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THE METHOD OF ROTHE AND TWO-SCALE CONVERGENCE IN NONLINEAR PROBLEMS*

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Abstract. Modelling of macroscopic behaviour of materials, consisting of several layers or components, cannot avoid their microstructural properties. This article demonstrates how the method of Rothe, described in the book of K. Rektorys *The Method of Discretization in Time*, together with the two-scale homogenization technique can be applied to the existence and convergence analysis of some strongly nonlinear time-dependent problems of this type.

Keywords: PDE's of evolution, method of Rothe, two-scale convergence, homogenization of periodic structures

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1. Introduction

This article arose as a modified and extended version of the author's contribution [25] at the International Conference in honour of the 80^{th} birthday of Professor Karel Rektorys. Its aim is to demonstrate how the seemingly very classical and simple method of discretization in time, whose basic idea (coming from the implicit Euler formula) was investigated by Rothe (cf. [20]), well-known to both mathematicians and physicists or engineers just from Rektorys' monograph [19], can be applied not only to the effective numerical analysis of standard problems, as described in [22], but, moreover, to the verification of existence and properties of solutions in rather complicated variational formulations, including those typical for the mechanics of composites and other materials consisting of several finely mixed constituents with a (quasi)periodic structure. Such materials are widely used in modern industry: e.g., it is well-known (for details see [23]) that the most effective way how the creep resistance of metals at high temperatures can be improved is their reinforcement by

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hard particles—such particles can precipitate in the matrix during the heat treatment of the material (as in the case of superalloys), can be added into the matrix or can precipitate in the matrix during solidification (as in the case of metal matrix composites).

From the practical point of view, the principal computational difficulty is that most significant material heterogeneities are very small (in our example the size of particles and their mutual distances are typically in micrometers) in comparison with the global dimension of a material sample in the laboratory or of a bearing element of some engineering construction (in meters). Standard software packages typically include some heuristic construction of "mean values" of material characteristics; unfortunately, their strange results, not observed in the nature yet, are able to illustrate the conflict between the "verification" and the "validation", introduced in [4]—the example in [25, p. 360], refers to non-realistic results even in case of a simple layered material. However, direct application of finite element or similar techniques can incorporate no correct information about the microstructure without extremely large, slow and expensive calculations. On the other hand, all reasonable calculations at the microstructural level require strict assumptions on the macroperiodicity of external loads, usually far from the practical ones.

The natural idea how to overcome this difficulty is to improve the "mean value" approach to avoid or simplify the computational microanalysis. This is the principal (and more than 25 years old—cf. [3]) idea of all the so-called homogenization techniques—beginning from the formal asymptotic expansion, adopted to the study of periodic problems, leading to the multiple-scale method (understood in sense of [7, p. 125]), and continuing to the method of oscillating test functions (cf. [7, p. 138]) and to more advanced approaches based on the G-convergence, the H-convergence and the Γ-convergence (corresponding references can be found in [16] and [24]). But certain disadvantages seem to be typical for any such approach: rather complicated definitions, making use of tricky test functions with no clear physical interpretation, and reader-unfriendly proofs of lemmas with numerous non-constructive steps generating non-trivial auxiliary problems (unlike, e.g., the method of Rothe, discussed above) cause that most physicists and engineers are not ready to accept them. Moreover, in particular for nonlinear problems the differences between various types of convergence are often not transparent (see the relations between the G-convergence and the H-convergence in [7, p. 243], for illustration).

An alternative approach to homogenization, based on the original notion of the so-called two-scale convergence, occurred in 1989; its basic definitions and lemmas (later published in essentially generalized forms—more references are in [24] again) can be found in [18] and [1]. This approach incorporates a compensated compactness phenomenon due to a particular (not very artificial) choice of test functions.

Moreover, this seems to be equivalent to another idea of [2], applying a certain transform of a spatial variable with respect to a hidden microstructural one. (The natural generalization of this approach, the s-scale convergence with some integer $s > 1$, it is known as the "reiterated homogenization" by [1, p. 1492], or as the "multi-scale convergence" by [15, p. 10]). Since we intend to demonstrate that the properties of two-scale limits can be understood without a deep study of special function spaces, measure theory, etc., we shall work, following [18], [1] and [2], with the Lebesgue measure on a domain Ω in the 3-dimensional Euclidean space $\mathbb{R}^3$ and with the Hausdorff measure on its boundary $\partial\Omega$; this gives us a chance to apply the standard notion of Lebesgue and Sobolev spaces and compare the properties of the two-scale convergence with the well-known strong or weak one. Nevertheless, we must mention the progress in last several years: for problems with particles of lower dimension (e.g. long fibers or thin plates) in composites as well as problems with cracks, cavitation and capillary channels (as those described in [8]) [6] replaces the Lebesgue measure by a certain class of periodic Radon measures (for practical application see [27]), [15] on base of the theory of Young measures introduces the so-called α-convergence, in a special case degenerating to the s-scale convergence ($s = 2$ in our considerations), and finds its relation to the Γ-convergence, [16] studies the properties of various finite element techniques for both micro- and macroapproximation in $\mathbb{R}^3$, etc.

2. Definition and properties of the two-scale convergence

Up to now, we have mentioned (using a macroscale) only a domain Ω in $\mathbb{R}^3$. To be able to study also the material microstructure, let us have a unit cube Y in $\mathbb{R}^3$. (The reason for such special choice is its formal simplicity—it is easy to replace, following [7, p. 173], a unit cube here by an arbitrary rectangular parallelepiped). For Cartesian coordinates in $\mathbb{R}^3$ we shall use the standard notation $x = (x_1, x_2, x_3)$ in Ω and $\partial\Omega$ and $y = (y_1, y_2, y_3)$ in Y without additional comments. The index $_\#$ will force the Y-periodicity, the index $_Y$ will emphasize that the operator in question is related only to the microstructural variable, defined on Y, the index $_\Omega$ the same related to the macrostructural variable, defined on Ω (needed in the proof of Theorem 2 only) and, finally, the index $_*$ will be used for a constant extension of a function from Ω to $\Omega \times Y$. The underlined symbol highlights sequences: namely, $\underline{v}^\varepsilon$ denotes a sequence of elements v^ε in an appropriate function space, corresponding to a sequence of positive numbers ε decreasing to zero. Let us remark that in proofs of Theorem 1 and Theorem 2 we will use also the hat symbol for sequences indexed by integers directly: e.g. $\hat{v}^k$ will mean a sequence of elements v^k for $k \in \{1, 2, \ldots\}$. Moreover, let us consider a real couple (p, q) with $p > 1$ and

$q = p/(p-1)$ and an integer dimension n (the number of unknown fields in applications). Then the following definitions introduce the two-scale convergence in $L^p(\Omega)^n$:

Definition 1. Let u^0 be an element of $L^p(\Omega \times Y)^n$. We say that a sequence $\underline{u}^\varepsilon$ from $L^p(\Omega)^n$ *(weakly) two-scale converges* to u^0 if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \cdot \psi(x, x/\varepsilon)\,\mathrm{d}x = \int_\Omega \int_Y u^0(x,y) \cdot \psi(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \forall\, \psi \in C_0^\infty(\Omega, C_\#^\infty(Y)^n);$$

briefly $\underline{u}^\varepsilon \rightharpoonup\!\!\rightharpoonup u^0$.

Definition 2. Let u^0 be an element of $L^p(\Omega \times Y)^n$. We say that a sequence $\underline{u}^\varepsilon$ from $L^p(\Omega)^n$ *strongly two-scale converges* to u^0 if $\underline{u}^\varepsilon \rightharpoonup\!\!\rightharpoonup u^0$ and, in addition,

$$\lim_{\varepsilon \to 0} \int_\Omega |u^\varepsilon(x)|^p\,\mathrm{d}x = \int_\Omega \int_Y |u^0(x,y)|^p\,\mathrm{d}y\,\mathrm{d}x;$$

briefly $\underline{u}^\varepsilon \twoheadrightarrow u^0$.

