

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decomposition (SVD) of the matrix $\mathbf{X}$. First, we use orthogonal transformations to triangularize the data matrix $\mathbf{X}$,

$$\widehat{\mathbf{X}} = \mathbf{X}\mathbf{Q}, \quad \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}, \qquad \widehat{\mathbf{X}} \text{ is lower triangular.}$$

Then we obtain

$$\widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}} = \mathbf{X}\mathbf{Q}(\mathbf{X}\mathbf{Q})^{\mathrm{T}} = \mathbf{X}\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}} = \mathbf{X}\mathbf{X}^{\mathrm{T}},$$

i.e. the correlation matrix of $\widehat{\mathbf{X}}$ is the same as the correlation matrix of the original $\mathbf{X}$ and we can apply SVD only on this lower triangular $m \times m$ matrix $\widehat{\mathbf{X}}$. Since $m \ll n$ we have reduced the problem significantly.

Let $\mathbf{X}^{\mathrm{new}}$ be an update of $\mathbf{X}$ which consists of selected original data and of some new measurements collected in a matrix $\mathbf{Z} \in \mathbb{R}^{m\times k}$. If we consider the following block structure of $\mathbf{X}$:

$$\mathbf{X} = (\mathbf{X}_1|\mathbf{Y}|\mathbf{X}_2),$$

where $\mathbf{X}_1 \in \mathbb{R}^{m\times n_1}$, $\mathbf{Y} \in \mathbb{R}^{m\times p}$, $\mathbf{X}_2 \in \mathbb{R}^{m\times n_2}$, $n_1 + p + n_2 = n$; $\mathbf{X}_1$, $\mathbf{X}_2$ remain in the new set of data, the submatrix $\mathbf{Y}$ of the original data matrix $\mathbf{X}$ is skipped, then the updated data matrix has the form

$$\mathbf{X}^{\mathrm{new}} = (\mathbf{X}_1|\mathbf{X}_2|\mathbf{Z}) \in \mathbb{R}^{m\times(n_1+n_2+k)}.$$

Its correlation matrix reads

$$\begin{aligned}\mathbf{X}^{\mathrm{new}}(\mathbf{X}^{\mathrm{new}})^{\mathrm{T}} = (\mathbf{X}_1|\mathbf{X}_2|\mathbf{Z})\begin{pmatrix}\mathbf{X}_1^{\mathrm{T}}\\ \mathbf{X}_2^{\mathrm{T}}\\ \mathbf{Z}^{\mathrm{T}}\end{pmatrix} &= \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} + \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}} + \mathbf{Z}\mathbf{Z}^{\mathrm{T}}\\ &= \mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{Y}\mathbf{Y}^{\mathrm{T}} + \mathbf{Z}\mathbf{Z}^{\mathrm{T}} = \widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}} - \mathbf{Y}\mathbf{Y}^{\mathrm{T}} + \mathbf{Z}\mathbf{Z}^{\mathrm{T}},\end{aligned}$$

i.e. we can form correlation matrices $\mathbf{Y}\mathbf{Y}^{\mathrm{T}}$ and $\mathbf{Z}\mathbf{Z}^{\mathrm{T}}$ and compute the eigenvalues of $\mathbf{X}^{\mathrm{new}}(\mathbf{X}^{\mathrm{new}})^{\mathrm{T}}$ by a suitable numerical method.

In this paper we present the so called hyperbolic singular value decomposition—another method for computing singular values of the update $\mathbf{X}^{\mathrm{new}}$.

Let us denote

$$\mathbf{C} = (\widehat{\mathbf{X}}|\mathbf{Y}|\mathbf{Z}) \in \mathbb{R}^{m\times(m+p+k)}.$$

Let $\mathbf{\Phi}$ be a diagonal matrix with blocks on diagonal equal to the identity or minus identity matrix. Namely,

$$\mathbf{\Phi} = \mathrm{diag}(\mathbf{I}_{m\times m}, -\mathbf{I}_{m\times p}, \mathbf{I}_{m\times k}).$$

Then

$$\mathbf{C}\mathbf{\Phi}\mathbf{C}^{\mathrm{T}} = \widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}} - \mathbf{Y}\mathbf{Y}^{\mathrm{T}} + \mathbf{Z}\mathbf{Z}^{\mathrm{T}} = \mathbf{X}^{\mathrm{new}}(\mathbf{X}^{\mathrm{new}})^{\mathrm{T}}.$$

Let $\mathbf{C}$ be decomposed into a product of matrices

$$\mathbf{C} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}, \tag{1.2}$$

where $\mathbf{U} \in \mathbb{R}^{m\times m}$ is orthogonal, $\mathbf{V} \in \mathbb{R}^{(m+p+k)\times(m+p+k)}$ has the property $\mathbf{V}^{\mathrm{T}}\boldsymbol{\Phi}\mathbf{V} = \widehat{\boldsymbol{\Phi}}$, $\widehat{\boldsymbol{\Phi}}$ is again a diagonal matrix with ± 1 on its diagonal (generally different from $\boldsymbol{\Phi}$) and $\mathbf{D} \in \mathbb{R}^{m\times(m+p+k)}$ is a diagonal matrix with m real positive diagonal entries. Then

$$\mathbf{X}^{\mathrm{new}}(\mathbf{X}^{\mathrm{new}})^{\mathrm{T}} = \mathbf{C}\boldsymbol{\Phi}\mathbf{C}^{\mathrm{T}} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}\boldsymbol{\Phi}\mathbf{V}\mathbf{D}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}} = \mathbf{U}\mathbf{D}\widehat{\boldsymbol{\Phi}}\mathbf{D}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}.$$

The diagonal entries of $\mathbf{D}\widehat{\boldsymbol{\Phi}}\mathbf{D}^{\mathrm{T}}$ are the eigenvalues of the updated correlation matrix $\mathbf{X}^{\mathrm{new}}(\mathbf{X}^{\mathrm{new}})^{\mathrm{T}}$ and the columns of $\mathbf{U}$ are the corresponding eigenvectors.

