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ALTERNATIVE APPROACHES TO THE TWO-SCALE CONVERGENCE*

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Abstract. Two-scale convergence is a special weak convergence used in homogenization theory. Besides the original definition by Nguetseng and Allaire two alternative definitions are introduced and compared. They enable us to weaken requirements on the admissibility of test functions $\psi(x, y)$. Properties and examples are added.

Keywords: two-scale convergence, weak convergence, homogenization

MSC 2000: 35B27

1. Introduction

The aim of the paper is to survey two-scale convergence and introduce some alternative approaches to it.

Two-scale convergence is a special kind of the weak convergence. It was developed for the homogenization theory in order to simplify the proofs. It overcomes difficulties resulting from properties of weakly converging sequences of periodic functions. In such sequences the weak limit does not keep the "information on oscillations" of the original functions. In some cases, the two-scale limit is able to conserve this information and thus, it makes limit procedures possible. It stands between the usual strong and weak convergences.

The concept was first introduced by Nguetseng [18] and then developed by Allaire [1] in early 90's. In the definition of two-scale convergence, the special so-called admissible test function is used. The widest set of these functions is not clear and thus it motivates alternative approaches. One of them is based on a two-scale

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transform, which changes a sequence of one variable functions into a sequence of two-variable functions. This transform was used by authors in homogenization of some problems in porous media [5] and it is suitable for an alternative definition of the two-scale convergence, see also [12], [16], [10]. Another one is based on the so-called inverse two-scale transform which defines a sequence of one-variable functions from the test function $\psi(x,y)$. This approach seems to be new.

The paper is organized as follows. Section 2 contains a survey of the definitions used, while Section 3 compares them. In the two-scale transform approach, we mention the case of boundary cubes exceeding Ω, which is usually neglected. Section 4 is devoted to the compactness property. Section 5 contains examples and some properties of two-scale convergence. Section 6 refers to some generalizations.

2. Definitions

Throughout the paper let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $Y = (0,1)^N$ the unit cube (a basic period). We will use periodic functions: a function v is said to be Y-periodic if $v(y+k) = v(y)$, $y \in \mathbb{R}^N$, $k \in \mathbb{Z}^N$. If the function v has more variables, we say that it is Y-periodic in y. Lebesgue spaces L^p are used with $p \in (1,\infty)$. The dual exponent is denoted by $q = p/(p-1)$. The symbol $\#$ is used for Y-periodic functions, e.g. $C_\#(Y)$ is the space of Y-periodic functions v such that $v|_{Y+y_0} \in C(Y+y_0)$, $\forall y_0 \in \mathbb{R}^N$. Further, we will consider a sequence of positive parameters $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ for $n \to \infty$. As usual, the subscript n will be omitted.

Let us begin with the classical definition by Nguetseng and Allaire which was introduced for the case of L^2:

Definition 2.1. We say that a sequence of functions $\{u_\varepsilon(x)\} \subset L^p(\Omega)$ two-scale converges to a limit $u_0(x,y) \in L^p(\Omega \times Y)$, if the relation

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{2.1}$$

holds for each $\psi(x,y) \in C_0^\infty[\Omega; C_\#^\infty(Y)]$ (the space of compactly supported infinitely differentiable functions with Y-periodic values in $C^\infty(\mathbb{R}^N)$). If, in addition,

$$\lim_{\varepsilon\to 0}\Big\|u_\varepsilon(x) - u_0\Big(x, \frac{x}{\varepsilon}\Big)\Big\|_{L^p(\Omega)} = 0,$$

we say that $\{u_\varepsilon\}$ two-scale converges strongly to $u_0(x,y)$.

Remark 2.2. In the vocabulary of homogenization the strong two-scale convergence is also called a corrector type result. Assumptions making two-scale convergence strong are discussed in Section 3. Although $\{u_\varepsilon\}$ is a sequence of N variables $x_1, \dots, x_N$, the limit is a function of $2N$ variables $x_1, \dots, x_N, y_1, \dots, y_N$. It enables us to describe the periodic behavior of u_ε better. Usually it is not emphasized, but the definition is connected with a fixed sequence of periods ε_n, i.e. for an extracted subsequence $\{u_{\varepsilon'}\}$ we should consider the same extracted subsequence $\{\varepsilon'\}$ of periods in the test function, see examples in Section 5.

Let us introduce an alternative definition. In [5], the authors dealt with a homogenization technique which was used for the description of a porous media. It is suitable for an alternative approach to the two-scale convergence. The idea is based on the so-called two-scale transform which changes a sequence of one-variable functions $\{u_\varepsilon(x)\}$ into a sequence of two-variable functions $\{\hat{u}_\varepsilon(x, y)\}$.

For each ε let us consider small non-overlapping cubes $C_\varepsilon^k = \varepsilon Y + \varepsilon k$, $k \in \mathbb{Z}^N$. Here, for sake of simplicity, we restrict ourselves to the domains Ω that can be decomposed into these cubes, i.e. $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$. The sequence $\{\hat{u}_\varepsilon(x, y)\}$ is defined by the relation

$$\hat{u}_\varepsilon(x, y) = u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) \quad x \in \Omega, \ \ y \in Y, \tag{2.2}$$

where $[x] = k$ is the vector of the greatest integers k_i less than or equal to x_i. On each cube $C_\varepsilon^k \times Y$ the function $\hat{u}_\varepsilon$ is constant in the variable x and as a function of y it is the function $u_\varepsilon(x)$ on C_ε^k transformed onto the unit cube Y. The alternative definition reads:

Definition 2.3. We say that a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converges to a function $u_0(x, y) \in L^p(\Omega \times Y)$, if

$$\lim_{\varepsilon \to 0} \int_\Omega \int_Y [\hat{u}_\varepsilon(x, y) - u_0(x, y)]\psi(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{2.3}$$

for every test function $\psi \in C_0^\infty[\Omega; C_\#^\infty(Y)]$. Moreover, if $\hat{u}_\varepsilon \to u_0$ in $L^p(\Omega \times Y)$ strongly, we say that $\{u_\varepsilon\}$ two-scale converges strongly.

Let us refer here to the recent paper [12], where the above mentioned approach is also used and is called periodic unfolding. Authors apply this approach to the case of periodic multi-scale problems.

The other alternative approach is based on the so-called inverse two-scale transform which makes a sequence $\{\overline{\psi}_\varepsilon(x)\}$ from a two-variable function $\psi(x, y)$. The functions $\overline{\psi}_\varepsilon$ are constructed as follows. Similarly, as in the previous transform, we consider non-overlapping cubes C_ε^k that cover the domain Ω (here the domain Ω need not be the union of the cubes C_ε^k, i.e. $\overline{\Omega} \subseteq \bigcup\limits_k \overline{C}_\varepsilon^k$). Outside the domain Ω we put $\psi(x,y) = 0$. Let us average the extended function $\psi(x,y)$ with respect to the first variable:

$$\overline{\psi}_\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{C_\varepsilon^k} \psi\Big(\xi, \frac{x}{\varepsilon}\Big)\,\mathrm{d}\xi, \quad x \in C_\varepsilon^k. \tag{2.4}$$

Definition 2.4. Let $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)$. We say that $\{u_\varepsilon\}$ two-scale converges to a function $u_0 \in L^p(\Omega \times Y)$, if

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\overline{\psi}_\varepsilon(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for each test function $\psi(x,y) \in C_0^\infty[\Omega; C_\#^\infty(Y)]$. Moreover, if

$$\lim_{\varepsilon\to 0} \|u_\varepsilon(x) - \overline{u}_0^\varepsilon(x)\|_{L^p(\Omega)} = 0,$$

we say that $\{u_\varepsilon\}$ two-scale converges strongly.

R e m a r k 2.5. The two-scale transform used in Definition 2.3 enables us to define two-scale and strong two-scale convergence more naturally with help of weak convergence with smooth test functions. On the other hand, Definition 2.4 is similar to the classical definition by Nguetseng and Allaire, but it differs from (2.1) by the choice of the test function on the left-hand side.

Why do we look for alternative approaches to the original one? In the definition we want to test the convergence with functions from a space as small as possible, thus, smooth functions are convenient. On the other hand, in applications the largest class is desirable. In Definition 2.1 we can not take the test function $\psi(x,y)$ from the whole space $L^q(\Omega \times Y)$, since it is not defined correctly on the zero-measure set $\{[x,y]\colon\ y = x/\varepsilon\}$ and thus the measurability of the composed function $\psi(x, x/\varepsilon)$ is not guaranteed. Moreover, the test functions must satisfy some convergence, which is not always obvious. The class of suitable test functions is discussed at the top of the following section. Such functions are called admissible. Further, we will see that the two mentioned alternative definitions avoid described problems.