Let us notice that we could replace $C_0^\infty(\Omega, C_\#^\infty(y)^n)$ by $L^p(\Omega, C_\#(Y)^n)$ in Definition 1, using the obvious density arguments. Moreover, it is possible to formulate Definition 2 with $\leqslant$ instead of $=$ because the remaining inequality is guaranteed by Definition 1 (for all details see [6, p. 1202]). The generalization of both the definitions to sequences in a space $L^p(\Omega)^{n\times 3}$ instead of $L^p(\Omega)^n$ is evident; (p,q) in both the definitions can be mutually exchanged or replaced by another analogous couple (e.g. (2,2)), too. The simplified notation $X := L^2(\partial\Omega)^n$ and $H := L^2(\Omega)^n$ will be also applied.

The most useful properties of the two-scale convergence can be expressed in the following four lemmas, whose first versions were published in [18] and [1] and whose complete proofs (even for much more general function spaces) are given in [26]:

Lemma 1. *If $\underline{u}^\varepsilon$ is bounded in $L^p(\Omega)^n$ then there exists such a $u^0 \in L^p(\Omega \times Y)^n$ that, up to a subsequence, $\underline{u}^\varepsilon \rightharpoonup\!\!\rightharpoonup u^0$.*

Lemma 2. *If $\underline{u}^\varepsilon \twoheadrightarrow u^0$ and $\underline{v}^\varepsilon \rightharpoonup\!\!\rightharpoonup v^0$, where $u^0 \in L^p(\Omega \times Y)^n$ and $v^0 \in L^p(\Omega \times Y)^n$, then also*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \cdot v^\varepsilon(x)\,\mathrm{d}x = \int_\Omega \int_Y u^0(x,y) \cdot v^0(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Lemma 3.

(i) *If* $\underline{u}^\varepsilon \to u$ *in* $L^p(\Omega)^n$ *then also* $\underline{u}^\varepsilon \twoheadrightarrow u_*$.

(ii) *If* $\underline{u}^\varepsilon \rightharpoonup\!\!\rightharpoonup u^0$ *then* $\underline{u}^\varepsilon \rightharpoonup u$ *for*

$$u(x) := \int_Y u^0(x,y)\,\mathrm{d}y.$$

Lemma 4. *If* $\underline{u}^\varepsilon$ *is bounded in* $L^p(\Omega)^n$ *and* $\nabla\underline{u}^\varepsilon$ *is bounded in* $L^p(\Omega)^{n\times 3}$ *then there exist such* $u \in W^{1,p}(\Omega)^n$ *and* $\tilde{u} \in L^p(\Omega, W^{1,p}_{\#}(Y)^n)$ *that, up to a subsequence,* $\underline{u}^\varepsilon \rightharpoonup\!\!\rightharpoonup u_*$ *and also* $\nabla\underline{u}^\varepsilon \rightharpoonup\!\!\rightharpoonup \nabla u_* + \nabla_Y \tilde{u}$.

The above mentioned lemmas are not sufficient to handle non-trivial time-dependent problems. The problems of evolution have been analyzed (even with slightly more general definition of the two-scale convergence, including time-integration) in [11] but, unfortunately, most conclusions cannot be directly applied to nonlinear formulations, where relatively simple "corrector-type results" are not available. To be able to make use of the properties of sequences of Rothe, we need another lemma.

For some positive final time T let us define a time interval $I = \{t \in \mathbb{R}_+ :\ t \leqslant T\}$. Let us select such positive ε that I can be covered (except the zero time) by a finite number $m := T/\varepsilon$ (m must be an integer) of subintervals $I = \{0\} \cup I_1^\varepsilon \cup \ldots \cup I_m^\varepsilon$ where

$$I_i^\varepsilon := \{t \in \mathbb{R}_+ :\ (i-1)\varepsilon < t \leqslant i\varepsilon\} \quad \forall\, i \in \{1,\ldots,m\}.$$

Let us consider a subspace V of a Sobolev space $W^{1,p}(\Omega)^n$, satisfying homogeneous Dirichlet boundary conditions on a certain part of $\partial\Omega$. Following [19, p. 166], using the formulae

$$u^\varepsilon(t) := u_i^\varepsilon + \frac{t - i\varepsilon}{\varepsilon}\,(u_i^\varepsilon - u_{i-1}^\varepsilon), \quad \bar{u}^\varepsilon(t) := u_i^\varepsilon \qquad \forall\, t \in I_i^\varepsilon \quad \forall\, i \in \{1,\ldots,m\}$$

for arbitrary $u_0^\varepsilon,\ldots,u_m^\varepsilon \in V$ (for $t=0$ both u^ε and $\bar{u}^\varepsilon$ can be set equal to u_0^ε), we are able to introduce two special abstract functions $u^\varepsilon, \bar{u}^\varepsilon :\ I \mapsto V$; u^ε is piecewise linear, $\bar{u}^\varepsilon$ is piecewise constant. (Clearly, u^ε, $\bar{u}^\varepsilon$, etc. are variable in Ω, too; this will be emphasized explicitly only in cases of potential misunderstanding). The set of all such u^ε will be denoted by $S^\varepsilon(I,V)$; $\bar{u}^\varepsilon$ can be always derived from u^ε uniquely. For $\varepsilon \to 0$, u^ε, $\bar{u}^\varepsilon$ and $\dot{u}^\varepsilon$ (a dot symbol is reserved for the derivatives by t everywhere) then generate the well-known sequences of Rothe.

Both V and H are reflexive Banach spaces. Moreover, in all considerations we will assume that a compact imbedding of V into H exists. The geometrical interpretation of such assumption (forcing the validity of the corresponding Sobolev imbedding theorem) is studied in [17, p. 62] in great detail. (To avoid technical difficulties, in the next section we will add some further requirements.) The following "two-scale modification" of the widely used result from [13, p. 25] is true:

Lemma 5. *If $\underline{u}^\varepsilon(t)$ is bounded in V and $\underline{\dot{u}}^\varepsilon(t)$ is bounded in H for any $t \in I$ then there exist such $u \in C(I,H) \cap L^\infty(I,V)$ with a time derivative $\dot{u} \in L^\infty(I,H)$ and such $\tilde{u} \in L^\infty(I, L^p(\Omega, W^{1,p}_\#(Y)^n))$ that, up to a subsequence,*
a) $\underline{\overline{u}}^\varepsilon(t) \rightharpoonup u(t)$ *in V for every $t \in I$,*
b) $\underline{\overline{u}}^\varepsilon \to u$ *in $C(I,H)$,*
c) $\underline{\overline{u}}^\varepsilon(t) \rightharpoonup\!\!\!\rightharpoonup u_*(t)$ *for every $t \in I$,*
d) $\nabla\underline{\overline{u}}^\varepsilon(t) \rightharpoonup\!\!\!\rightharpoonup \nabla u_*(t) + \nabla_Y \tilde{u}(t)$ *for every $t \in I$,*
e) $\underline{\dot{u}}^\varepsilon(t) \rightharpoonup\!\!\!\rightharpoonup \dot{u}_*(t)$ *for every $t \in I$.*

Proof. Since V is reflexive, the assertion a) with $u \in L^\infty(I,V)$ follows from the Eberlein-Shmul'yan theorem (see [10, p. 197]). The assertions c) and d) are simple consequences of Lemma 4; Lemma 3 forces that the two-scale limit of a sequence $\underline{\overline{u}}^\varepsilon(t)$ and its strong limit in H must coincide for each $t \in I$. Thanks to the estimates

$$\max_{\tau\in I} \|u^\varepsilon(\tau)\|V = \max_{\tau\in I} \|\overline{u}^\varepsilon(\tau)\|_V \leqslant C, \quad \|u^\varepsilon(t) - \overline{u}^\varepsilon(t)\|_H \leqslant \varepsilon \max_{\tau\in I} \|\dot{u}^\varepsilon(\tau)\|_H \leqslant C\varepsilon \quad \forall\, t \in I$$

for some positive constant C (independent of ε) the same convergence properties are guaranteed for both $\underline{u}^\varepsilon$ and $\underline{\overline{u}}^\varepsilon$. Moreover, we have

$$\|u^\varepsilon(t) - u^\varepsilon(t')\|_H \leqslant |t-t'| \max_{\tau\in I} \|\dot{u}^\varepsilon(\tau)\|_H \leqslant C|t-t'| \quad \forall\, t,t' \in I.$$

Thus, the compactness of the imbedding of V into H and the Arzelà-Ascoli theorem (see [13, p. 24] and [14, p. 35 and 44]) imply the assertion b), which guarantees $u \in C(I,H)$, too.