We conclude that if we are able to perform the decomposition (1.2) we obtain the eigensystem of the updated correlation matrix $\mathbf{X}^{\mathrm{new}}(\mathbf{X}^{\mathrm{new}})^{\mathrm{T}}$.

Throughout the paper we suppose that all vectors and matrices are allowed to have complex elements. In Section 2, we resume properties of so called hyperexchange and hypernormal matrices. Then the definitions of hyperbolic Givens and hyperbolic Householder transformation matrices are given. Their properties are proved in detail in [5]. Appropriate algorithms in MATLAB can be found in [6].

Section 3 contains an algorithm of the hyperbolic singular value decomposition (HSVD). Its idea is similar to that of the SVD algorithm by Golub and Kahan (see [3], § 8.6), only all reductions of the matrices involved are performed by making use of the hyperbolic transformation matrices.

2. Hyperbolic transformations

Throughout the paper we use the notation

$$\operatorname{sgn} z = \begin{cases} \dfrac{\overline{z}}{|z|} & \text{for } z \in \mathbb{C}\setminus\{0\}, \\ 0 & \text{if } z = 0, \end{cases} \tag{2.1}$$

which is again a complex number. This implies $z \operatorname{sgn} z = |z|$. We also use $\|\mathbf{x}\|$ for the ordinary Euclidean norm of a vector $\mathbf{x} \in \mathbb{C}^n$, and we denote the first unit vector by

$$\mathbf{e}_1 = (1, 0, \ldots, 0)^{\mathrm{T}} \in \mathbb{R}^n.$$

The operation $\mathbf{V}^{\mathrm{H}}$ means the transposition of the matrix $\mathbf{V}$ and the complex conjugation of all its elements. Thus, if $\mathbf{V}$ is real then $\mathbf{V}^{\mathrm{H}} = \mathbf{V}^{\mathrm{T}}$, where $\mathbf{V}^{\mathrm{T}}$ is the transpose of $\mathbf{V}$.

Definition 2.1. Let $\mathbf{\Phi}, \widehat{\mathbf{\Phi}}$ be two $m \times m$ diagonal matrices with diagonal entries ± 1. An $m \times m$ matrix $\mathbf{V}$, generally with complex elements, satisfying

$$\mathbf{V}^{\mathrm{H}} \mathbf{\Phi} \mathbf{V} = \widehat{\mathbf{\Phi}}, \tag{2.2}$$

will be called a *hyperexchange matrix with respect to* $\mathbf{\Phi}$. If $\mathbf{\Phi} = \widehat{\mathbf{\Phi}}$, $\mathbf{V}$ satisfying (2.2) will be called *hypernormal with respect to* $\mathbf{\Phi}$.

Remark 2.1. The terms *hypernormal* and also *hyperexchange matrix with respect to* $\mathbf{\Phi}$ occur in [1]. The term *pseudo-orthogonal with respect to* $\mathbf{\Phi}$ is also in use. It can be found in [3].

Example 2.1. Let $\mathbf{P}$ be a permutation matrix. Then $\mathbf{P}^{\mathrm{T}} \mathbf{\Phi} \mathbf{P} = \widehat{\mathbf{\Phi}}$, i.e. $\mathbf{P}$ is the hyperexchange matrix with respect to $\mathbf{\Phi}$.

Hypernormal or hyperexchange matrices are always regular, their determinants are equal to 1 or -1 and their inversions can be computed very easily: if $\mathbf{V}$ is a hyperexchange matrix with respect to $\mathbf{\Phi}$, i.e. $\mathbf{V}$ fulfils (2.2), then

$$\mathbf{V}^{-1} = \widehat{\mathbf{\Phi}} \mathbf{V}^{\mathrm{H}} \mathbf{\Phi}. \tag{2.3}$$

For a given diagonal matrix $\mathbf{\Phi} = \operatorname{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n)$, $\varphi_j = \pm 1$ for $j = 1, \ldots, n$, and a given vector $\mathbf{x} \in \mathbb{C}^n$, we define

$$\|\mathbf{x}\|_{\mathbf{\Phi}} = \operatorname{sgn}(\mathbf{x}^{\mathrm{H}} \mathbf{\Phi} \mathbf{x}) |\mathbf{x}^{\mathrm{H}} \mathbf{\Phi} \mathbf{x}|^{1/2}. \tag{2.4}$$

We shall call $\|\mathbf{x}\|_{\mathbf{\Phi}}$ the *hyperbolic energy* of the vector $\mathbf{x}$. It should be mentioned that $\|\mathbf{x}\|_{\mathbf{\Phi}}$, in general, is not a norm.