The main goal of this section is to prove equivalence of the definitions. It means that each of the definitions yields the same two-scale limits.

First of all we proceed with the notion of admissible test function mentioned above. Since the widest set of suitable test functions ψ is not clear (we do not know the minimal conditions making these functions regular enough), the following characterization is useful:

Definition 3.1. A Y-periodic (in y) test function $\psi(x,y) \in L^q(\Omega \times Y)$ is said to be admissible, if

$$\lim_{\varepsilon\to 0}\left\|\psi\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^q(\Omega)} = \|\psi(x,y)\|_{L^q(\Omega\times Y)} \tag{3.1}$$

and for a separable subspace $X \subseteq L^q(\Omega \times Y)$

$$\left\|\psi\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^q(\Omega)} \leqslant \|\psi(x,y)\|_X. \tag{3.2}$$

Let us emphasize that there exist Y-periodic functions in y which do not satisfy the convergence (3.1) even if the measurability of the composed function $\psi(x,x/\varepsilon)$ is guaranteed, see [1]. Allaire also showed (for $p=2$) that spaces, e.g. $C[\overline{\Omega};C_\#(Y)]$, $L^2[\Omega;C_\#(Y)]$ or $L^2_\#[Y;C(\overline{\Omega})]$, are made up from the admissible functions. All these spaces are separable and their elements are functions continuous at least in one variable. Such functions are Carathéodory, which is the sufficient condition for measurability of the composed function $\psi(x,x/\varepsilon)$, and moreover, they satisfy (3.1) and (3.2). These two properties are needful in the proof of the compactness property, see Section 4. If ψ belongs to these spaces, we also have

$$\lim_{\varepsilon\to 0}\int_\Omega \psi\left(x,\frac{x}{\varepsilon}\right)\,\mathrm{d}x = \int_\Omega\int_Y \psi(x,y)\,\mathrm{d}x\,\mathrm{d}y; \tag{3.3}$$

for details, see [1]. This relation is natural: taking the stationary sequence $\{u_\varepsilon = 1\}$, (2.1) and (3.3) yield the two-scale limit $u_0(x,y)=1$.

In the alternative Definition 2.3 the situation is simplified. Due to the density of smooth functions in L^q, the convergence (2.3) implies also the weak convergence in $L^p(\Omega\times Y)$, i.e. the space of test functions can be enlarged to the whole $L^q(\Omega\times Y)$.

The special choice of the test function in Definition 2.4 is also motivated by the effort to enlarge the class of test functions. This choice is convenient due to the following lemma:

Lemma 3.2. *Let $\psi \in L^q(\Omega \times Y)$ be a Y-periodic function and let $\overline{\psi}_\varepsilon$ be defined by (2.4). Then we have*

$$\|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} \leqslant \|\psi(x,y)\|_{L^q(\Omega\times Y)}, \tag{3.4}$$

$$\lim_{\varepsilon\to 0} \|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} = \|\psi(x,y)\|_{L^q(\Omega\times Y)}. \tag{3.5}$$

Proof. For ε fixed, let us consider the minimal number of the cubes C_ε^k that cover the domain Ω. For these cubes we denote $\tilde{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$. The transform in the Lebesgue integral and the Fubini theorem yield

$$\begin{aligned}\int_\Omega \int_Y |\psi(x,y)|^q \, \mathrm{d}x \, \mathrm{d}y &= \int_{\tilde{\Omega}} \int_Y |\psi(x,y)|^q \, \mathrm{d}x \, \mathrm{d}y = \sum_k \int_Y \left[\int_{C_\varepsilon^k} |\psi(\xi,y)|^q \, \mathrm{d}\xi \right] \mathrm{d}y \\ &= \sum_k \int_{C_\varepsilon^k} \left[\frac{1}{\varepsilon^N} \int_{C_\varepsilon^k} \left| \psi\Big(\xi, \frac{x}{\varepsilon}\Big) \right|^q \mathrm{d}\xi \right] \mathrm{d}x = \sum_k \int_{C_\varepsilon^k} |\overline{\psi}_\varepsilon(x)|^q \, \mathrm{d}x \\ &= \int_{\tilde{\Omega}} |\overline{\psi}_\varepsilon(x)|^q \, \mathrm{d}x \geqslant \int_\Omega |\overline{\psi}_\varepsilon(x)|^q \, \mathrm{d}x.\end{aligned}$$

Thus, (3.4) is verified. Moreover, we have

$$\int_{\tilde{\Omega}\setminus\Omega} |\overline{\psi}_\varepsilon(x)|^q \, \mathrm{d}x \leqslant \sum_k \int_{C_\varepsilon^{k,b}} |\overline{\psi}_\varepsilon(x)|^q \, \mathrm{d}x = \sum_k \int_{C_\varepsilon^{k,b}} \left[\int_Y \psi(x,y) \, \mathrm{d}y \right] \mathrm{d}x,$$

where $C_\varepsilon^{k,b}$ are cubes in the boundary layer. Since their number is proportional to $M\varepsilon^{1-N}$ (M is the constant following from the surface integral), we have

$$\operatorname*{meas}_N \Big(\bigcup_k C_\varepsilon^{k,b} \Big) \leqslant M\varepsilon^{1-N}\varepsilon^N = M\varepsilon.$$

The absolute continuity of the Lebesgue integral yields

$$\sum_k \int_{C_\varepsilon^{k,b}} \left[\int_Y \psi(x,y) \, \mathrm{d}y \right] \mathrm{d}x \to 0 \quad \text{for } \varepsilon \to 0,$$

which implies (3.5). □

Remark 3.3. If the union of the cubes gives the whole domain Ω, i.e. $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$, then we have

$$\|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} = \|\psi(x,y)\|_{L^q(\Omega\times Y)}.$$

Lemma 3.2 says that every test function $\psi \in L^q(\Omega \times Y)$ can be called admissible (compare with properties in the Definition 3.1). Thus, the space of the test functions can be enlarged to the whole $L^q(\Omega \times Y)$.

The following lemma specifies functions $\hat{u}_\varepsilon$ used in the two-scale transform approach (have on mind that Ω is considered to be a union of the cubes C_ε^k).

Lemma 3.4. *Let $u_\varepsilon(x) \in L^p(\Omega)$ and let $\hat{u}_\varepsilon$ be defined by (2.2). Then $\hat{u}_\varepsilon(x,y) \in L^p(\Omega \times Y)$ and*

$$\|u_\varepsilon\|_{L^p(\Omega)} = \|\hat{u}_\varepsilon(x,y)\|_{L^p(\Omega\times Y)}.$$

P r o o f. Similarly to the previous proof, we compute

$$\begin{aligned}\int_\Omega |u_\varepsilon(x)|^p \,\mathrm{d}x &= \sum_k \int_{C_\varepsilon^k} |u_\varepsilon(x)|^p \,\mathrm{d}x = \sum_k \varepsilon^N \int_Y |\hat{u}_\varepsilon(x,y)|^p \,\mathrm{d}y \\ &= \sum_k \int_{C_\varepsilon^k} \mathrm{d}x \int_Y |\hat{u}_\varepsilon(x,y)|^p \,\mathrm{d}y = \int_\Omega \int_Y |\hat{u}_\varepsilon(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

□

R e m a r k 3.5. The situation is more complicated in the case of cubes exceeding the domain Ω. The two-scale transform defined by (2.2) works well on the cubes C_ε^k, i.e. a function u_ε defined on C_ε^k is transformed into a function $\hat{u}_\varepsilon$ defined on $C_\varepsilon^k \times Y$. Near the boundary, where $C_\varepsilon^k \cap \Omega \neq C_\varepsilon^k$, it can cause difficulties. As in the proof of Lemma 3.2, let us consider the minimal number of the cubes C_ε^k covering Ω. The union $S_\varepsilon = \big(\bigcup_k \overline{C}_\varepsilon^k\big) \setminus \Omega$ is of a positive measure. In case of a "good" boundary we have $\operatorname{meas}_N S_\varepsilon \to 0$ (as $\varepsilon \to 0$), but $\|\hat{u}_\varepsilon\|$ cannot be estimated by $\|u_\varepsilon\|$ as the following example shows:

Let us take $\Omega = (0,a)$, $a \in \mathbb{R}$, and a sequence of periods ε such that the interval $(0,a)$ cannot be expressed as the union of the small intervals $I_\varepsilon^k = (\varepsilon k, \varepsilon(k+1))$, $k \in \mathbb{Z}$. We define a sequence $\{u_\varepsilon\} \subset L^1(\Omega)$ by

$$u_\varepsilon(x) = \begin{cases} 0, & x \in (0, a-\varepsilon^2), \\ \varepsilon^{-2}, & x \in (a-\varepsilon^2, a). \end{cases}$$

Thus, the intervals I_ε^k exceed the interval $(0,a)$ by $\varepsilon - \varepsilon^2$ (on this small part we put $u_\varepsilon = 0$). Obviously, the $L^1(\Omega)$ norm $\|u_\varepsilon(x)\|_{L^1(\Omega)} = 1$, while $\|\hat{u}_\varepsilon(x,y)\|_{L^1(\Omega\times Y)} = \varepsilon^2$, i.e. $\|u_\varepsilon\| \not\approx \|\hat{u}_\varepsilon\|$ (we have $\|u_\varepsilon\| \geqslant \|\hat{u}_\varepsilon\|$ only).