Lemma 1 yields the assertion e) with some $u'(t) \in L^2(\Omega \times Y)^n$ instead of $\dot{u}(t)$. Integrating the relation

$$\lim_{\varepsilon\to 0} \int_\Omega \dot{u}^\varepsilon(x,t)\cdot\psi(x,x/\varepsilon)\,\mathrm{d}x = \int_\Omega\int_Y u'(x,y,t)\cdot\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x \;\; \forall\, \psi \in C_0^\infty(\Omega, C_\#^\infty(Y)^n)$$

in time, we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega (u^\varepsilon(x,t) - u_0(x))\cdot\psi(x,x/\varepsilon)\,\mathrm{d}x = \int_\Omega\int_Y \left(\int_0^t u'(x,y,\tau)\,\mathrm{d}\tau\right)\cdot\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

The convergence $u^\varepsilon(t) \rightharpoonup\!\!\!\rightharpoonup u_*(t)$ (see the assertion c)) then gives

$$u(x,t) = u_0(x) + \int_0^t u'(x,y,\tau)\,\mathrm{d}\tau$$

for almost every $x \in \Omega$ and $y \in Y$; this enables us to identify u' with $\dot{u}_*$ only. $\square$

3. Formulation and analysis of a model problem

To make the notation as simple as possible, for arbitrary positive integers l, r, s let us introduce some special classes of functions. Let $\mathrm{Car}^l(\Omega, Y, \mathbb{R}^r)^s$ be a class of all functions $\zeta\colon \Omega \times Y \times \mathbb{R}^r \mapsto \mathbb{R}^s$ (later we will have in particular $l \in \{2, p\}$, $r \in \{n, n \times 3 + n\}$ and $s \in \{n, n \times 3)\}$) with the following properties:

A) The function ζ is Y-periodic. Moreover, for almost every $x \in \Omega$ and $y \in Y$ and for each $w \in \mathbb{R}^s$ the growth condition $|\zeta(x, y, w)| \leqslant K_\zeta(1 + |w|^{l-1})$ is satisfied for some positive constant K_ζ.

B) The function $\zeta_\varphi(x, y) := \zeta(x, y, \varphi(x, y))$ applied to every x and y from A) for any $\varphi \in L^p(\Omega, C_\#(Y)^n)$ defines a continuous mapping $\zeta_\varphi\colon L^p(\Omega, C_\#(Y)^r) \mapsto L^q(\Omega, C_\#(Y)^s)$.

(In many cases the Nemytskiĭ operators from [21, p. 36] and [10, p. 75] are useful to verify such properties.) A class $\mathrm{Car}^2(\partial\Omega, \mathbb{R}^3)$ can be introduced in the same way with small changes: only $l = 2$ and $r = s = n$ will be needed, Ω is substituted by $\partial\Omega$ (the 2-dimensional Hausdorff measure σ is considered here instead of the standard 3-dimensional Lebesgue measure), in both A) and B) the variable y is missing, the requirement on periodicity in A) disappears.

Making use of A) and B), let us assume that $a \in \mathrm{Car}^p(\Omega, Y, \mathbb{R}^{n\times 3+n})^{n\times 3}$, $b \in L^\infty(\Omega \times Y)^n$ (in most applications "material characteristics" a and b are time-independent explicitly) and $f(t) \in \mathrm{Car}^2(\Omega, Y, \mathbb{R}^3)^n$, $g(t) \in \mathrm{Car}^2(\partial\Omega, \mathbb{R}^3)^n$ for any $t \in I$ (the process of evolution is driven by "external loads" g and "internal loads" f). Let some $u_0 \in V$ be given. Now we are ready to formulate two closely related problems:

Problem 1. For an arbitrary $\varepsilon \in \mathbb{R}_+$ find such $u^\varepsilon \in S^\varepsilon(I, V)$ that

$$
\begin{aligned}
(1)\qquad & \int_\Omega b(x, x/\varepsilon)\dot{u}^\varepsilon(x, t) \cdot v(x)\,\mathrm{d}x + \int_\Omega a(x, x/\varepsilon, \bar{u}^\varepsilon(x, t), \nabla\bar{u}^\varepsilon(x, t)) \cdot \nabla v(x)\,\mathrm{d}x \\
& + \int_{\partial\Omega} g(x, \bar{u}^\varepsilon(x, t), \bar{t}^\varepsilon) \cdot v(x)\,\mathrm{d}\sigma(x) = \int_\Omega f(x, x/\varepsilon, \bar{u}^\varepsilon(x, t), \bar{t}^\varepsilon) \cdot v(x)\,\mathrm{d}x
\end{aligned}
$$

for all $v \in V$ and at every time $t \in I$ (represented by its piecewise constant approximation $\bar{t}^\varepsilon$) if $u^\varepsilon(x, 0) = u_0(x)$ for almost every $x \in \Omega$.

Problem 2. Find such $u \in C(I, H) \cap L^\infty(I, V)$ with a time derivative $\dot{u} \in L^\infty(I, H)$ and such $\tilde{u} \in L^\infty(I, L^p(\Omega, W^{1,p}_\#(Y)^n))$ that

$$
\begin{aligned}
(2)\qquad & \int_0^t \int_\Omega \tilde{b}(x)(u(x, \tau) - u_0(x)) \cdot v(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
& + \int_0^t \int_\Omega \int_Y a(x, y, u(x, \tau), \nabla u(x, \tau) + \nabla_Y \tilde{u}(x, y, \tau), \tau) \cdot \nabla v(x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \int_{\partial\Omega} g(x,u(x,\tau),\tau)\cdot v(x)\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_0^t \int_\Omega \int_Y f(x,y,u(x,\tau),\tau)\cdot v(x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
$$

for all $v\in V$ and at every time $t\in I$ if $u(x,0)=u_0(x)$ for almost every $x\in\Omega$; the notation

$$\tilde{b}(x) := \int_Y b(x,y)\,\mathrm{d}y$$

is used here.

Let us notice that (2) can be obtained by integration in time from

$$
\begin{aligned}
(3)\qquad &\int_\Omega \bar{b}(x)\dot{u}(x,t)\cdot v(x)\,\mathrm{d}x \\
&\quad+\int_\Omega\int_Y a(x,y,u(x,t),\nabla u(x,t)+\nabla_Y\tilde{u}(x,y,t),t)\cdot\nabla v(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad+\int_{\partial\Omega} g(x,u(x,t),t)\cdot v(x)\,\mathrm{d}\sigma(x) \\
&= \int_\Omega\int_Y f(x,y,u(x,t),t)\cdot v(x)\,\mathrm{d}y\,\mathrm{d}x;
\end{aligned}
$$

in this way (1) seems to be a suitable “time-discretized representation” of (2). Moreover, (1) can be rewritten, step-by-step for $i\in\{1,\dots,m\}$, in its alternative form

$$
\begin{aligned}
(4)\qquad &\varepsilon^{-1}\int_\Omega b(x,x/\varepsilon)(u_i^\varepsilon(x)-u_{i-1}^\varepsilon(x))\cdot v(x)\,\mathrm{d}x \\
&\quad+\int_\Omega a(x,x/\varepsilon,u_i^\varepsilon(x),\nabla u_i^\varepsilon(x))\cdot\nabla v(x)\,\mathrm{d}x \\
&\quad+\int_{\partial\Omega} g(x,u_i^\varepsilon(x),i\varepsilon)\cdot v(x)\,\mathrm{d}\sigma(x) \\
&= \int_\Omega f(x,x/\varepsilon,u_i^\varepsilon(x),i\varepsilon)\cdot v(x)\,\mathrm{d}x
\end{aligned}
$$

for all $v\in V$; the algorithm starts with $u_0^\varepsilon(x):=u_0(x)$. Thus, to solve Problem 1 means to verify the correctness of the suggested algorithm. The analysis of the limit behavior of $\underline{u}^\varepsilon$, $\overline{u}^\varepsilon$ and $\underline{\dot{u}}^\varepsilon$ from (1), based on Lemma 5, should be helpful for solving Problem 2.