2.1. Hyperbolic Givens transformation.

Definition 2.2. Let $\mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{C}^2$ be such that

$$|x_1| > |x_2| > 0. \tag{2.5}$$

Let us define

$$\varphi = \operatorname{arctanh} \left| \frac{x_2}{x_1} \right|, \quad \text{i.e. } \tanh \varphi = \left| \frac{x_2}{x_1} \right| < 1,$$
$$c_h = \cosh \varphi, \quad s_h = \sinh \varphi.$$

Then the matrix

$$\mathbf{G} = \begin{pmatrix} c_h & -\operatorname{sgn}(\overline{x}_1) \operatorname{sgn}(x_2) s_h \\ -\operatorname{sgn}(x_1) \operatorname{sgn}(\overline{x}_2) s_h & c_h \end{pmatrix} \tag{2.6}$$

is called the matrix of a *hyperbolic Givens rotation.* It depends on φ.

Since $\cosh^2\varphi - \sinh^2\varphi = 1$, we obtain

$$\mathbf{G} = \frac{1}{\varrho}\begin{pmatrix} |x_1| & -\dfrac{x_1\overline{x}_2}{|x_1|} \\ -\dfrac{\overline{x}_1 x_2}{|x_1|} & |x_1| \end{pmatrix}, \tag{2.7}$$

where $\varrho = \sqrt{|x_1|^2 - |x_2|^2} > 0$ and $\mathbf{x}$ satisfies condition (2.5).

For a vector $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ such that two different components x_p, x_q of $\mathbf{x}$ satisfy

$$|x_p| > |x_q| > 0,$$

the matrix of a hyperbolic Givens rotation has the form

$$\mathbf{G}(p,q,\varphi) = \begin{array}{cc} \left(\begin{array}{ccccccccccc} 1 &&&&&&&&&& 0 \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& c_h & \cdots & \cdots & \cdots & -\sigma\cdot s_h &&& \\ &&& \vdots & 1 &&& \vdots &&& \\ &&& \vdots && \ddots && \vdots &&& \\ &&& \vdots &&& 1 & \vdots &&& \\ &&& -\overline{\sigma}\cdot s_h & \cdots & \cdots & \cdots & c_h &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ 0 &&&&&&&&&& 1 \end{array}\right) & \begin{array}{l} \\ \\ \\ \longleftarrow p \\ \\ \\ \\ \longleftarrow q \\ \\ \\ \\ \end{array} \\ \begin{array}{ccccccccccc} &&& \uparrow &&&& \uparrow &&& \\ &&& p &&&& q &&& \end{array} & \end{array},$$

where

$$c_h = \cosh\varphi, \quad s_h = \sinh\varphi,$$
$$\varphi = \operatorname{arctanh}\left|\frac{x_q}{x_p}\right|, \quad \sigma = \operatorname{sgn}(\overline{x_p})\operatorname{sgn}(x_q).$$

The main properties of the classical and hyperbolic Givens rotation matrices for a vector $\mathbf{x} \in \mathbb{C}^2$ are summarized in Tab. 1.

Givens transformation ($\mathbf{x} \neq \mathbf{0}$)	Hyperbolic Givens transformation ($\lvert x_1\rvert > \lvert x_2\rvert > 0$)
$\mathbf{F} = \begin{pmatrix} c & -\overline{s} \\ s & c \end{pmatrix}$ $c \in \mathbb{R},\ \ c^2 + \lvert s\rvert^2 = 1$	$\mathbf{G} = \begin{pmatrix} c_h & -\sigma s_h \\ -\overline{\sigma} s_h & c_h \end{pmatrix},\ \sigma = \operatorname{sgn}(\overline{x_1}) \operatorname{sgn}(x_2)$ $c_h, s_h \in \mathbb{R},\ \ c_h^2 - s_h^2 = 1$
$c = \lvert x_1\rvert / \lVert\mathbf{x}\rVert,\ \ s = -\operatorname{sgn}(x_1) x_2 / \lVert\mathbf{x}\rVert$	$c_h = \lvert x_1\rvert/\varrho, \quad s_h = \lvert x_2\rvert/\varrho$ $\varrho = \sqrt{\lvert x_1\rvert^2 - \lvert x_2\rvert^2}$
$\mathbf{F}^{\mathrm{H}}\mathbf{F} = \mathbf{I}$ $\mathbf{F}\mathbf{x} = \begin{pmatrix} \operatorname{sgn}(\overline{x_1}) \lVert\mathbf{x}\rVert \\ 0 \end{pmatrix}$ $\lVert\mathbf{F}\mathbf{x}\rVert = \lVert\mathbf{x}\rVert$	$\mathbf{G}^{\mathrm{H}}\mathbf{\Phi}\mathbf{G} = \mathbf{\Phi}$ $\mathbf{\Phi} = \operatorname{diag}(\varphi_1, \varphi_2),\ \ \varphi_1 = \pm 1,\ \ \varphi_2 = -\varphi_1$ $\mathbf{G}\mathbf{x} = \begin{pmatrix} \operatorname{sgn}(\overline{x_1})\varrho \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_1 \operatorname{sgn}(\overline{x_1}) \lVert\mathbf{x}\rVert_{\mathbf{\Phi}} \\ 0 \end{pmatrix}$ $\lVert\mathbf{G}\mathbf{x}\rVert_{\mathbf{\Phi}} = \lVert\mathbf{x}\rVert_{\mathbf{\Phi}}$
$\lVert\mathbf{x}\rVert = (\mathbf{x}^{\mathrm{H}}\mathbf{x})^{1/2}$	$\lVert\mathbf{x}\rVert_{\mathbf{\Phi}} = \operatorname{sgn}(\mathbf{x}^{\mathrm{H}}\mathbf{\Phi}\mathbf{x}) \lvert\mathbf{x}^{\mathrm{H}}\mathbf{\Phi}\mathbf{x}\rvert^{1/2}$

Table 1. Classical and hyperbolic Givens rotation in $\mathbb{C}^2$.