By the transform we want to conserve the norms of u_ε and $\hat{u}_\varepsilon$ even if the cubes exceed Ω. It is not difficult in 1D, since it is sufficient to re-scale with the actual length of the boundary segment $C_\varepsilon^k \cap \Omega$ instead of ε. In higher dimensions it is more difficult.

Theorem 3.6. *Let us assume $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$ and let $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge (in Nguetseng-Allaire sense) to a function $u_0(x,y)$. Then $\{u_\varepsilon\}$ two-scale converges to u_0 in the sense of Definition 2.3 and Definition 2.4, too.*

Proof. Let us assume $\{u_\varepsilon\}$ two-scale converges to u_0, but $\{\hat{u}_\varepsilon\}$ converges weakly to $\tilde{u}_0$. Then we have

$$I_\varepsilon \equiv \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], \frac{x}{\varepsilon}\Big)\,\mathrm{d}x + \delta_\varepsilon,$$

where

$$\delta_\varepsilon = \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x - \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], \frac{x}{\varepsilon}\Big)\,\mathrm{d}x.$$

The two-scale transform yields

$$\begin{aligned} I_\varepsilon &= \varepsilon^N \int_Y u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], \Big[\frac{x}{\varepsilon}\Big] + y\Big)\,\mathrm{d}y + \delta_\varepsilon \\ &= \int_{C_\varepsilon^k} \mathrm{d}x \int_Y u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], y\Big)\,\mathrm{d}y + \delta_\varepsilon \\ &= \int_{C_\varepsilon^k}\int_Y \hat{u}_\varepsilon(x,y)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], y\Big)\,\mathrm{d}x\,\mathrm{d}y + \delta_\varepsilon = \hat{I}_\varepsilon + \delta_\varepsilon + \hat{\delta}_\varepsilon, \end{aligned}$$

where

$$\hat{I}_\varepsilon \equiv \int_{C_\varepsilon^k}\int_Y \hat{u}_\varepsilon(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\hat{\delta}_\varepsilon = \int_{C_\varepsilon^k}\int_Y \hat{u}_\varepsilon(x,y)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], y\Big)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega\int_Y \hat{u}_\varepsilon(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since $u_\varepsilon \overset{2-s}{\rightharpoonup} u_0$ and $\hat{u}_\varepsilon \rightharpoonup \tilde{u}_0$, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \delta_\varepsilon &= \int_{C_\varepsilon^k}\int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{C_\varepsilon^k}\int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \\ \lim_{\varepsilon\to 0} \hat{\delta}_\varepsilon &= \int_{C_\varepsilon^k}\int_Y \tilde{u}_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{C_\varepsilon^k}\int_Y \tilde{u}_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0}(I_\varepsilon - \hat{I}_\varepsilon) = \int_{C_\varepsilon^k}\int_Y (u_0(x,y) - \tilde{u}_0(x,y))\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

which implies $u_0 = \tilde{u}_0$ a.e. in $L^p(\Omega \times Y)$. A similar argument can be used in the case of the inverse two-scale transform definition. □

R e m a r k 3.7. We can establish even a stronger property. Under the assumption in Theorem 3.6, we have

$$\int_\Omega u_\varepsilon(x)\psi\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega\int_Y \hat{u}_\varepsilon(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Let us deal with the strong two-scale convergence. The weak convergence $u_\varepsilon \rightharpoonup u$ equipped with the additional condition $\|u_\varepsilon\| \to \|u\|$ is also strong, i.e. $\|u_\varepsilon - u\| \to 0$. The following theorem introduces similar additional assumptions that strengthen two-scale convergence into the strong one (in the case of Nguetseng-Allaire definition).

Theorem 3.8. *A sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converges strongly to a limit u_0, if and only if $\{u_\varepsilon\}$ two-scale converges to u_0 and the relations*

$$\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0(x,y)\|_{L^p(\Omega\times Y)},$$
$$\Big\|u_0\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^p(\Omega)} \to \|u_0(x,y)\|_{L^p(\Omega\times Y)}$$

hold.

P r o o f. More details and the proof can be found in [15], [1]. □

R e m a r k 3.9. In the two-scale transform approach, the weak convergence of the sequence $\{\hat{u}_\varepsilon\}$ plays the role of the two-scale convergence. Hence, $\{u_\varepsilon\}$ two-scale converges strongly, if and only if $\{\hat{u}_\varepsilon\}$ converges weakly to u_0 and $\|\hat{u}_\varepsilon\|_{L^p(\Omega\times Y)} \to \|u_0\|_{L^p(\Omega\times Y)}$.

A similar result holds for the inverse two-scale transform approach. As in Theorem 3.8, the additional assumptions $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$, $\|\bar{u}_0^\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$ strengthen two-scale convergence into the strong one. But due to Lemma 3.2, each function $u_0 \in L^p(\Omega\times Y)$ satisfies the convergence $\|\bar{u}_0^\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$. Thus, we have

Lemma 3.10. *A sequence $\{u_\varepsilon\}$ two-scale converges strongly (in the sense of Definition 2.4), if and only if it two-scale converges to u_0 and $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$.*

Theorem 3.11. *Let $\overline{\Omega} = \bigcup\limits_k \overline{C}_\varepsilon^k$ and let a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge strongly to a limit u_0 according to Nguetseng-Allaire's definition. Then $\{u_\varepsilon\}$ also two-scale converges strongly to u_0 in the sense of Definition 2.1 and Definition 2.3.*

P r o o f. The result is a direct consequence of Theorem 3.6, Theorem 3.8, Lemma 3.4, Lemma 3.2 and Remark 3.9. □

R e m a r k 3.12. Theorems 3.6, 3.11 together show the equivalence of the definitions used, i.e., all of them yield the same limits.

4. Compactness

The two-scale convergence can be used in applications due to the following compactness property.

Theorem 4.1. *A bounded sequence $\{u_\varepsilon\}$ in $L^p(\Omega)$ is compact with respect to the two-scale convergence, i.e. there exists an extracted subsequence $\{u_{\varepsilon'}\}$ two-scale converging to a function $u_0 \in L^p(\Omega \times Y)$.*

Allaire's proof in [1] is carried out for the test functions from $L^2[\Omega; C_\#(Y)]$. It is based on the properties of the dual space to $L^2[\Omega; C_\#(Y)]$. This space is not so transparent, since it is represented by $L^2[\Omega; M_\#(Y)]$, where $M_\#(Y)$ is the space of Y-periodic Radon measures.

In the alternative approach based on the two-scale transform, the situation is more simplified, since the two-scale compactness follows directly from the weak compactness of bounded sequences in $L^p(\Omega \times Y)$ (a closed ball is compact with respect to the weak convergence).

Let us prove a modification of the theorem for the case of two-scale convergence based on the inverse two-scale transform.