To avoid technical difficulties, let us assume $p\geqslant 2$ and accept some additional requirements:

C) Dirichlet boundary conditions prescribed on $\partial\Omega$ force the equivalence of norms $\|v\|_V$, $\|\nabla v\|_{L^p(\Omega)^{n\times 3}}$ and $\|\nabla v\|_{L^p(\Omega)^{n\times 3}}+\|v\|_X$ for all $v\in V$ (this refers to the generalized Friedrichs inequality and to the trace theorem—cf. [17, pp. 216 and 222]).

D) The estimates

$$a(x,y,z,\theta)\cdot\theta \geqslant \kappa|\theta|^p, \quad (a(x,y,z,\theta)-a(x,y,z,\theta'))\cdot(\theta-\theta') \geqslant 0$$

are true for almost every $x\in\Omega$ and $y\in Y$, for all $z\in\mathbb{R}^n$ and for arbitrary θ, $\theta'\in\mathbb{R}^{n\times 3}$; κ in the first estimate is a positive real constant. If $\theta\neq\theta'$ then the second estimate holds also with $>$ instead of $\geqslant$.

E) The estimates

$$b(x,y)z\cdot z \geqslant \beta|z|^2, \quad |b(x,y)| \leqslant \beta'$$

are valid for almost every $x\in\Omega$ and $y\in Y$ and for all $z\in\mathbb{R}^n$ with some positive constants β, β'.

F) The estimate

$$(g(x,z,t)-g(x,z',t))\cdot(z-z') \geqslant 0$$

holds for almost every $x\in\Omega$ and for all $z,z'\in\mathbb{R}^n$ at each time $t\in I$.

G) There exists such $\dot{u}_0\in L^2(\Omega\times Y)^n$ (substituting the unknown initial time derivative—this fact is emphasized by a dot symbol here) that the equation (description of a stationary initial state)

$$\begin{aligned}\int_\Omega\int_Y b(x,y)\dot{u}_0(x)\cdot v(x)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\int_Y a(x,y,u_0(x),\nabla u_0(x))\cdot\nabla v(x)\,\mathrm{d}y\,\mathrm{d}x\\ + \int_{\partial\Omega} g(x,u_0(x),0)\cdot v(x)\,\mathrm{d}\sigma(x) = \int_\Omega\int_Y f(x,y,u_0(x),0)\cdot v(x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

is true for any $v\in V$.

H) The symmetry

$$b(x,y)z\cdot z' = b(x,y)z'\cdot z$$

is preserved for almost every $x\in\Omega$ and $y\in Y$ and for all $z,z'\in\mathbb{R}^n$.

I) The estimates

$$\begin{aligned}|f(x,y,z,t)-f(x,y,z',t)| &\leqslant \xi_f|z-z'|,\\ |f(x,y,z,t)-f(x,y,z,t')| &\leqslant \xi_f|t-t'|\end{aligned}$$

are true for almost every $x\in\Omega$ and $y\in Y$ and for all $z,z'\in\mathbb{R}^n$ at arbitrary times $t,t'\in I$ with a positive constant ξ_f.

J) The imbedding of V into X (guaranteed by C)) is compact (cf. [17, p. 221]) and the estimates

$$\begin{aligned}|g(x,z,t)-g(x,z',t)| &\leqslant \xi_g|z-z'|,\\ |g(x,z,t)-g(x,z,t')| &\leqslant \xi_g|t-t'|\end{aligned}$$

are true for almost every $x \in \partial\Omega$ and for all $z, z' \in \mathbb{R}^n$ at arbitrary times $t, t' \in I$ with a positive constant ξ_g.

Theorem 1. *Let the assumptions* C), D), E) *and* F) *be satisfied. Then Problem* 1 *has a solution.*

P r o o f. Let us introduce the notation of scalar products in H

$$(B^\varepsilon u, v) := \int_\Omega b(x, x/\varepsilon) u(x) \cdot v(x) \,\mathrm{d}\sigma(x),$$
$$(F_i^\varepsilon u, v) := \int_\Omega f(x, x/\varepsilon, u(x), i\varepsilon) \cdot v(x) \,\mathrm{d}x$$

and of dualities between V and $V^\star$

$$\langle A^\varepsilon u, v\rangle := \int_\Omega a(x, x/\varepsilon, u(x), \nabla u(x)) \cdot \nabla v(x) \,\mathrm{d}x,$$
$$\langle G_i^\varepsilon u, v\rangle := \int_{\partial\Omega} g(x, u(x), i\varepsilon) \cdot v(x) \,\mathrm{d}\sigma(x),$$
$$\langle T_i^\varepsilon u, v\rangle := \varepsilon^{-1}(B^\varepsilon u - B^\varepsilon u_{i-1}, v) + \langle A^\varepsilon u, v\rangle + \langle G_i^\varepsilon u, v\rangle - (F_i^\varepsilon u, v)$$

for any $u, v \in V$ $(i \in \{1, \dots, m\})$. Using this notation, (4) assumes the form $\langle T_i^\varepsilon u_i^\varepsilon, v\rangle = 0$ for all $v \in V$. By [9, p. 279], if

i) T_i^ε is coercive,
ii) T_i^ε is demicontinuous,
iii) T_i^ε is bounded and
iv) the estimate

$$\limsup_{k\to\infty} \langle T_i^\varepsilon v^k - T_i^\varepsilon v, v^k - v\rangle \leqslant 0 \tag{5}$$

for an arbitrary sequence $\hat{v}^k$ in V together with $\hat{v}^k \rightharpoonup v \in V$ forces $\hat{v}^k \to v$

then T_i^ε is also surjective; this would imply the existence of a solution $u_i^\varepsilon \in V$ of (4) immediately. Consequently, it remains to verify the four conditions i), ii), iii), iv).

i) Directly from the definition of T_i^ε (with $u = v$) we have

$$\langle T_i^\varepsilon v, v\rangle = \varepsilon^{-1}(B^\varepsilon v, v) + \langle A^\varepsilon v\rangle + \langle G_i^\varepsilon v, v\rangle - (F_i^\varepsilon v, v) - \varepsilon^{-1}(B^\varepsilon v, u_{i-1}^\varepsilon)$$

where the assumpions D) and E) imply

$$\langle A^\varepsilon v, v\rangle \geqslant \kappa \|\nabla v\|^p_{L^p(\Omega)^{n\times 3}} \geqslant \kappa' \|v\|^p_V, \quad (B^\varepsilon v, v) \geqslant \beta \|v\|^2_H;$$

the existence of a new positive constant κ' here follows from the assumption C). The assumption F) with respect to the assumption A) for any $\delta \in \mathbb{R}_+$ (o denotes a zero vector in $\mathbb{R}^n$) gives similarly

$$\begin{aligned}\langle G_i^\varepsilon v, v\rangle &= \langle G_i^\varepsilon v - G_i^\varepsilon o, v\rangle + \langle G_i^\varepsilon o, v\rangle \geqslant \langle G_i^\varepsilon o, v\rangle \geqslant -K_g\|v\|_X \\ &\geqslant -\frac{K_g}{4\delta}\,\sigma(\partial\Omega) - K_g\delta\|v\|_X^2 \geqslant -\frac{K_g}{4\delta}\,\sigma(\partial\Omega) - K_g'\delta\|v\|_V^2 \\ &\geqslant -\frac{K_g}{4\delta}\,\sigma(\partial\Omega) - K_g'\delta(1+\|v\|_V^p);\end{aligned}$$

the existence of a new positive constant K_g' here follows from the assumption C). The assumption A) yields

$$(F_i^\varepsilon v, v) \leqslant K_f\|v\|_{L(\Omega)^n} + K_f\|v\|_H^2 \leqslant \frac{K_f}{4}\operatorname{vol}\Omega + 2K_f\|v\|_H^2.$$