Hence, the matrix $\mathbf{G}$ from Definition 2.2 is Hermitian and hypernormal with respect to $\mathbf{\Phi}$. After its action on the vector $\mathbf{x}$ the second component x_2 vanishes, but the hyperbolic energy of the vector $\mathbf{x}$ is preserved.

Remark 2.2. The hyperbolic Givens transformation matrix will reduce to the ordinary Givens rotation in the case of $\mathbf{\Phi} = \pm\mathbf{I}$.

If the vector $\mathbf{x} \in \mathbb{C}^2$ satisfies (instead of the condition (2.5)) the condition

$$\lvert x_2\rvert > \lvert x_1\rvert > 0, \tag{2.8}$$

we define

$$\mathbf{G} = \frac{1}{\varrho} \begin{pmatrix} \lvert x_2\rvert & -\dfrac{x_2\overline{x}_1}{\lvert x_2\rvert} \\ -\dfrac{\overline{x}_2 x_1}{\lvert x_2\rvert} & \lvert x_2\rvert \end{pmatrix}, \tag{2.9}$$

where $\varrho = \sqrt{\lvert x_2\rvert^2 - \lvert x_1\rvert^2} > 0$.

Let $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the permutation matrix and $\mathbf{\Phi} = \operatorname{diag}(\varphi_1, \varphi_2)$, where $\varphi_1 = \pm 1$, $\varphi_2 = -\varphi_1$ is the given diagonal matrix. Then the matrix $\mathbf{GP}$ is hyperexchange with respect to both $\mathbf{\Phi}$ and $-\mathbf{\Phi}$,

$$(\mathbf{GP})\mathbf{x} = (\operatorname{sgn}(\overline{x_2})\varrho, 0)^{\mathrm{T}}$$

and the hyperbolic energy of the vector $\mathbf{x}$ is preserved, i.e.

$$(\mathbf{GPx})^{\mathrm{H}}(\mathbf{P}^{\mathrm{T}}\mathbf{\Phi}\mathbf{P})(\mathbf{GPx}) = \mathbf{x}^{\mathrm{H}}\mathbf{\Phi}\mathbf{x}.$$

The most complicated case occurs when the hyperbolic energy of a given vector $\mathbf{x} \in \mathbb{C}^2$ with respect to a given diagonal matrix $\mathbf{\Phi}$ is equal to zero. It can be shown that then there exist a complex matrix $\mathbf{M} \in \mathbb{C}^{2\times 2}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{2\times 2}$ such that the second component of the vector $\mathbf{Mx}$ is equal to zero and the matrix $\mathbf{MD}$ is hyperexchange with respect to $\mathbf{\Phi}$. In particular,

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{2}}{2} & u\sqrt{2} \\ \overline{u} & 1 \end{pmatrix}, \quad u = -\frac{x_1}{x_2}, \quad \mathbf{D} = \operatorname{diag}(\sqrt{2}, 1). \tag{2.10}$$

2.2. Hyperbolic Householder transformation.

Definition 2.3. Let $\mathbf{\Phi} = \operatorname{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n), \varphi_i = \pm 1, \mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ be given and, moreover,

$$\|\mathbf{x}\|_{\mathbf{\Phi}} \neq 0, \quad \operatorname{sgn}(\|\mathbf{x}\|_{\mathbf{\Phi}}) = \varphi_1. \tag{2.11}$$

Then the matrix

$$\mathbf{H} = \mathbf{\Phi} - \beta(\mathbf{\Phi}\mathbf{b})(\mathbf{\Phi}\mathbf{b})^{\mathrm{H}}, \tag{2.12}$$

where

$$\begin{aligned} \mathbf{b} &= \mathbf{x} + \operatorname{sgn}(\overline{x_1})|\,\|\mathbf{x}\|_{\mathbf{\Phi}}|\mathbf{e}_1, \\ \beta &= \big(\|\mathbf{x}\|_{\mathbf{\Phi}}(|\,\|\mathbf{x}\|_{\mathbf{\Phi}}| + |x_1|)\big)^{-1}, \end{aligned}$$

is called the *hyperbolic Householder matrix* for transformation of the given vector $\mathbf{x}$.

Remark 2.3. If $\mathbf{\Phi} = \mathbf{I}$, the identity matrix, then $\|\mathbf{x}\|_{\mathbf{\Phi}} = \|\mathbf{x}\|$ and the hyperbolic Householder matrix is just the classical one. A similar situation is in the case $\mathbf{\Phi} = -\mathbf{I}$.