P r o o f. The boundedness, Hölder's inequality and Lemma 3.2 yield

$$\left| \int_\Omega u_\varepsilon(x) \overline{\psi}_\varepsilon(x) \, \mathrm{d}x \right| \leqslant C \|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} \leqslant C \|\psi(x,y)\|_{L^q(\Omega\times Y)}. \tag{4.1}$$

In view of (4.1), u_ε represents a bounded linear functional U_ε on $L^q(\Omega \times Y)$ defined by

$$\langle U_\varepsilon, \psi \rangle = \int_\Omega u_\varepsilon(x) \overline{\psi}_\varepsilon(x) \, \mathrm{d}x. \tag{4.2}$$

Since $L^q(\Omega \times Y)$ is separable, there exists U_0 such that an extracted subsequence $U_{\varepsilon'}$ converges $*$-weakly to U_0, i.e.

$$U_{\varepsilon'} \overset{*}{\rightharpoonup} U_0 \quad \text{ in } \; [L^q(\Omega \times Y)]^*.$$

This relation and Lemma 3.2 yield

$$|\langle U_0, \psi \rangle| = \lim_{\varepsilon\to 0} \int_\Omega u_{\varepsilon'}(x) \overline{\psi}_{\varepsilon'}(x) \, \mathrm{d}x \leqslant C \lim_{\varepsilon\to 0} \|\overline{\psi}_{\varepsilon'}(x)\|_{L^q(\Omega)} = C\|\psi(x,y)\|_{L^q(\Omega\times Y)}.$$

Thus, U_0 is also a bounded linear functional. By the Riesz representation theorem there exists a function $u_0 \in L^p(\Omega \times Y)$ such that

$$\langle U_0, \psi \rangle = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\lim_{\varepsilon' \to 0} \int_\Omega u_{\varepsilon'}(x)\overline{\psi}_{\varepsilon'}(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which is the desired result. □

5. Examples and properties

Let us introduce a few typical examples of two-scale convergent sequences.

Example 5.1. (i) Let $a(y)$ be a Y-periodic bounded function with zero mean value $\int_Y a(y)\,\mathrm{d}y = 0$ and let $b_1(x), b_2(x) \in L^p(\Omega)$. Then the sequence $\{u_\varepsilon\}$ defined by $u_\varepsilon(x) = b_1(x)a(x/\varepsilon) + b_2(x)$ converges weakly to $b_2(x)$ and it two-scale converges (strongly) to $b_1(x)a(y) + b_2(x)$. We see that the weak limit is the function b_2 only. It says nothing on the periodic behaviour of the functions u_ε. On the other hand, in the two scale limit the information on "oscillations" is kept. This loss of information in the weak limit causes some "unpleasant" properties mentioned in Introduction, e.g. taking two weakly converging sequences $u_\varepsilon \rightharpoonup u$, $v_\varepsilon \rightharpoonup v$ does not imply $u_\varepsilon v_\varepsilon \rightharpoonup uv$, etc. The example shows that the weak limit is the average of the two-scale limit with respect to y. This is a direct consequence of the definition, if we take a test function ψ depending on the variable x only.

(ii) Let us consider the same functions $a(y)$, $b_1(x)$, $b_2(x)$, but another sequence $\{v_\varepsilon\}$ defined by $v_\varepsilon(x) = b_1(x)a(x/\varepsilon^2) + b_2(x)$. Then the two-scale and weak limits coincide, which means that the two-scale limit is constant in the variable y. In this case, the information on oscillations is not kept. Similarly, taking a sequence given by $w_\varepsilon(x) = b_1(x)a(cx\varepsilon) + b_2$ with c irrational, the two-scale limit equals to b_2 only. It is the consequence of the fact that diminishing of the periods in sequences is not in the resonance with the periods in the test function.

The sequences from the previous two examples point to an interesting fact. An extracted subsequence of weakly converging sequence converges to the same limit. In the case of two-scale convergence we must consider convergence also with respect to subsequences of periods. Otherwise the limits may differ (e.g. the sequence $\{v_\varepsilon\}$ from the example above can be considered an extracted subsequence from $\{u_\varepsilon\}$).

Example 5.2. Let $\{u_\varepsilon\} \subset L^p(\Omega)$ be a sequence satisfying convergence $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0(x,y)\|_{L^p(\Omega\times Y)}$. Then u_ε need not two-scale converge to u_0. Let $u(y)$ be a Y-periodic function. Let us consider functions $u_0(x,y) = u(y)$ and $\tilde{u}_0 = u(y-1/2)$. Since $u(y)$ is periodic, we have $\|u_0\|_{L^p(\Omega\times Y)} = \|\tilde{u}_0\|_{L^p(\Omega\times Y)}$. Let $\{u_\varepsilon\}$ be the sequence defined by $u_\varepsilon(x) = \tilde{u}_0(x/\varepsilon)$. Then $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$, but $\{u_\varepsilon\}$ two-scale converges to $\tilde{u}_0$.

Theorem 5.3. *Let $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge to $u_0(x,y) \in L^p(\Omega\times Y)$ and let it converge weakly to $u(x)$. Then*

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega)} \geqslant \|u_0\|_{L^p(\Omega\times Y)} \geqslant \|u\|_{L^p(\Omega)}. \tag{5.1}$$

Proof. The first inequality (5.1) can be proved with help of Young's inequality and the definition of the two-scale convergence. The second inequality follows from Hölder's inequality and can be interpreted as follows: two-scale limit conserves more information on a periodic behaviour of $\{u_\varepsilon\}$ than the usual weak limit. □

Example 5.4. Let us consider the sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ from Example 5.1. Denoting the two-scale limit by u_0 and the weak limit by u, we have $\lim \|u_\varepsilon\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega\times Y)} > \|u\|_{L^p(\Omega)}$, $\lim \|v_\varepsilon\|_{L^p(\Omega)} > \|u_0\|_{L^p(\Omega\times Y)} = \|u\|_{L^p(\Omega)}$ and finally the sum $u_\varepsilon + v_\varepsilon$ satisfies sharp inequalities.

Theorem 5.3 also implies: every sequence $\{u_\varepsilon\}$ strongly convergent to a function u two-scale converges to $u_0(x,y) = u(x)$, too. The following theorem on a limit procedure is useful in applications:

Theorem 5.5. *Let $\{u_\varepsilon^1\} \subset L^{p_1}(\Omega)$, ..., $\{u_\varepsilon^m\} \subset L^{p_m}(\Omega)$ be sequences two-scale converging strongly to limits $u_0^1(x,y), \dots, u_0^m(x,y)$, $p_1, \dots, p_m, r \in \langle 1, \infty)$ and let $f(x,\xi_1,\dots,\xi_m)$ be a Carathéodory function satisfying a growth condition*

$$|f(x,\xi_1,\dots,\xi_m)| \leqslant g(x) + C\sum_{i=1}^m |\xi_i|^{p_i/r},$$

where $g \in L^r(\Omega)$ and C is a positive constant. Then

$$f(x,u_\varepsilon^1(x),\dots,u_\varepsilon^m(x)) \to \int_Y f(x,u_0^1(x,y),\dots,u_0^m(x,y))\,\mathrm{d}y \quad \textit{in } L^r(\Omega).$$

Proof. This relation can be obtained by use of the alternative approach. Since the assumptions of the theorem on Nemytskij operators are satisfied, we have $f(x,\hat{u}_\varepsilon^1,\dots,\hat{u}_\varepsilon^m) \to f(x,u_0^1,\dots,u_0^m)$ (this mapping is continuous). On the other hand, we know that the weak and here also the strong L^p limit is the average of the two-scale limit (with respect to y). □

Remark 5.6. The relation

$$u_\varepsilon^1 \dots u_\varepsilon^m \to \int_Y u_0^1(x,y)\dots u_0^m(x,y)\,\mathrm{d}y$$

is a special case of this theorem. This situation often occurs in proofs. The convergence changes into weak as one of $\{u_\varepsilon^i\}$ two-scale converges (not strongly) only.

6. Concluding remarks

We have surveyed some phenomena in the two-scale convergence. Two alternative definitions to the usual one were discussed. They make it possible weaken the requirements making test functions admissible. The definition based on the two-scale transform is more straightforward in the case of the two-scale compactness property or strong two-scale convergence, while the definition based on the inverse two-scale transform is closer to the original one, but it differs by the construction of the left-hand side test function.

In the text above we have considered the basic period to be the unit cube. It can be replaced by an arbitrary block $Y \subset \mathbb{R}^N$, but in the appropriate relations, the term $1/|Y|$ would appear.

The theory for time depending sequences used in evolution equations is completed in [15]. Considering problems in some porous media is popular in many papers, see e.g. [2], [10], [13]. For this analysis it is useful to define two-scale convergence more generally with respect to measures [7], [21], [4], [19]. The extension from the periodic case to the almost periodic one can be found in [9] and to the stochastic case in [8], [19]. Further, the two-scale convergence can be generalized to the so-called multiscale convergence, where multiple separated scales of oscillations are considered, see e.g. [3], [6], [11].

Two-scale convergence has been applied in many other papers, see e.g. [13], [10], [11], [16], [14]. Many other references can be found in the survey paper on two-scale convergence [17].

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ALTERNATIVE APPROACHES TO THE TWO-SCALE CONVERGENCE*

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Abstract. Two-scale convergence is a special weak convergence used in homogenization theory. Besides the original definition by Nguetseng and Allaire two alternative definitions are introduced and compared. They enable us to weaken requirements on the admissibility of test functions $\psi(x, y)$. Properties and examples are added.