Making use of $u_{i-1}^\varepsilon \in V$, prepared from the preceding time step, we obtain

$$(B^\varepsilon v, u_{i-1}^\varepsilon) \leqslant \beta'\|v\|_H\|u_{i-1}^\varepsilon\|_H^2 \leqslant \beta'\|v\|_H^2 + \frac{\beta'}{4}\|u_{i-1}^\varepsilon\|_H^2.$$

Thus, for ε and δ small enough we can conclude

$$\langle T_i^\varepsilon v, v\rangle \geqslant \kappa_1\|v\|_V^p + \frac{\kappa_2}{\varepsilon}\|v\|_H^2 - \kappa_0$$

with some positive constants κ_0, κ_1 and κ_2; this makes the coerciveness of T_i^ε evident.

ii) We have to prove that for any sequence $\hat{u}^k$ in V the strong convergence $\hat{u}^k \to u \in V$ yields

$$\lim_{k\to\infty}\langle T_i^\varepsilon \hat{u}^k - T_i^\varepsilon u, v\rangle = 0 \quad \text{for } v \in V$$

(cf. [9, p. 270]). However, we can write

$$\begin{aligned}\langle T_i^\varepsilon \hat{u}^k - T_i^\varepsilon u, v\rangle &= \varepsilon^{-1}(B^\varepsilon \hat{u}^k - B^\varepsilon u, v) + \langle A^\varepsilon \hat{u}^k - A^\varepsilon u, v\rangle \\ &\quad + \langle G_i^\varepsilon \hat{u}^k - G_i^\varepsilon u, v\rangle - (F_i^\varepsilon \hat{u}^k - F_i^\varepsilon u, v) \\ &= \int_\Omega b(x, x/\varepsilon)(u^k(x) - u(x))\cdot v(x)\,\mathrm{d}x \\ &\quad + \int_\Omega (a(x, x/\varepsilon, u^k(x), \nabla u^k(x)) - a(x, x/\varepsilon, u(x), \nabla u(x)))\cdot\nabla v(x)\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} (g(x, u^k(x), i\varepsilon) - g(x, u(x), i\varepsilon))\cdot v(x)\,\mathrm{d}\sigma \\ &\quad - \int_\Omega (f(x, x/\varepsilon, u^k(x), i\varepsilon) - f(x, x/\varepsilon, u(x), i\varepsilon))\cdot v(x)\,\mathrm{d}x\end{aligned}$$

and with help of the Hölder inequality we are able to derive all the expected convergence results. For $k \to \infty$ the first integral goes to zero, as $\hat{u}^k \to u$ in H; the same is true for the last integral where the assumption B) applied to f is available. We have $\hat{u}^k \to u$ also in X and $\nabla \hat{u}^k \to \nabla u$ in $L^p(\Omega)^{n\times 3}$, which together with the assumption B) applied to g and a yields identical results for the third and second integrals, too.

iii) Let us construct estimates

$$\begin{aligned}
\langle A^\varepsilon u, v\rangle &\leqslant K_a\|\nabla v\|_{L(\Omega)^{n\times 3}} + K_a(\|\nabla u\|^{p-1}_{L^p(\Omega)^{n\times 3}} + \|u\|^{p-1}_{L^p(\Omega)^n})\|\nabla v\|_{L^p(\Omega)^n} \\
&\leqslant K_a((\operatorname{vol}\Omega)^{(p-1)/p} + \|u\|_V^{p-1})\|v\|_V, \\
\langle G_i^\varepsilon u, v &\leqslant K_g\|\nabla v\|_{(\partial\Omega)^n} + K_g\|u\|_X\|v\|_X \leqslant ((\sigma(\Omega))^{1/2} + \|u\|_X)\|v\|_X, \\
(F_i^\varepsilon u, v) &\leqslant K_f\|v\|_{L(\Omega)^n} + K_g\|u\|_H\|v\|_H \leqslant ((\operatorname{vol}\Omega)^{1/2} + \|u\|_H)\|v\|_H
\end{aligned}$$

and by the assumption E) also

$$(B^\varepsilon u - B^\varepsilon u_{i-1}^\varepsilon, v) \leqslant (\|B^\varepsilon u\|_H + \|B^\varepsilon u_{i-1}^\varepsilon\|_H)\|v\|_H \leqslant (\beta'\|u\|_H + \|B^\varepsilon u_{i-1}^\varepsilon\|_H)\|v\|_H.$$

The boundedness of T_i^ε then comes from the assumption C) and from the fact that a (compact) imbedding of V into H exists.

iv) Let $\hat{v}^k$ be an arbitrary sequence in V such that $\hat{v}^k \rightharpoonup v \in V$. Let us study the expression

$$\begin{aligned}
&\langle T_i^\varepsilon v^k - T_i^\varepsilon v, v^k - v\rangle \\
&\quad= \frac{1}{\varepsilon}(B^\varepsilon v^k - B^\varepsilon v, v^k - v) + \langle A^\varepsilon v^k - A^\varepsilon v, v^k - v\rangle \\
&\qquad+ \langle G_i^\varepsilon v^k - G_i^\varepsilon v, v^k - v\rangle - (F_i^\varepsilon v^k - F_i^\varepsilon v, v^k - v) \\
&\quad= \int_\Omega b(x, x/\varepsilon)(v^k(x) - v(x))\cdot(v^k(x) - v(x))\,\mathrm{d}x \\
&\qquad+ \int_\Omega (a(x, x/\varepsilon, v^k(x), \nabla v^k(x)) - a(x, x/\varepsilon, v(x), \nabla v(x)))\cdot(v^k(x) - v(x))\,\mathrm{d}x \\
&\qquad+ \int_\Omega (g(x, v^k(x), i\varepsilon) - g(x, v(x), i\varepsilon))\cdot(v^k(x) - v(x))\,\mathrm{d}\sigma(x) \\
&\qquad- \int_\Omega (f(x, x/\varepsilon, v^k(x), i\varepsilon) - f(x, x/\varepsilon, v(x), i\varepsilon))\cdot(v^k(x) - v(x))\,\mathrm{d}x.
\end{aligned}$$

For $k \to \infty$ the first and last integrals vanish (because $\hat{v}^k \to v$ in H). Using the assumption F) as a lower estimate we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\langle T_i^\varepsilon v^k - T_i^\varepsilon v, v^k - v\rangle \\
&\geqslant \limsup_{k\to\infty}\int_\Omega (a(x, x/\varepsilon, v^k(x), \nabla v^k(x)) - a(x, x/\varepsilon, v(x), \nabla v(x)))\cdot(v^k(x) - v(x))\,\mathrm{d}x.
\end{aligned}$$

If $\hat{v}^k \not\to v$ in V then this estimate together with (5) leads according to the assumption D) (whose second part has to be used here for the first time) to the contradiction $0 > 0$; therefore $\hat{v}^k \to v$ in V. □

Theorem 2. *Let the assumptions* C), D), E), F), G), H), I) *and* J) *be satisfied. Then Problem* 2 *has a solution.*

Proof. Using the brief notation from the proof of Theorem 1, we are able to convert (4) for any $i \in \{1, \dots, m\}$ into the form

$$\varepsilon^{-1}(B^\varepsilon(u_i^\varepsilon - u_{i-1}^\varepsilon), v) + \langle A^\varepsilon u_i^\varepsilon, v\rangle + \langle G_i^\varepsilon u_i^\varepsilon, v\rangle = (F_i^\varepsilon u_i^\varepsilon, v) \tag{6}$$

for all $v \in V$. Thanks to the assumption G) the same holds with $u_{-1}^\varepsilon := u_0^\varepsilon - \varepsilon \dot{u}_0^\varepsilon$ for a zero index instead of i. If, in particular, $v = (u_i^\varepsilon - u_{i-1}^\varepsilon)/\varepsilon$ then the difference between (6) and the same equation with all indices reduced by 1 gives

$$\begin{aligned}&\varepsilon^{-2}(B^\varepsilon(u_i^\varepsilon - 2u_{i-1}^\varepsilon + u_{i-2}^\varepsilon), u_i^\varepsilon - u_{i-1}^\varepsilon) + \varepsilon^{-1}\langle A^\varepsilon(u_i^\varepsilon - u_{i-1}^\varepsilon), u_i^\varepsilon - u_{i-1}^\varepsilon\rangle\\ &+\varepsilon^{-1}\langle G_i^\varepsilon u_i^\varepsilon - G_{i-1}^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon - u_{i-1}^\varepsilon\rangle = \varepsilon^{-1}(F_i^\varepsilon u_i^\varepsilon - F_{i-1}^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon - u_{i-1}^\varepsilon).\end{aligned}$$