Hence, the matrix $\mathbf{H}$ from Definition 2.3 is Hermitian and hypernormal with respect to $\mathbf{\Phi}$. After its action on a vector $\mathbf{x}$ all its components except the first vanish. The hyperbolic energy of the vector $\mathbf{x}$ is preserved.

Householder transformation	Hyperbolic Householder transformation (assumption: $\operatorname{sgn}(\|\mathbf{x}\|_{\mathbf{\Phi}}) = \varphi_1$)
$\|\mathbf{x}\| = (\mathbf{x}^{\mathrm{H}}\mathbf{x})^{1/2}$ $\mathbf{Mx} = -\operatorname{sgn}(\overline{x_1})\,\|\mathbf{x}\|\mathbf{e_1}$ $\|\mathbf{Mx}\| = \|\mathbf{x}\|$	$\|\mathbf{x}\|_{\mathbf{\Phi}} = \operatorname{sgn}(\mathbf{x}^{\mathrm{H}}\mathbf{\Phi}\mathbf{x})\,\lvert\mathbf{x}^{\mathrm{H}}\mathbf{\Phi}\mathbf{x}\rvert^{1/2}$ $\mathbf{Hx} = -\operatorname{sgn}(\overline{x_1})\,\|\mathbf{x}\|_{\mathbf{\Phi}}\mathbf{e_1}$ $\|\mathbf{Hx}\|_{\mathbf{\Phi}} = \|\mathbf{x}\|_{\mathbf{\Phi}}$
$\mathbf{M} = \mathbf{I} - \beta\mathbf{u}\mathbf{u}^{\mathrm{H}}$, where $\mathbf{u} = \mathbf{x} + \operatorname{sgn}(\overline{x_1})\,\|\mathbf{x}\|\mathbf{e_1}$ $\beta = \big(\|\mathbf{x}\|(\|\mathbf{x}\| + \lvert x_1\rvert)\big)^{-1} = 2/(\mathbf{u}^{\mathrm{H}}\mathbf{u})$	$\mathbf{H} = \mathbf{\Phi} - \beta(\mathbf{\Phi b})(\mathbf{\Phi b})^{\mathrm{H}}$, where $\mathbf{b} = \mathbf{x} + \operatorname{sgn}(\overline{x_1})\lvert\,\|\mathbf{x}\|_{\mathbf{\Phi}}\rvert\mathbf{e_1}$ $\beta = \big(\|\mathbf{x}\|_{\mathbf{\Phi}}(\lvert\,\|\mathbf{x}\|_{\mathbf{\Phi}}\rvert + \lvert x_1\rvert)\big)^{-1} = 2/(\mathbf{b}^{\mathrm{H}}\mathbf{\Phi b})$
$\|\mathbf{x}\| = 0 \;\Leftrightarrow\; \mathbf{x} = \mathbf{0}$	$\|\mathbf{x}\|_{\mathbf{\Phi}} = 0 \;\not\Rightarrow\; \mathbf{x} = \mathbf{0}$

Table 2. Classical and hyperbolic Householder transformation.

If the condition (2.11) is not valid, i.e. if

$$\operatorname{sgn}(\|\mathbf{x}\|_{\mathbf{\Phi}}) = -\varphi_1, \tag{2.13}$$

then we define a permutation matrix $\mathbf{P}$ which permutes the first row of the matrix $\mathbf{\Phi}$ with any row with a diagonal entry equal to $-\varphi_1$. We apply the same $\mathbf{P}$ to the vector $\mathbf{x}$. Then

$$\begin{gathered}\mathbf{P\Phi P} = \widetilde{\mathbf{\Phi}} = \operatorname{diag}(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n), \quad \widetilde{\varphi}_1 = \operatorname{sgn}(\|\mathbf{x}\|_{\mathbf{\Phi}}),\\ (\mathbf{Px})^{\mathrm{H}}(\mathbf{P\Phi P})(\mathbf{Px}) = \mathbf{x}^{\mathrm{H}}\mathbf{P}^{\mathrm{T}}\mathbf{P\Phi PPx} = \mathbf{x}^{\mathrm{H}}\mathbf{\Phi x},\end{gathered}$$

i.e. the vector $\mathbf{Px}$ fulfils the condition (2.11).

If a given vector $\mathbf{x} \in \mathbb{C}^n$ has its hyperbolic energy equal to zero with respect to a given diagonal matrix $\mathbf{\Phi}$ then there exist a matrix $\mathbf{U} \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ such that all components of the vector $\mathbf{Ux}$ are zero except the first and the matrix $\mathbf{UD}$ is hyperexchange with respect to $\mathbf{\Phi}$; see Theorem 4.3 in [5].

3. Hyperbolic singular value decomposition (HSVD)

3.1. Existence of HSVD.

In the following, we assume that for a given $\mathbf{A} \in \mathbb{C}^{n\times m}$ and a given $\mathbf{\Phi} = \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$, $\varphi_i = \pm 1$, the matrix $\mathbf{A\Phi A}^{\mathrm{H}}$ has a full rank, i.e. $\operatorname{rank}(\mathbf{A\Phi A}^{\mathrm{H}}) = \min(n, m)$. Let us note that a similar theorem is also valid without this assumption, but then it is necessary to introduce diagonal matrices $\mathbf{\Phi}$ with diagonal entries ± 1 and 0 and to extend the definition of the hyperexchange and hypernormal matrices in this sense. For the proof of Theorem 3.1, see [8]. The proof of the general existence theorem can be found in [1], [7].