Keywords: two-scale convergence, weak convergence, homogenization

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1. Introduction

The aim of the paper is to survey two-scale convergence and introduce some alternative approaches to it.

Two-scale convergence is a special kind of the weak convergence. It was developed for the homogenization theory in order to simplify the proofs. It overcomes difficulties resulting from properties of weakly converging sequences of periodic functions. In such sequences the weak limit does not keep the "information on oscillations" of the original functions. In some cases, the two-scale limit is able to conserve this information and thus, it makes limit procedures possible. It stands between the usual strong and weak convergences.

The concept was first introduced by Nguetseng [18] and then developed by Allaire [1] in early 90's. In the definition of two-scale convergence, the special so-called admissible test function is used. The widest set of these functions is not clear and thus it motivates alternative approaches. One of them is based on a two-scale

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transform, which changes a sequence of one variable functions into a sequence of two-variable functions. This transform was used by authors in homogenization of some problems in porous media [5] and it is suitable for an alternative definition of the two-scale convergence, see also [12], [16], [10]. Another one is based on the so-called inverse two-scale transform which defines a sequence of one-variable functions from the test function $\psi(x, y)$. This approach seems to be new.

The paper is organized as follows. Section 2 contains a survey of the definitions used, while Section 3 compares them. In the two-scale transform approach, we mention the case of boundary cubes exceeding Ω, which is usually neglected. Section 4 is devoted to the compactness property. Section 5 contains examples and some properties of two-scale convergence. Section 6 refers to some generalizations.

2. Definitions

Throughout the paper let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $Y = (0,1)^N$ the unit cube (a basic period). We will use periodic functions: a function v is said to be Y-periodic if $v(y + k) = v(y)$, $y \in \mathbb{R}^N$, $k \in \mathbb{Z}^N$. If the function v has more variables, we say that it is Y-periodic in y. Lebesgue spaces L^p are used with $p \in (1, \infty)$. The dual exponent is denoted by $q = p/(p-1)$. The symbol $\#$ is used for Y-periodic functions, e.g. $C_{\#}(Y)$ is the space of Y-periodic functions v such that $v|_{Y+y_0} \in C(Y + y_0)$, $\forall y_0 \in \mathbb{R}^N$. Further, we will consider a sequence of positive parameters $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ for $n \to \infty$. As usual, the subscript n will be omitted.

Let us begin with the classical definition by Nguetseng and Allaire which was introduced for the case of L^2:

Definition 2.1. We say that a sequence of functions $\{u_\varepsilon(x)\} \subset L^p(\Omega)$ two-scale converges to a limit $u_0(x, y) \in L^p(\Omega \times Y)$, if the relation

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \psi\Big(x, \frac{x}{\varepsilon}\Big) \,\mathrm{d}x = \int_\Omega \int_Y u_0(x, y) \psi(x, y) \,\mathrm{d}x \,\mathrm{d}y \tag{2.1}$$

holds for each $\psi(x, y) \in C_0^\infty[\Omega; C_{\#}^\infty(Y)]$ (the space of compactly supported infinitely differentiable functions with Y-periodic values in $C^\infty(\mathbb{R}^N)$). If, in addition,

$$\lim_{\varepsilon \to 0} \Big\| u_\varepsilon(x) - u_0\Big(x, \frac{x}{\varepsilon}\Big) \Big\|_{L^p(\Omega)} = 0,$$

we say that $\{u_\varepsilon\}$ two-scale converges strongly to $u_0(x, y)$.

R e m a r k 2.2. In the vocabulary of homogenization the strong two-scale convergence is also called a corrector type result. Assumptions making two-scale convergence strong are discussed in Section 3. Although $\{u_\varepsilon\}$ is a sequence of N variables $x_1, \ldots, x_N$, the limit is a function of $2N$ variables $x_1, \ldots, x_N, y_1, \ldots, y_N$. It enables us to describe the periodic behavior of u_ε better. Usually it is not emphasized, but the definition is connected with a fixed sequence of periods ε_n, i.e. for an extracted subsequence $\{u_{\varepsilon'}\}$ we should consider the same extracted subsequence $\{\varepsilon'\}$ of periods in the test function, see examples in Section 5.

Let us introduce an alternative definition. In [5], the authors dealt with a homogenization technique which was used for the description of a porous media. It is suitable for an alternative approach to the two-scale convergence. The idea is based on the so-called two-scale transform which changes a sequence of one-variable functions $\{u_\varepsilon(x)\}$ into a sequence of two-variable functions $\{\hat{u}_\varepsilon(x,y)\}$.

For each ε let us consider small non-overlapping cubes $C_\varepsilon^k = \varepsilon Y + \varepsilon k$, $k \in \mathbb{Z}^N$. Here, for sake of simplicity, we restrict ourselves to the domains Ω that can be decomposed into these cubes, i.e. $\overline{\Omega} = \bigcup\limits_k \overline{C}_\varepsilon^k$. The sequence $\{\hat{u}_\varepsilon(x,y)\}$ is defined by the relation

$$\hat{u}_\varepsilon(x,y) = u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) \quad x \in \Omega,\ \ y \in Y, \tag{2.2}$$

where $[x] = k$ is the vector of the greatest integers k_i less than or equal to x_i. On each cube $C_\varepsilon^k \times Y$ the function $\hat{u}_\varepsilon$ is constant in the variable x and as a function of y it is the function $u_\varepsilon(x)$ on C_ε^k transformed onto the unit cube Y. The alternative definition reads:

Definition 2.3. We say that a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converges to a function $u_0(x,y) \in L^p(\Omega \times Y)$, if

$$\lim_{\varepsilon \to 0} \int_\Omega \int_Y [\hat{u}_\varepsilon(x,y) - u_0(x,y)]\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{2.3}$$

for every test function $\psi \in C_0^\infty[\Omega; C_\#^\infty(Y)]$. Moreover, if $\hat{u}_\varepsilon \to u_0$ in $L^p(\Omega \times Y)$ strongly, we say that $\{u_\varepsilon\}$ two-scale converges strongly.

Let us refer here to the recent paper [12], where the above mentioned approach is also used and is called periodic unfolding. Authors apply this approach to the case of periodic multi-scale problems.

The other alternative approach is based on the so-called inverse two-scale transform which makes a sequence $\{\overline{\psi}_\varepsilon(x)\}$ from a two-variable function $\psi(x,y)$. The functions $\overline{\psi}_\varepsilon$ are constructed as follows. Similarly, as in the previous transform, we consider non-overlapping cubes C_ε^k that cover the domain Ω (here the domain Ω need not be the union of the cubes C_ε^k, i.e. $\overline{\Omega} \subseteq \bigcup\limits_k \overline{C}_\varepsilon^k$). Outside the domain Ω we put $\psi(x,y) = 0$. Let us average the extended function $\psi(x,y)$ with respect to the first variable:

$$\overline{\psi}_\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{C_\varepsilon^k} \psi\Big(\xi, \frac{x}{\varepsilon}\Big)\,\mathrm{d}\xi, \quad x \in C_\varepsilon^k. \tag{2.4}$$

Definition 2.4. Let $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)$. We say that $\{u_\varepsilon\}$ two-scale converges to a function $u_0 \in L^p(\Omega \times Y)$, if

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\overline{\psi}_\varepsilon(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for each test function $\psi(x,y) \in C_0^\infty[\Omega; C_\#^\infty(Y)]$. Moreover, if

$$\lim_{\varepsilon\to 0} \|u_\varepsilon(x) - \overline{u}_0^\varepsilon(x)\|_{L^p(\Omega)} = 0,$$

we say that $\{u_\varepsilon\}$ two-scale converges strongly.

Remark 2.5. The two-scale transform used in Definition 2.3 enables us to define two-scale and strong two-scale convergence more naturally with help of weak convergence with smooth test functions. On the other hand, Definition 2.4 is similar to the classical definition by Nguetseng and Allaire, but it differs from (2.1) by the choice of the test function on the left-hand side.

Why do we look for alternative approaches to the original one? In the definition we want to test the convergence with functions from a space as small as possible, thus, smooth functions are convenient. On the other hand, in applications the largest class is desirable. In Definition 2.1 we can not take the test function $\psi(x,y)$ from the whole space $L^q(\Omega \times Y)$, since it is not defined correctly on the zero-measure set $\{[x,y]\colon\ y = x/\varepsilon\}$ and thus the measurability of the composed function $\psi(x, x/\varepsilon)$ is not guaranteed. Moreover, the test functions must satisfy some convergence, which is not always obvious. The class of suitable test functions is discussed at the top of the following section. Such functions are called admissible. Further, we will see that the two mentioned alternative definitions avoid described problems.