Since the assumption H) forces

$$(B^\varepsilon(w - w'), w) = \frac{1}{2}(B^\varepsilon w, w) - \frac{1}{2}(B^\varepsilon w', w') + \frac{1}{2}(B^\varepsilon(w - w'), w - w') \quad \forall\, w, w' \in V,$$

the sum of all such equations for $j \in \{1, \dots, i\}$ is

$$\begin{aligned}&\frac{1}{2}\varepsilon^{-2}(B^\varepsilon(u_j^\varepsilon - u_{j-1}^\varepsilon), u_j^\varepsilon - u_{j-1}^\varepsilon)\\ &\qquad + \frac{1}{2}\varepsilon^{-2}\sum_{i=1}^{j}(B^\varepsilon(u_i^\varepsilon - 2u_{i-1}^\varepsilon + u_{i-2}^\varepsilon), u_i^\varepsilon - 2u_{i-1}^\varepsilon + u_{i-2}^\varepsilon)\\ &\qquad + \varepsilon^{-1}\sum_{i-1}^{j}\langle A^\varepsilon(u_i^\varepsilon - u_{i-1}^\varepsilon), u_i^\varepsilon - u_{i-1}^\varepsilon\rangle + \varepsilon^{-1}\sum_{i=1}^{j}\langle G_i^\varepsilon u_i^\varepsilon - G_{i-1}^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon - u_{i-1}^\varepsilon\rangle\\ &\quad = \varepsilon^{-1}\sum_{i=1}^{j}(F_i^\varepsilon u_i^\varepsilon - F_{i-1}^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon - u_{i-1}^\varepsilon) + \frac{1}{2}(B^\varepsilon \dot{u}_0^\varepsilon, \dot{u}_0^\varepsilon).\end{aligned}$$

But all left-hand-side additive terms except the first are non-negative; thus the estimate

$$\begin{aligned}&\varepsilon^{-2}(B^\varepsilon(u_j^\varepsilon - u_{j-1}^\varepsilon), u_j^\varepsilon - u_{j-1}^\varepsilon)\\ &\qquad \leqslant 2\varepsilon^{-1}\sum_{i=1}^{j}(F_i^\varepsilon u_i^\varepsilon - F_i^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon - u_{i-1}^\varepsilon) + \varepsilon^{-1}(B^\varepsilon \dot{u}_0^\varepsilon, \dot{u}_0^\varepsilon)\end{aligned}$$

is possible. For its particular terms the assumption E) yields

$$(B^\varepsilon(u_j^\varepsilon - u_{j-1}^\varepsilon), u_j^\varepsilon - u_{j-1}^\varepsilon) \geqslant \beta\|u_j^\varepsilon - u_{j-1}^\varepsilon\|_H^2, \quad (B^\varepsilon \dot{u}_0^\varepsilon, \dot{u}_0^\varepsilon) \leqslant \beta'\|u_0^\varepsilon\|_H^2$$

and similarly the assumption I) gives

$$\begin{aligned}(F_i^\varepsilon u_i^\varepsilon - F_{i-1}^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon - u_{i-1}^\varepsilon) &\leqslant \xi\|u_i^\varepsilon - u_{i-1}^\varepsilon\|_H^2 + \xi\varepsilon\|u_i^\varepsilon - u_{i-1}^\varepsilon\|_{L(\partial\Omega)^n}\\ &\leqslant \xi\|u_i^\varepsilon - u_{i-1}^\varepsilon\|_H^2 + \xi\varepsilon(\operatorname{vol}\Omega)^{1/2}\|u_i^\varepsilon - u_{i-1}^\varepsilon\|_H\\ &\leqslant 2\xi\|u_i^\varepsilon - u_{i-1}^\varepsilon\|_H^2 + \frac{1}{4}\xi\varepsilon^2\operatorname{vol}\Omega.\end{aligned}$$

Clearly $j\varepsilon \leqslant T$; in this way for a sufficiently small ε we obtain

$$\|(u_j^\varepsilon - u_{j-1}^\varepsilon)/\varepsilon\|_H \leqslant c$$

with a positive constant c. Thus $\underline{\dot{u}}^\varepsilon(t)$ is bounded in H for any $t \in I$, which is the second assumption of Lemma 5. Another natural consequence of this estimate is

$$\|u_j^\varepsilon\|_H \leqslant \varepsilon\sum_{i=1}^{j}\|(u_j^\varepsilon - u_{j-1}^\varepsilon)/\varepsilon\|_H \leqslant cT$$

(which means that $\underline{\overline{u}}^\varepsilon(t)$ is bounded in H, too).

Repeating the same argument with a special choice $v = u_i^\varepsilon$ in (6), we obtain

$$\langle A^\varepsilon u_i^\varepsilon, u_i^\varepsilon\rangle + \langle G_i^\varepsilon u_i^\varepsilon, u_i^\varepsilon\rangle = (F_i^\varepsilon u_i^\varepsilon, u_i^\varepsilon) - \varepsilon^{-1}(B^\varepsilon u_i^\varepsilon - B^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon).$$

However, the assumption E) implies

$$-\varepsilon^{-1}(B^\varepsilon u_i^\varepsilon - B^\varepsilon u_{i-1}^\varepsilon, u_i^\varepsilon) \leqslant \beta'\|(u_i^\varepsilon - u_{i-1}^\varepsilon)/\varepsilon\|_H\|u_i^\varepsilon\|_H \leqslant \beta' c^2 T$$

and part i) of the proof of Theorem 1 gives

$$\begin{aligned}&\langle A^\varepsilon u_i^\varepsilon, u_i^\varepsilon\rangle \geqslant \kappa'\|u_i^\varepsilon\|_V^p,\\ &\langle G_i^\varepsilon u_i^\varepsilon, u_i^\varepsilon\rangle \geqslant -\frac{K_g}{4\delta}\,\sigma(\partial\Omega) - K_g'\delta(1 + \|u_i^\varepsilon\|_V^p),\\ &(F_i^\varepsilon u_i^\varepsilon, u_i^\varepsilon) \leqslant \frac{K_f}{4}\operatorname{vol}\Omega + 2K_f\|u_i^\varepsilon\|_H^2 \leqslant \frac{K_f}{4}\operatorname{vol}\Omega + 2K_f c^2 T^2;\end{aligned}$$

for δ small enough this yields

$$\|u_i^\varepsilon\|_V \leqslant c'$$

with a positive constant c', i.e. the boundedness of $\underline{\overline{u}}^\varepsilon(t)$ in V for any $t \in I$, which is the first assumption of Lemma 5.