Theorem 3.1. *Let* $\mathbf{\Phi}$ *be an* $m \times m$ *diagonal matrix with diagonal entries* ± 1. *Let* $\mathbf{A} \in \mathbb{C}^{n\times m}$ *be an* $n \times m$ *rectangular matrix such that* $\operatorname{rank}(\mathbf{A\Phi A}^{\mathrm{H}}) = \min(n, m)$. *Then there exist matrices* $\mathbf{U} \in \mathbb{C}^{n\times n}$, $\mathbf{V} \in \mathbb{C}^{m\times m}$ *and* $\mathbf{S} \in \mathbb{C}^{n\times m}$ *such that* $\mathbf{U}$ *is a unitary matrix,* $\mathbf{V}$ *is a hyperexchange matrix with respect to* $\mathbf{\Phi}$, *i.e.* $\mathbf{V}^{\mathrm{H}}\mathbf{\Phi V} = \widehat{\mathbf{\Phi}}$, *where* $\widehat{\mathbf{\Phi}}$ *is again an* $m \times m$ *diagonal matrix with diagonal entries* ± 1 *(generally different from* $\mathbf{\Phi}$*), and* $\mathbf{S}$ *is a rectangular diagonal matrix with real positive diagonal entries. For these matrices, the following equation holds:*

$$\mathbf{A} = \mathbf{USV}^{\mathrm{H}}. \tag{3.1}$$

Definition 3.1. The decomposition (3.1) is called the hyperbolic singular value decomposition of the matrix $\mathbf{A}$. Diagonal elements of the matrix $\mathbf{S}$ are called the hyperbolic singular values of the matrix $\mathbf{A}$.

Remark 3.1. Since $\mathbf{A\Phi A}^{\mathrm{H}} = \mathbf{US}\widehat{\mathbf{\Phi}}\mathbf{S}^{\mathrm{T}}\mathbf{U}^{\mathrm{H}}$ and $\mathbf{U}$ is a unitary matrix, the diagonal elements of the matrix $\mathbf{S}\widehat{\mathbf{\Phi}}\mathbf{S}^{\mathrm{T}}$ are the eigenvalues of the matrix $\mathbf{A\Phi A}^{\mathrm{H}}$ and the columns of the matrix $\mathbf{U}$ are the corresponding eigenvectors.

3.2. An algorithm of HSVD.

The classical algorithm for the singular value decomposition (SVD) of a given rectangular matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$, $n > m$ (see for example [3]) consists of two steps. By using Householder transformations, the given rectangular matrix is reduced to the upper bidiagonal form. Then this bidiagonal form is iteratively diagonalized by a sequence of Givens transformations.

The hyperbolic singular value decomposition works similarly. Let a rectangular matrix $\mathbf{A} \in \mathbb{C}^{n\times m}$ be given. Let $\mathbf{\Phi}$ be an $m \times m$ diagonal matrix with entries ± 1 and let the $\operatorname{rank}(\mathbf{A\Phi A}^{\mathrm{H}}) = \min(n, m) = l$. Our aim is to find a hyperbolic singular value decomposition (3.1). From this formula we see that

$$\mathbf{S} = \mathbf{U}^{\mathrm{H}}\mathbf{A}(\mathbf{V}^{\mathrm{H}})^{-1}.$$

We substitute the exact form of $(\mathbf{V}^{\mathrm{H}})^{-1}$ (see (2.3)) and obtain

$$\mathbf{S} = \mathbf{U}^{\mathrm{H}}\mathbf{A\Phi V}\widehat{\mathbf{\Phi}}. \tag{3.2}$$

Let us denote $\widehat{\mathbf{S}} = \mathbf{S}\widehat{\mathbf{\Phi}}$, $\widehat{\mathbf{A}} = \mathbf{A\Phi}$. Then the formula (3.2) has the form

$$\widehat{\mathbf{S}} = \mathbf{U}^{\mathrm{H}}\widehat{\mathbf{A}}\mathbf{V}. \tag{3.3}$$

The idea is simple now. We will transform the matrix $\widehat{\mathbf{A}}$ to the diagonal form $\widehat{\mathbf{S}}$ using orthogonal transformations from the left and hyperbolic transformations from the right.

R e m a r k 3.2. By using (3.3) we obtain

$$\widehat{\mathbf{S}}\widehat{\mathbf{\Phi}}\widehat{\mathbf{S}}^{\mathrm{H}} = \mathbf{U}^{\mathrm{H}}\widehat{\mathbf{A}}\mathbf{V}\widehat{\mathbf{\Phi}}\mathbf{V}^{\mathrm{H}}\widehat{\mathbf{A}}^{\mathrm{H}}\mathbf{U} = \mathbf{U}^{\mathrm{H}}\widehat{\mathbf{A}}\mathbf{\Phi}\widehat{\mathbf{A}}^{\mathrm{H}}\mathbf{U} = \mathbf{U}^{\mathrm{H}}\mathbf{A}\mathbf{\Phi}\mathbf{A}^{\mathrm{H}}\mathbf{U}, \tag{3.4}$$

therefore the eigenvalues of the diagonal matrix $\widehat{\mathbf{S}}\widehat{\mathbf{\Phi}}\widehat{\mathbf{S}}^{\mathrm{T}}$ are eigenvalues of $\mathbf{A}\mathbf{\Phi}\mathbf{A}^{\mathrm{H}}$, i.e. hyperbolic singular values of $\mathbf{A}$.