3. Comparison of the definitions

The main goal of this section is to prove equivalence of the definitions. It means that each of the definitions yields the same two-scale limits.

First of all we proceed with the notion of admissible test function mentioned above. Since the widest set of suitable test functions ψ is not clear (we do not know the minimal conditions making these functions regular enough), the following characterization is useful:

Definition 3.1. A Y-periodic (in y) test function $\psi(x,y) \in L^q(\Omega \times Y)$ is said to be admissible, if

$$\lim_{\varepsilon\to 0}\Big\|\psi\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^q(\Omega)} = \|\psi(x,y)\|_{L^q(\Omega\times Y)} \tag{3.1}$$

and for a separable subspace $X \subseteq L^q(\Omega \times Y)$

$$\Big\|\psi\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^q(\Omega)} \leqslant \|\psi(x,y)\|_X. \tag{3.2}$$

Let us emphasize that there exist Y-periodic functions in y which do not satisfy the convergence (3.1) even if the measurability of the composed function $\psi(x,x/\varepsilon)$ is guaranteed, see [1]. Allaire also showed (for $p=2$) that spaces, e.g. $C[\overline{\Omega};C_\#(Y)]$, $L^2[\Omega;C_\#(Y)]$ or $L^2_\#[Y;C(\overline{\Omega})]$, are made up from the admissible functions. All these spaces are separable and their elements are functions continuous at least in one variable. Such functions are Carathéodory, which is the sufficient condition for measurability of the composed function $\psi(x,x/\varepsilon)$, and moreover, they satisfy (3.1) and (3.2). These two properties are needful in the proof of the compactness property, see Section 4. If ψ belongs to these spaces, we also have

$$\lim_{\varepsilon\to 0}\int_\Omega \psi\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega\int_Y \psi(x,y)\,\mathrm{d}x\,\mathrm{d}y; \tag{3.3}$$

for details, see [1]. This relation is natural: taking the stationary sequence $\{u_\varepsilon = 1\}$, (2.1) and (3.3) yield the two-scale limit $u_0(x,y)=1$.

In the alternative Definition 2.3 the situation is simplified. Due to the density of smooth functions in L^q, the convergence (2.3) implies also the weak convergence in $L^p(\Omega\times Y)$, i.e. the space of test functions can be enlarged to the whole $L^q(\Omega\times Y)$.

The special choice of the test function in Definition 2.4 is also motivated by the effort to enlarge the class of test functions. This choice is convenient due to the following lemma:

Lemma 3.2. *Let $\psi \in L^q(\Omega \times Y)$ be a Y-periodic function and let $\overline{\psi}_\varepsilon$ be defined by* (2.4). *Then we have*

$$\|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} \leqslant \|\psi(x,y)\|_{L^q(\Omega\times Y)}, \tag{3.4}$$

$$\lim_{\varepsilon\to 0} \|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} = \|\psi(x,y)\|_{L^q(\Omega\times Y)}. \tag{3.5}$$

Proof. For ε fixed, let us consider the minimal number of the cubes C_ε^k that cover the domain Ω. For these cubes we denote $\tilde{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$. The transform in the Lebesgue integral and the Fubini theorem yield

$$\begin{aligned}\int_\Omega \int_Y |\psi(x,y)|^q \,\mathrm{d}x\,\mathrm{d}y &= \int_{\tilde{\Omega}} \int_Y |\psi(x,y)|^q \,\mathrm{d}x\,\mathrm{d}y = \sum_k \int_Y \left[\int_{C_\varepsilon^k} |\psi(\xi,y)|^q \,\mathrm{d}\xi\right] \mathrm{d}y \\ &= \sum_k \int_{C_\varepsilon^k} \left[\frac{1}{\varepsilon^N} \int_{C_\varepsilon^k} \left|\psi\Big(\xi, \frac{x}{\varepsilon}\Big)\right|^q \mathrm{d}\xi\right] \mathrm{d}x = \sum_k \int_{C_\varepsilon^k} |\overline{\psi}_\varepsilon(x)|^q \,\mathrm{d}x \\ &= \int_{\tilde{\Omega}} |\overline{\psi}_\varepsilon(x)|^q \,\mathrm{d}x \geqslant \int_\Omega |\overline{\psi}_\varepsilon(x)|^q \,\mathrm{d}x.\end{aligned}$$

Thus, (3.4) is verified. Moreover, we have

$$\int_{\tilde{\Omega}\setminus\Omega} |\overline{\psi}_\varepsilon(x)|^q \,\mathrm{d}x \leqslant \sum_k \int_{C_\varepsilon^{k,b}} |\overline{\psi}_\varepsilon(x)|^q \,\mathrm{d}x = \sum_k \int_{C_\varepsilon^{k,b}} \left[\int_Y \psi(x,y)\,\mathrm{d}y\right] \mathrm{d}x,$$

where $C_\varepsilon^{k,b}$ are cubes in the boundary layer. Since their number is proportional to $M\varepsilon^{1-N}$ (M is the constant following from the surface integral), we have

$$\operatorname*{meas}_N\Big(\bigcup_k C_\varepsilon^{k,b}\Big) \leqslant M\varepsilon^{1-N}\varepsilon^N = M\varepsilon.$$

The absolute continuity of the Lebesgue integral yields

$$\sum_k \int_{C_\varepsilon^{k,b}} \left[\int_Y \psi(x,y)\,\mathrm{d}y\right] \mathrm{d}x \to 0 \quad \text{for } \varepsilon \to 0,$$

which implies (3.5). □

Remark 3.3. If the union of the cubes gives the whole domain Ω, i.e. $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$, then we have

$$\|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} = \|\psi(x,y)\|_{L^q(\Omega\times Y)}.$$

Lemma 3.2 says that every test function $\psi \in L^q(\Omega \times Y)$ can be called admissible (compare with properties in the Definition 3.1). Thus, the space of the test functions can be enlarged to the whole $L^q(\Omega \times Y)$.

The following lemma specifies functions $\hat{u}_\varepsilon$ used in the two-scale transform approach (have on mind that Ω is considered to be a union of the cubes C_ε^k).

Lemma 3.4. *Let* $u_\varepsilon(x) \in L^p(\Omega)$ *and let* $\hat{u}_\varepsilon$ *be defined by* (2.2). *Then* $\hat{u}_\varepsilon(x,y) \in L^p(\Omega \times Y)$ *and*

$$\|u_\varepsilon\|_{L^p(\Omega)} = \|\hat{u}_\varepsilon(x,y)\|_{L^p(\Omega\times Y)}.$$

Proof. Similarly to the previous proof, we compute

$$\begin{aligned}\int_\Omega |u_\varepsilon(x)|^p \,\mathrm{d}x &= \sum_k \int_{C_\varepsilon^k} |u_\varepsilon(x)|^p \,\mathrm{d}x = \sum_k \varepsilon^N \int_Y |\hat{u}_\varepsilon(x,y)|^p \,\mathrm{d}y \\ &= \sum_k \int_{C_\varepsilon^k} \mathrm{d}x \int_Y |\hat{u}_\varepsilon(x,y)|^p \,\mathrm{d}y = \int_\Omega \int_Y |\hat{u}_\varepsilon(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

□

Remark 3.5. The situation is more complicated in the case of cubes exceeding the domain Ω. The two-scale transform defined by (2.2) works well on the cubes C_ε^k, i.e. a function u_ε defined on C_ε^k is transformed into a function $\hat{u}_\varepsilon$ defined on $C_\varepsilon^k \times Y$. Near the boundary, where $C_\varepsilon^k \cap \Omega \neq C_\varepsilon^k$, it can cause difficulties. As in the proof of Lemma 3.2, let us consider the minimal number of the cubes C_ε^k covering Ω. The union $S_\varepsilon = \big(\bigcup_k \overline{C}_\varepsilon^k\big) \setminus \Omega$ is of a positive measure. In case of a "good" boundary we have $\operatorname{meas}_N S_\varepsilon \to 0$ (as $\varepsilon \to 0$), but $\|\hat{u}_\varepsilon\|$ cannot be estimated by $\|u_\varepsilon\|$ as the following example shows:

Let us take $\Omega = (0,a)$, $a \in \mathbb{R}$, and a sequence of periods ε such that the interval $(0,a)$ cannot be expressed as the union of the small intervals $I_\varepsilon^k = (\varepsilon k, \varepsilon(k+1))$, $k \in \mathbb{Z}$. We define a sequence $\{u_\varepsilon\} \subset L^1(\Omega)$ by

$$u_\varepsilon(x) = \begin{cases} 0, & x \in (0, a-\varepsilon^2), \\ \varepsilon^{-2}, & x \in (a-\varepsilon^2, a). \end{cases}$$

Thus, the intervals I_ε^k exceed the interval $(0,a)$ by $\varepsilon - \varepsilon^2$ (on this small part we put $u_\varepsilon = 0$). Obviously, the $L^1(\Omega)$ norm $\|u_\varepsilon(x)\|_{L^1(\Omega)} = 1$, while $\|\hat{u}_\varepsilon(x,y)\|_{L^1(\Omega\times Y)} = \varepsilon^2$, i.e. $\|u_\varepsilon\| \not\approx \|\hat{u}_\varepsilon\|$ (we have $\|u_\varepsilon\| \geqslant \|\hat{u}_\varepsilon\|$ only).