By Lemma 5 there exist a certain $u \in C(I,H) \cap L^\infty(I,V)$ with a time derivative $\dot{u} \in L^\infty(I,H)$ and a certain $\tilde{u} \in L^\infty(I, L^p(\Omega, W^{1,p}_{\#}(Y)^n))$ attained as limits of $\underline{\overline{u}}^\varepsilon$ and $\underline{\dot{u}}^\varepsilon$ in the sense of the assertions a), b), c), d) and e). It remains to prove that these limits satisfy (3) (and consequently (2)). Let us choose an arbitrary $v \in V$ and introduce $\omega^\varepsilon(x) := b(x, x/\varepsilon)v(x)$ and $\omega(x,y) := b(x,y)v(x)$; evidently $\omega^\varepsilon \twoheadrightarrow \omega$. The assertion e) of Lemma 5 in accordance with Lemma 2 and with respect to the assumption H) results in

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int_\Omega b(x,x/\varepsilon)\dot{u}^\varepsilon(x,t)\cdot v(x)\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega \omega^\varepsilon(x)\cdot \dot{u}^\varepsilon(x,t)\,\mathrm{d}x \\
= \int_\Omega\int_Y \omega(x,y)\cdot\dot{u}(x,t)\,\mathrm{d}y\,\mathrm{d}x &= \int_\Omega \tilde{b}(x)\dot{u}(x,t)\cdot v(x)\,\mathrm{d}x;
\end{aligned}
$$

this shows that the first integral in (1) tends to the corresponding one in (3) for $\varepsilon \to 0$. The same for the last (right-hand side) integral follows from the relation

$$
\begin{aligned}
&\int_\Omega\int_Y f(x,y,u(x,t),t)\cdot v(x)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad= \lim_{\varepsilon\to 0}\int_\Omega f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),\overline{t}^\varepsilon)\cdot v(x)\,\mathrm{d}x \\
&\qquad+ \lim_{\varepsilon\to 0}\int_\Omega (f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),t) - f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),\overline{t}^\varepsilon))\cdot v(x)\,\mathrm{d}x \\
&\qquad+ \lim_{\varepsilon\to 0}\int_\Omega (f(x,x/\varepsilon,u(x,t),t) - f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),t))\cdot v(x)\,\mathrm{d}x,
\end{aligned}
$$

and from the fact that by the assumption I) the last two limits vanish (with respect to the assertion b) of Lemma 5), while for the third integral thus follows from the similar relation

$$
\begin{aligned}
&\int_{\partial\Omega} g(x,u(x,t),t)\cdot v(x)\,\mathrm{d}x \\
&\quad= \lim_{\varepsilon\to 0}\int_{\partial\Omega} g(x,\overline{u}^\varepsilon(x,t),\overline{t}^\varepsilon)\cdot v(x)\,\mathrm{d}x \\
&\qquad+ \lim_{\varepsilon\to 0}\int_{\partial\Omega} (g(x,\overline{u}^\varepsilon(x,t),t) - g(x,\overline{u}^\varepsilon(x,t),\overline{(t)}^\varepsilon))\cdot v(x)\,\mathrm{d}x \\
&\qquad+ \lim_{\varepsilon\to 0}\int_{\partial\Omega} (g(x,u(x,t),t) - g(x,\overline{u}^\varepsilon(x,t),t))\cdot v(x)\,\mathrm{d}x
\end{aligned}
$$

and from the assumption J) (thanks to the convergence $\underline{\overline{u}}^\varepsilon(t) \to u(t)$ in X) by which the last two limits vanish again. Since for any $t \in I$

$$
\alpha^\varepsilon(x,t) := a(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),\nabla\overline{u}^\varepsilon(x,t))
$$

generates a bounded sequence in $L^q(\Omega)^{n\times 3}$ and consequently by Lemma 1 $\underline{\alpha}^\varepsilon(t) \rightharpoonup\!\!\rightharpoonup \alpha^0(t)$ for some $\alpha^0(t) \in L^p(\Omega, C_\#(Y)^{n\times 3})$, only the last (but rather extensive) step must be done to complete the proof: to verify that

$$\alpha^0(x,y,t) = a(x,y,u(x),t), \nabla u(x,t) + \nabla_Y \tilde{u}(x,y,t)) \tag{7}$$

for almost every $x \in \Omega$ and $y \in Y$ (independently of $t \in I$).

Let us consider

$$\begin{aligned} u^0_\nabla(x,y,t) &:= \nabla u(x,t) + \nabla_Y \varphi^k(x,y,t) + \delta\varphi(x,y,t),\\ u^\varepsilon_\nabla(x,t) &:= \nabla u(x,t) + \nabla_Y \varphi^k(x,x/\varepsilon,t) + \delta\varphi(x,x/\varepsilon,t) \end{aligned}$$

where $\delta \in \mathbb{R}_+$, $\varphi(t) \in C_0^\infty(\Omega, C_\#^\infty(y)^{n\times 3})$ and $\varphi^k(t) \in C_0^\infty(\Omega, C_\#^\infty(Y)^n)$ (for a fixed k now—this will be specified later). Moreover, let us introduce the brief notation

$$a_\nabla(x,y,w(x,t)) := a(x,y,\overline{u}^\varepsilon(x,t), w(x,t), u^0_\nabla(x,y,t))$$

with $w = \overline{u}^\varepsilon$ or $w = u$. From the assumption D) we have

$$\begin{aligned} \int_\Omega &(a(x,x/\varepsilon,\overline{u}^\varepsilon(x,t), \nabla\overline{u}^\varepsilon(x,t)) - a(x,x/\varepsilon,\overline{u}^\varepsilon(x,t), u^\varepsilon_\nabla(x,t)))\\ &\cdot (\nabla\overline{u}^\varepsilon(x,t) - u^\varepsilon_\nabla(x,t))\,\mathrm{d}x \geqslant 0 \end{aligned}$$

which with help of (1) (with $v(x) = \overline{u}^\varepsilon(x,t)$) assumes the form

$$\begin{aligned} &\int_\Omega f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),t)\cdot\overline{u}^\varepsilon(x,t)\,\mathrm{d}x - \int_{\partial\Omega} g(x,\overline{u}^\varepsilon(x,t),t)\cdot\overline{u}^\varepsilon(x,t)\,\mathrm{d}\sigma(x)\\ &- \int_\Omega b(x,x/\varepsilon)\dot{u}^\varepsilon(x,t)\cdot\overline{u}^\varepsilon(x,t)\,\mathrm{d}x - \int_\Omega a_\nabla(x,x/\varepsilon,\overline{u}^\varepsilon(x,t))\cdot\nabla\overline{u}^\varepsilon(x,t)\,\mathrm{d}x\\ &+ \int_\Omega a_\nabla(x,x/\varepsilon,\overline{u}^\varepsilon(x,t))\cdot u^\varepsilon_\nabla(x,t)\,\mathrm{d}x - \int_\Omega \alpha^\varepsilon(x,t)\cdot u^\varepsilon_\nabla(x,t)\,\mathrm{d}x\,\mathrm{d}\tau \geqslant 0. \end{aligned}$$

But we know that $\underline{\overline{u}}^\varepsilon(t) \to u(t)$ both in H and X again; thus the assumption A) applied to f and g is sufficient to guarantee

$$\begin{aligned} \int_\Omega f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t),t)\cdot\overline{u}^\varepsilon(x,t)\,\mathrm{d}x &= \int_\Omega\int_Y f(x,y,u(x,t),t)\cdot u(x,t)\,\mathrm{d}y\,\mathrm{d}x,\\ \int_{\partial\Omega} g(x,\overline{u}^\varepsilon(x,t),t)\cdot\overline{u}^\varepsilon(x,t)\,\mathrm{d}\sigma(x) &= \int_{\partial\Omega} g(x,u(x,t),t)\cdot u(x,t)\,\mathrm{d}\sigma(x). \end{aligned}$$

The assertion e) of Lemma 5 with respect to Lemma 2 (the assumption H) has to be applied again) gives also

$$\int_\Omega b(x,x/\varepsilon)\dot{u}^\varepsilon(x,t)\cdot\overline{u}^\varepsilon(x,t)\,\mathrm{d}x = \int_\Omega \tilde{b}(x)\dot{u}(x,t)\cdot u(x,t)\,\mathrm{d}y\,\mathrm{d}x.$$