Following the classical SVD algorithm, we divide our HSVD algorithm into two steps. In the first, we will reduce the matrix $\widehat{\mathbf{A}} \in \mathbb{C}^{n\times m}$ into the lower (for $n \leqslant m$) or upper (for $n > m$) bidiagonal matrix $\mathbf{B}$.

Algorithm 3.1 a ($n \leqslant m$).
Set

$$\mathbf{A}_0 = \widehat{\mathbf{A}}, \quad \mathbf{\Phi}_0 = \mathbf{\Phi}, \qquad l = \min(n, m) = n.$$

For $k = 1, 2, \ldots, l-2$, generate a sequence of matrices

$$\mathbf{A}_k = \mathbf{M}_k\mathbf{A}_{k-1}\mathbf{H}_k,$$

where $\mathbf{H}_k$ is the hyperbolic Householder matrix which annihilates the $(k, k+1), \ldots, (k, m)$ elements of the k-th row of $\mathbf{A}_{k-1}$. $\mathbf{M}_k$ is the Householder transformation matrix which annihilates the elements $(k+2, k), \ldots, (n, k)$ of the k-th column of the matrix $\mathbf{A}_{k-1}\mathbf{H}_k$. Moreover, we set ($\mathbf{H}_k$ is a hyperexchange matrix)

$$\mathbf{\Phi}_k = \mathbf{H}_k^{\mathrm{H}}\mathbf{\Phi}_{k-1}\mathbf{H}_k.$$

Then

1. If $n < m$, we have to continue and for $k = l-1, l$ generate matrices

$$\mathbf{A}_k = \mathbf{A}_{k-1}\mathbf{H}_k,$$

where $\mathbf{H}_k$ is the hyperbolic Householder matrix which annihilates the $(k, k+1), \ldots, (k, m)$ elements of $\mathbf{A}_{k-1}$. Moreover,

$$\mathbf{\Phi}_k = \mathbf{H}_k^{\mathrm{H}}\mathbf{\Phi}_{k-1}\mathbf{H}_k.$$

We set

$$\mathbf{B} = \mathbf{A}_l, \quad \tilde{\mathbf{\Phi}} = \mathbf{\Phi}_l.$$

2. If $n = m$, we generate

$$\mathbf{A}_{l-1} = \mathbf{A}_{l-2}\mathbf{H}_{l-1},$$

where $\mathbf{H}_{l-1}$ is the hyperbolic Householder matrix which annihilates the $(l-1, l)$-th element of $\mathbf{A}_{l-2}$. We set

$$\mathbf{B} = \mathbf{A}_{l-1}, \quad \tilde{\mathbf{\Phi}} = \mathbf{\Phi}_{l-1} = \mathbf{H}_{l-1}^{\mathrm{H}}\mathbf{\Phi}_{l-2}\mathbf{H}_{l-1}.$$

Algorithm 3.1 b $(n > m)$.
Set

$$\mathbf{A}_0 = \widehat{\mathbf{A}}, \quad \mathbf{\Phi}_0 = \mathbf{\Phi}, \quad l = \min(n, m) = m.$$

For $k = 1, 2, \ldots, l-2$, generate a sequence of matrices

$$\mathbf{A}_k = \mathbf{M}_k\mathbf{A}_{k-1}\mathbf{H}_k,$$

where $\mathbf{H}_k$ is the hyperbolic Householder matrix which annihilates the $(k, k+2)$, $\ldots, (k, m)$ elements of the k-th row of $\mathbf{A}_{k-1}$. $\mathbf{M}_k$ is the Householder transformation matrix which annihilates the elements $(k+1, k), \ldots, (n, k)$ of the k-th column of the matrix $\mathbf{A}_{k-1}\mathbf{H}_k$. Moreover, we set ($\mathbf{H}_k$ is a hyperexchange matrix)

$$\mathbf{\Phi}_k = \mathbf{H}_k^{\mathrm{H}}\mathbf{\Phi}_{k-1}\mathbf{H}_k.$$

For $k = l-1$, l generate matrices

$$\mathbf{A}_k = \mathbf{M}_k\mathbf{A}_{k-1},$$

where $\mathbf{M}_k$ is the Householder matrix which annihilates the $(k+1, k), \ldots, (n, k)$ elements of $\mathbf{A}_{k-1}$. We set

$$\mathbf{B} = \mathbf{A}_l, \quad \tilde{\mathbf{\Phi}} = \mathbf{\Phi}_{l-2}.$$

In general, after l steps we obtain the matrix $\mathbf{B}$ in the lower (by Algorithm 3.1 a) or upper (by Algorithm 3.1 b) bidiagonal form:

$$\mathbf{B} = \mathbf{M}^{\mathrm{H}}\widehat{\mathbf{A}}\mathbf{H}, \tag{3.5}$$

where

$$\begin{aligned}
\mathbf{M} &= \mathbf{M}_1\mathbf{M}_2\ldots\mathbf{M}_{l-2} && \text{for} \quad n \leqslant m,\\
\mathbf{M} &= \mathbf{M}_1\mathbf{M}_2\ldots\mathbf{M}_{l} && \text{for} \quad n > m,\\
\mathbf{H} &= \mathbf{H}_1\mathbf{H}_2\ldots\mathbf{H}_{l-2} && \text{for} \quad n > m,\\
\mathbf{H} &= \mathbf{H}_1\mathbf{H}_2\ldots\mathbf{H}_{l-1} && \text{for} \quad n = m,\\
\mathbf{H} &= \mathbf{H}_1\mathbf{H}_2\ldots\mathbf{H}_{l} && \text{for} \quad n < m.
\end{aligned}$$