By the transform we want to conserve the norms of u_ε and $\hat{u}_\varepsilon$ even if the cubes exceed Ω. It is not difficult in 1D, since it is sufficient to re-scale with the actual length of the boundary segment $C_\varepsilon^k \cap \Omega$ instead of ε. In higher dimensions it is more difficult.

Theorem 3.6. *Let us assume* $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$ *and let* $\{u_\varepsilon\} \subset L^p(\Omega)$ *two-scale converge (in Nguetseng-Allaire sense) to a function* $u_0(x,y)$*. Then* $\{u_\varepsilon\}$ *two-scale converges to* u_0 *in the sense of Definition 2.3 and Definition 2.4, too.*

Proof. Let us assume $\{u_\varepsilon\}$ two-scale converges to u_0, but $\{\hat{u}_\varepsilon\}$ converges weakly to $\tilde{u}_0$. Then we have

$$I_\varepsilon \equiv \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], \frac{x}{\varepsilon}\Big)\,\mathrm{d}x + \delta_\varepsilon,$$

where

$$\delta_\varepsilon = \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x - \int_{C_\varepsilon^k} u_\varepsilon(x)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], \frac{x}{\varepsilon}\Big)\,\mathrm{d}x.$$

The two-scale transform yields

$$\begin{aligned} I_\varepsilon &= \varepsilon^N \int_Y u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], \Big[\frac{x}{\varepsilon}\Big] + y\Big)\,\mathrm{d}y + \delta_\varepsilon \\ &= \int_{C_\varepsilon^k} \mathrm{d}x \int_Y u_\varepsilon\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], y\Big)\,\mathrm{d}y + \delta_\varepsilon \\ &= \int_{C_\varepsilon^k}\int_Y \hat{u}_\varepsilon(x,y)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], y\Big)\,\mathrm{d}x\,\mathrm{d}y + \delta_\varepsilon = \hat{I}_\varepsilon + \delta_\varepsilon + \hat{\delta}_\varepsilon, \end{aligned}$$

where

$$\hat{I}_\varepsilon \equiv \int_{C_\varepsilon^k}\int_Y \hat{u}_\varepsilon(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\hat{\delta}_\varepsilon = \int_{C_\varepsilon^k}\int_Y \hat{u}_\varepsilon(x,y)\psi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big], y\Big)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega\int_Y \hat{u}_\varepsilon(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since $u_\varepsilon \overset{2-s}{\rightharpoonup} u_0$ and $\hat{u}_\varepsilon \rightharpoonup \tilde{u}_0$, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \delta_\varepsilon &= \int_{C_\varepsilon^k}\int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{C_\varepsilon^k}\int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \\ \lim_{\varepsilon\to 0} \hat{\delta}_\varepsilon &= \int_{C_\varepsilon^k}\int_Y \tilde{u}_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{C_\varepsilon^k}\int_Y \tilde{u}_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0}(I_\varepsilon - \hat{I}_\varepsilon) = \int_{C_\varepsilon^k}\int_Y (u_0(x,y) - \tilde{u}_0(x,y))\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

which implies $u_0 = \tilde{u}_0$ a.e. in $L^p(\Omega \times Y)$. A similar argument can be used in the case of the inverse two-scale transform definition. □

R e m a r k 3.7. We can establish even a stronger property. Under the assumption in Theorem 3.6, we have

$$\int_\Omega u_\varepsilon(x)\psi\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_Y \hat{u}_\varepsilon(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Let us deal with the strong two-scale convergence. The weak convergence $u_\varepsilon \rightharpoonup u$ equipped with the additional condition $\|u_\varepsilon\| \to \|u\|$ is also strong, i.e. $\|u_\varepsilon - u\| \to 0$. The following theorem introduces similar additional assumptions that strengthen two-scale convergence into the strong one (in the case of Nguetseng-Allaire definition).

Theorem 3.8. *A sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converges strongly to a limit u_0, if and only if $\{u_\varepsilon\}$ two-scale converges to u_0 and the relations*

$$\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0(x,y)\|_{L^p(\Omega\times Y)},$$
$$\Big\|u_0\Big(x, \frac{x}{\varepsilon}\Big)\Big\|_{L^p(\Omega)} \to \|u_0(x,y)\|_{L^p(\Omega\times Y)}$$

hold.

P r o o f. More details and the proof can be found in [15], [1]. □

R e m a r k 3.9. In the two-scale transform approach, the weak convergence of the sequence $\{\hat{u}_\varepsilon\}$ plays the role of the two-scale convergence. Hence, $\{u_\varepsilon\}$ two-scale converges strongly, if and only if $\{\hat{u}_\varepsilon\}$ converges weakly to u_0 and $\|\hat{u}_\varepsilon\|_{L^p(\Omega\times Y)} \to \|u_0\|_{L^p(\Omega\times Y)}$.

A similar result holds for the inverse two-scale transform approach. As in Theorem 3.8, the additional assumptions $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$, $\|\overline{u}_0^\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$ strengthen two-scale convergence into the strong one. But due to Lemma 3.2, each function $u_0 \in L^p(\Omega \times Y)$ satisfies the convergence $\|\overline{u}_0^\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$. Thus, we have

Lemma 3.10. *A sequence $\{u_\varepsilon\}$ two-scale converges strongly (in the sense of Definition 2.4), if and only if it two-scale converges to u_0 and $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$.*

Theorem 3.11. *Let $\overline{\Omega} = \bigcup_k \overline{C}_\varepsilon^k$ and let a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge strongly to a limit u_0 according to Nguetseng-Allaire's definition. Then $\{u_\varepsilon\}$ also two-scale converges strongly to u_0 in the sense of Definition 2.1 and Definition 2.3.*

P r o o f. The result is a direct consequence of Theorem 3.6, Theorem 3.8, Lemma 3.4, Lemma 3.2 and Remark 3.9. □

R e m a r k 3.12. Theorems 3.6, 3.11 together show the equivalence of the definitions used, i.e., all of them yield the same limits.

4. Compactness

The two-scale convergence can be used in applications due to the following compactness property.

Theorem 4.1. *A bounded sequence $\{u_\varepsilon\}$ in $L^p(\Omega)$ is compact with respect to the two-scale convergence, i.e. there exists an extracted subsequence $\{u_{\varepsilon'}\}$ two-scale converging to a function $u_0 \in L^p(\Omega \times Y)$.*

Allaire's proof in [1] is carried out for the test functions from $L^2[\Omega; C_\#(Y)]$. It is based on the properties of the dual space to $L^2[\Omega; C_\#(Y)]$. This space is not so transparent, since it is represented by $L^2[\Omega; M_\#(Y)]$, where $M_\#(Y)$ is the space of Y-periodic Radon measures.

In the alternative approach based on the two-scale transform, the situation is more simplified, since the two-scale compactness follows directly from the weak compactness of bounded sequences in $L^p(\Omega \times Y)$ (a closed ball is compact with respect to the weak convergence).

Let us prove a modification of the theorem for the case of two-scale convergence based on the inverse two-scale transform.