In this way we obtain

$$\int_\Omega\int_Y f(x,y,u(x,t),t)\cdot u(x,t)\,\mathrm{d}y\,\mathrm{d}x - \int_{\partial\Omega} g(x,u(x,t),t)\cdot u(x,t)\,\mathrm{d}\sigma(x)$$
$$-\int_\Omega \tilde b(x)\dot u(x,t)\cdot u(x,t)\,\mathrm{d}x - \int_\Omega\int_Y a_\triangledown(x,y,u(x,t))\cdot(\nabla u(x,t)+\nabla_Y u_\triangledown(x,y,t))\,\mathrm{d}y\,\mathrm{d}x$$
$$+\int_\Omega\int_Y (a_\triangledown(x,y,u(x,t))-\alpha^0(x,y,t))\cdot u^0_\triangledown(x,y,t)\,\mathrm{d}y\,\mathrm{d}x \geqslant 0$$

and in another order (using δ, φ and φ^k)

$$\begin{aligned}\int_\Omega\int_Y f(x,y,u(x,t))\cdot u(x,t)\,\mathrm{d}y\,\mathrm{d}x &- \int_{\partial\Omega} g(x,u(x,t),t)\cdot u(x,t)\,\mathrm{d}\sigma(x)\\
&-\int_\Omega \tilde b(x)\dot u(x,t)\cdot u(x,t)\,\mathrm{d}x - \int_\Omega\int_Y \alpha_0(x,y,t)\cdot\nabla u(x,t)\,\mathrm{d}y\,\mathrm{d}x\\
&-\int_\Omega\int_Y \alpha_0(x,y,t)\cdot\nabla\varphi^k(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\\
&+\int_\Omega\int_Y a_\triangledown(x,y,u(x,t))\cdot\nabla_Y(\varphi^k(x,y,t)-\tilde u(x,y,t))\,\mathrm{d}y\,\mathrm{d}x\\
&+\delta\int_\Omega\int_Y (a_\triangledown(x,y,u(x,t))-\alpha_0(x,y,t))\cdot\varphi(x,y,t)\,\mathrm{d}y\,\mathrm{d}x \geqslant 0.\end{aligned}$$

However, repeating the above mentioned convergence arguments we have by (1)

$$\begin{aligned}\int_\Omega\int_Y f(x,y,u(x,t))\cdot u(x,t)\,\mathrm{d}y\,\mathrm{d}x &- \int_{\partial\Omega} g(x,u(x,t),t)\cdot u(x,t)\,\mathrm{d}\sigma(x)\\
&-\int_\Omega \tilde b(x)\dot u(x,t)\cdot u(x,t)\,\mathrm{d}x - \int_\Omega\int_Y \alpha^0(x,y,t)\cdot\nabla u(x,t)\,\mathrm{d}y\,\mathrm{d}x\\
=\ &\lim_{\varepsilon\to0}\int_\Omega f(x,x/\varepsilon,\overline{u}^\varepsilon(x,t))\cdot u(x,t)\,\mathrm{d}x - \lim_{\varepsilon\to0}\int_{\partial\Omega} g(x,\overline{u}^\varepsilon(x,t))\cdot u(x,t)\,\mathrm{d}\sigma(x)\\
&-\lim_{\varepsilon\to0}\int_\Omega b(x,x/\varepsilon)\dot u^\varepsilon(x,t)\cdot u(x,t)\,\mathrm{d}x - \lim_{\varepsilon\to0}\int_\Omega \alpha^\varepsilon(x,t)\cdot\nabla u(x,t)\,\mathrm{d}x = 0\end{aligned}$$

and the first four integrals in the previous inequality vanish. Also the fifth can be removed: we have

$$\begin{aligned}\int_\Omega\int_Y \alpha^0(x,y,t)\cdot\nabla_Y\varphi^k(x,y,t)\,\mathrm{d}y\,\mathrm{d}x &= \lim_{\varepsilon\to0}\int_\Omega \alpha^\varepsilon(x,t)\cdot\nabla_Y\varphi^k(x,x/\varepsilon,t)\,\mathrm{d}x\\
&= \lim_{\varepsilon\to0}\varepsilon\int_\Omega \alpha^\varepsilon(x,t)\cdot(\nabla\varphi^k(x,x/\varepsilon,t)-\nabla_\Omega\varphi^k(x,x/\varepsilon,t)) = 0.\end{aligned}$$

In particular, it is always possible to choose $\hat\varphi^k(t)\to\tilde u(t)$ in $L^p(\Omega\times Y)^n$; the limit passage $k\to\infty$ then removes the sixth integral. Finally, divided by δ, the result reduces to

$$\lim_{k\to\infty}\int_\Omega\int_Y (a(x,y,u(x,t),\nabla u(x,t)+\nabla_Y\varphi^k(x,y,t)+\delta\varphi(x,y,t))-\alpha^0(x,y,t))\\ \cdot\varphi(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\geqslant 0;$$

this can be rewritten (thanks to the assumption A) applied to a) as

$$\int_\Omega\int_Y \eta(x,y,t)\cdot\varphi(x,y,t)\,\mathrm{d}y\,\mathrm{d}x\geqslant 0 \tag{8}$$

with $\eta(x,y,t):=a(x,y,u(x,t)+\nabla_Y\tilde{u}(x,y,t))-\alpha^0(x,y,t)$; to derive this result, we have put $\delta=1/k$. In $L^p(\Omega\times Y)^{n\times 3}$ let us study a sequence $\hat{\eta}^k(t)$ of elements from $C_0^\infty(\Omega,C_\#^\infty(Y)^n)$ with the strong limit η. If the norm of each element of such sequence in $L^p(\Omega\times Y)^{n\times 3}$ is zero then the norm of η must be zero, too, and (7) is satisfied for almost every $x\in\Omega$ and $y\in Y$. But the well-known Minty trick (cf. [10, p. 261]) $\varphi(x,y,t)=\varrho|\eta^k(x,y,t)|^{q-1}\operatorname{sgn}\eta^k(x,y,t)$ with $\varrho\in\{-1,1\}$, applied to (8), gives

$$\varrho\int_\Omega\int_Y|\eta^k(x,y,t)|^q\,\mathrm{d}y\,\mathrm{d}x\geqslant 0;$$

this forces $\eta^k(t)$ to be a zero point of $L^p(\Omega\times Y)^n$. □

4. Conclusions and generalizations

In Theorem 1 (handling the time-discretized formulation) and Theorem 2 (studying its limit behavior) we have demonstrated how the coupling of the method of Rothe and of the two-scale convergence technique can be useful in the analysis of a certain class of strongly nonlinear problems of evolution, occurring in continuum mechanics and engineering applications in many situations when the proper information about the periodic material structure is available and should not be lost. Most assumed properties, identified by letters from A) to J), can be verified in practical cases without great difficulties (both those on the geometrical configurations and those on the choice of prescribed functions, i.e. "loads" and "material characteristics"). Nevertheless, some of them could be weakened; the main reason for ignoring such possibilities was the effort, proclaimed in the introduction, to keep this text readable and user-friendly for more people than for a rather exclusive society (as classified by numerous physicists and engineers) of specialists in mathematical homogenization and similar techniques. Let us also remark that no Gronwall arguments (cf. [13, p. 29]) or other special tricks from the theory of difference equations were needed in proofs explicitly, but, regardless of this fact, the asymptotic behavior of solution for $T\to\infty$ is not evident and would require a proper additional analysis. Another disadvantage is that only special parabolic problems were taken into considerations; therefore the presented theory covers e.g. the conduction of heat of solids as a very special case with $n = 1$ (with exactly one unknown field of temperature—see [5, p. 172]), but not directly the classical wave or telegraph equations and (in more practical cases) the problems of dynamical viscoelasticity or elastoplasticity where the second time derivatives cannot be removed. Thus, there are many occasions and themes for the research in near future.

Let us close this paper with a short reference to what can happen if some of the assumed properties is violated. The first candidate for this may be the very strict assumption H); more generally, we would prefer to have something like $b(x, x/\varepsilon, \overline{u}^{\varepsilon}(x,t))$ instead of $b(x, x/\varepsilon)\,\overline{u}^{\varepsilon}(x,t)$ in (1). But such formulation (with some physically reasonable requirements on b) enlarges both the extent of proofs and the amount of theoretical preliminaries rapidly even in all "classical" considerations with no proper microstructural arguments (as a basis for their generalization like Theorem 1): probably the best example is the problem of the simultaneous heat and moisture transport in porous media (for the separate moisture transport see the mathematical analysis [12], the complete physical background is discussed in [8]), involving (apart from the seemingly effective numerical solvers) many open existence and convergence questions yet.

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