The matrix $\mathbf{M}$ is a unitary matrix, the matrix $\mathbf{H}$ is a hyperexchange matrix with respect to the original $\mathbf{\Phi}$:

$$\mathbf{H}^{\mathrm{H}}\mathbf{\Phi}\mathbf{H} = \tilde{\mathbf{\Phi}}.$$

Remark 3.3. The resulting matrix $\mathbf{B} \in \mathbb{C}^{l\times l}$ in (3.5) (we have omitted the zero rows and columns) has the forms

$$\begin{array}{cc} n \leqslant m & n > m \\ \mathbf{B} = \begin{pmatrix} x & 0 & & \dots & 0 \\ x & x & 0 & & 0 \\ 0 & x & x & \ddots & 0 \\ & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x & x \end{pmatrix}, & \mathbf{B} = \begin{pmatrix} x & x & 0 & \dots & 0 \\ 0 & x & x & & 0 \\ 0 & & \ddots & \ddots & 0 \\ & & & x & x \\ 0 & \dots & & 0 & x \end{pmatrix}. \end{array}$$

In the second step of the HSVD algorithm we will iteratively diagonalize the bidiagonal matrix $\mathbf{B}$.

Algorithm 3.2.
Set

$$\mathbf{B}_1 = \mathbf{B}, \quad \mathbf{\Phi}_1 = \tilde{\mathbf{\Phi}}.$$

For $k = 1, 2, \dots$ construct a sequence of matrices

$$\mathbf{B}_{k+1} = \mathbf{G}_k^{\mathrm{H}}\mathbf{B}_k\mathbf{T}_k, \tag{3.6}$$

where $\mathbf{G}_k$ is a unitary matrix,

$$\mathbf{G}_k = \mathbf{G}_{1,2}^{(k)}\mathbf{G}_{2,3}^{(k)}\dots\mathbf{G}_{l-1,l}^{(k)},$$

and $\mathbf{G}_{j,j+1}^{(k)}$, $j = 1, \dots, l-1$ are the matrices of the Givens rotation which are chosen in such a way that for $k \to \infty$ the sequence of matrices $\mathbf{B}_k$ converges to a diagonal matrix. The matrix $\mathbf{T}_k$ is a hyperexchange matrix of the (classical or hyperbolic) Givens rotation which ensures that the matrix $\mathbf{B}_{k+1}$ remains in the lower bidiagonal form.
We set

$$\mathbf{\Phi}_{k+1} = \mathbf{T}_k^{\mathrm{H}}\mathbf{\Phi}_k\mathbf{T}_k.$$

If the matrix $\mathbf{B}_{k+1}\mathbf{\Phi}_{k+1}\mathbf{B}_{k+1}^{\mathrm{H}}$ is not yet in the diagonal form (with the prescribed precision), we enlarge k and continue at (3.6). If it is already diagonal, we set

$$\widehat{\mathbf{S}} = \mathbf{B}_{k+1}, \qquad \widehat{\mathbf{\Phi}} = \mathbf{\Phi}_{k+1},$$
$$\mathbf{G} = \mathbf{G}_1\mathbf{G}_2\dots\mathbf{G}_{k+1}, \quad \mathbf{T} = \mathbf{T}_1\mathbf{T}_2\dots\mathbf{T}_{k+1}.$$

R e m a r k 3.4. The definition of the matrices of Givens rotations $\mathbf{G}_{j,j+1}$, $j = 1,\dots,l-1$ is based on a variant of the QR algorithm with shifts. The strategy is the same as in the case of the classical SVD, see [9].

By performing Algorithm 3.2 we obtain the decomposition

$$\widehat{\mathbf{S}} = \mathbf{G}^{\mathrm{H}}\mathbf{B}\mathbf{T},$$

where $\mathbf{G}$ is a unitary matrix, $\mathbf{T}$ is a hyperexchange matrix:

$$\mathbf{T}^{\mathrm{H}}\tilde{\mathbf{\Phi}}\mathbf{T} = \widehat{\mathbf{\Phi}}.$$

Hyperbolic singular values of the original matrix $\mathbf{A}$ are the diagonal entries of the matrix $\widehat{\mathbf{S}}\widehat{\mathbf{\Phi}}\widehat{\mathbf{S}}^{\mathrm{T}}$ (see (3.4)).

Finally, let us mention some applications of the HSVD algorithm. By using the hyperbolic transformations approach, some problems can be solved concerning regression updating, see [2]. In various time-series problems, one is interested in changing the relationship between variables. A regression model with a fixed number of terms, as it moves over a series of data, generates a "window" on the sample, with a new observation added and an old one deleted, as the window moves to the next point in the series.

The hyperbolic singular value decomposition has also a number of signal processing applications, e.g. recursive digital filtering techniques, see [4].

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