P r o o f. The boundedness, Hölder's inequality and Lemma 3.2 yield

$$\left| \int_\Omega u_\varepsilon(x) \overline{\psi}_\varepsilon(x) \, \mathrm{d}x \right| \leqslant C \|\overline{\psi}_\varepsilon(x)\|_{L^q(\Omega)} \leqslant C \|\psi(x,y)\|_{L^q(\Omega \times Y)}. \tag{4.1}$$

In view of (4.1), u_ε represents a bounded linear functional U_ε on $L^q(\Omega \times Y)$ defined by

$$\langle U_\varepsilon, \psi \rangle = \int_\Omega u_\varepsilon(x) \overline{\psi}_\varepsilon(x) \, \mathrm{d}x. \tag{4.2}$$

Since $L^q(\Omega \times Y)$ is separable, there exists U_0 such that an extracted subsequence $U_{\varepsilon'}$ converges $*$-weakly to U_0, i.e.

$$U_{\varepsilon'} \overset{*}{\rightharpoonup} U_0 \qquad \text{in } \ [L^q(\Omega \times Y)]^*.$$

This relation and Lemma 3.2 yield

$$|\langle U_0, \psi \rangle| = \lim_{\varepsilon \to 0} \int_\Omega u_{\varepsilon'}(x) \overline{\psi}_{\varepsilon'}(x) \, \mathrm{d}x \leqslant C \lim_{\varepsilon \to 0} \|\overline{\psi}_{\varepsilon'}(x)\|_{L^q(\Omega)} = C \|\psi(x,y)\|_{L^q(\Omega \times Y)}.$$

Thus, U_0 is also a bounded linear functional. By the Riesz representation theorem there exists a function $u_0 \in L^p(\Omega \times Y)$ such that

$$\langle U_0, \psi \rangle = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\lim_{\varepsilon' \to 0} \int_\Omega u_{\varepsilon'}(x)\overline{\psi}_{\varepsilon'}(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

which is the desired result. □

5. Examples and properties

Let us introduce a few typical examples of two-scale convergent sequences.

Example 5.1. (i) Let $a(y)$ be a Y-periodic bounded function with zero mean value $\int_Y a(y)\,\mathrm{d}y = 0$ and let $b_1(x), b_2(x) \in L^p(\Omega)$. Then the sequence $\{u_\varepsilon\}$ defined by $u_\varepsilon(x) = b_1(x)a(x/\varepsilon) + b_2(x)$ converges weakly to $b_2(x)$ and it two-scale converges (strongly) to $b_1(x)a(y) + b_2(x)$. We see that the weak limit is the function b_2 only. It says nothing on the periodic behaviour of the functions u_ε. On the other hand, in the two scale limit the information on "oscillations" is kept. This loss of information in the weak limit causes some "unpleasant" properties mentioned in Introduction, e.g. taking two weakly converging sequences $u_\varepsilon \rightharpoonup u$, $v_\varepsilon \rightharpoonup v$ does not imply $u_\varepsilon v_\varepsilon \rightharpoonup uv$, etc. The example shows that the weak limit is the average of the two-scale limit with respect to y. This is a direct consequence of the definition, if we take a test function ψ depending on the variable x only.

(ii) Let us consider the same functions $a(y)$, $b_1(x)$, $b_2(x)$, but another sequence $\{v_\varepsilon\}$ defined by $v_\varepsilon(x) = b_1(x)a(x/\varepsilon^2) + b_2(x)$. Then the two-scale and weak limits coincide, which means that the two-scale limit is constant in the variable y. In this case, the information on oscillations is not kept. Similarly, taking a sequence given by $w_\varepsilon(x) = b_1(x)a(cx\varepsilon) + b_2$ with c irrational, the two-scale limit equals to b_2 only. It is the consequence of the fact that diminishing of the periods in sequences is not in the resonance with the periods in the test function.

The sequences from the previous two examples point to an interesting fact. An extracted subsequence of weakly converging sequence converges to the same limit. In the case of two-scale convergence we must consider convergence also with respect to subsequences of periods. Otherwise the limits may differ (e.g. the sequence $\{v_\varepsilon\}$ from the example above can be considered an extracted subsequence from $\{u_\varepsilon\}$).

Example 5.2. Let $\{u_\varepsilon\} \subset L^p(\Omega)$ be a sequence satisfying convergence $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0(x,y)\|_{L^p(\Omega\times Y)}$. Then u_ε need not two-scale converge to u_0. Let $u(y)$ be a Y-periodic function. Let us consider functions $u_0(x,y) = u(y)$ and $\tilde{u}_0 = u(y-1/2)$. Since $u(y)$ is periodic, we have $\|u_0\|_{L^p(\Omega\times Y)} = \|\tilde{u}_0\|_{L^p(\Omega\times Y)}$. Let $\{u_\varepsilon\}$ be the sequence defined by $u_\varepsilon(x) = \tilde{u}_0(x/\varepsilon)$. Then $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times Y)}$, but $\{u_\varepsilon\}$ two-scale converges to $\tilde{u}_0$.

Theorem 5.3. *Let $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge to $u_0(x,y) \in L^p(\Omega\times Y)$ and let it converge weakly to $u(x)$. Then*

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega)} \geqslant \|u_0\|_{L^p(\Omega\times Y)} \geqslant \|u\|_{L^p(\Omega)}. \tag{5.1}$$

Proof. The first inequality (5.1) can be proved with help of Young's inequality and the definition of the two-scale convergence. The second inequality follows from Hölder's inequality and can be interpreted as follows: two-scale limit conserves more information on a periodic behaviour of $\{u_\varepsilon\}$ than the usual weak limit. $\square$

Example 5.4. Let us consider the sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ from Example 5.1. Denoting the two-scale limit by u_0 and the weak limit by u, we have $\lim \|u_\varepsilon\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega\times Y)} > \|u\|_{L^p(\Omega)}$, $\lim \|v_\varepsilon\|_{L^p(\Omega)} > \|u_0\|_{L^p(\Omega\times Y)} = \|u\|_{L^p(\Omega)}$ and finally the sum $u_\varepsilon + v_\varepsilon$ satisfies sharp inequalities.

Theorem 5.3 also implies: every sequence $\{u_\varepsilon\}$ strongly convergent to a function u two-scale converges to $u_0(x,y) = u(x)$, too. The following theorem on a limit procedure is useful in applications:

Theorem 5.5. *Let $\{u^1_\varepsilon\} \subset L^{p_1}(\Omega)$, …, $\{u^m_\varepsilon\} \subset L^{p_m}(\Omega)$ be sequences two-scale converging strongly to limits $u^1_0(x,y),\dots,u^m_0(x,y)$, $p_1,\dots,p_m, r \in \langle 1,\infty)$ and let $f(x,\xi_1,\dots,\xi_m)$ be a Carathéodory function satisfying a growth condition*

$$|f(x,\xi_1,\dots,\xi_m)| \leqslant g(x) + C\sum_{i=1}^m |\xi_i|^{p_i/r},$$

where $g \in L^r(\Omega)$ and C is a positive constant. Then

$$f(x,u^1_\varepsilon(x),\dots,u^m_\varepsilon(x)) \to \int_Y f(x,u^1_0(x,y),\dots,u^m_0(x,y))\,\mathrm{d}y \quad \textit{in } L^r(\Omega).$$

Proof. This relation can be obtained by use of the alternative approach. Since the assumptions of the theorem on Nemytskij operators are satisfied, we have $f(x,\hat{u}^1_\varepsilon,\dots,\hat{u}^m_\varepsilon) \to f(x,u^1_0,\dots,u^m_0)$ (this mapping is continuous). On the other hand, we know that the weak and here also the strong L^p limit is the average of the two-scale limit (with respect to y). $\square$

R e m a r k 5.6. The relation

$$u_\varepsilon^1 \dots u_\varepsilon^m \to \int_Y u_0^1(x,y)\dots u_0^m(x,y)\,\mathrm{d}y$$

is a special case of this theorem. This situation often occurs in proofs. The convergence changes into weak as one of $\{u_\varepsilon^i\}$ two-scale converges (not strongly) only.

6. Concluding remarks

We have surveyed some phenomena in the two-scale convergence. Two alternative definitions to the usual one were discussed. They make it possible weaken the requirements making test functions admissible. The definition based on the two-scale transform is more straightforward in the case of the two-scale compactness property or strong two-scale convergence, while the definition based on the inverse two-scale transform is closer to the original one, but it differs by the construction of the left-hand side test function.

In the text above we have considered the basic period to be the unit cube. It can be replaced by an arbitrary block $Y \subset \mathbb{R}^N$, but in the appropriate relations, the term $1/|Y|$ would appear.

The theory for time depending sequences used in evolution equations is completed in [15]. Considering problems in some porous media is popular in many papers, see e.g. [2], [10], [13]. For this analysis it is useful to define two-scale convergence more generally with respect to measures [7], [21], [4], [19]. The extension from the periodic case to the almost periodic one can be found in [9] and to the stochastic case in [8], [19]. Further, the two-scale convergence can be generalized to the so-called multiscale convergence, where multiple separated scales of oscillations are considered, see e.g. [3], [6], [11].

Two-scale convergence has been applied in many other papers, see e.g. [13], [10], [11], [16], [14]. Many other references can be found in the survey paper on two-scale convergence [17].

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