Radjesvarane Alexandre; Hassan Taha

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# PROPAGATION OF ELECTROMAGNETIC WAVES IN NON-HOMOGENEOUS MEDIA

Radjesvarane Alexandre, Hassan Taha, Orleans Cedex

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*Abstract.* We consider electromagnetic waves propagating in a periodic medium characterized by two small scales. We perform the corresponding homogenization process, relying on the modelling by Maxwell partial differential equations.

*Keywords*: Maxwell equations, homogenization, two-scale convergence, oscillating test functions

*MSC 2000*: 35Q60, 35B27

## 1. Introduction

Homogenization results for Maxwell equations, by the classical method of asymptotic expansions or more recently by two-scale convergence, are well known, see for instance [1], [3], [4], [5], [6], [19]. Let us first recall Maxwell equations framework, see [7], [8], [9]. Let $\Omega \subset \mathbb{R}^3$ be a physical domain. Then the electromagnetic propagation of waves in $\Omega$ is described by four ($\mathbb{R}^3$) vector valued functions $D$, $E$, $B$, $H$ of $(x,t) \in \Omega \times \mathbb{R}$. Here $D$ is the electric induction, $E$ the electric field, $B$ the magnetic induction and $H$ the magnetic field. Introducing furthermore the charge density $\varrho = \varrho(t,x)$ and the current density $J = J(t,x)$ of charges inside $\Omega$, one has Maxwell equations in the form

$$
\begin{aligned}
&\text{(i)} && -\frac{\partial D}{\partial t} + \operatorname{rot} H = J && \text{Ampere law,}\\
&\text{(ii)} && \frac{\partial B}{\partial t} + \operatorname{rot} E = 0 && \text{Faraday law,}\\
&\text{(iii)} && \operatorname{div} D = \varrho && \text{Gauss electrical law,}\\
&\text{(iv)} && \operatorname{div} B = 0 && \text{Gauss magnetic law.}
\end{aligned} \tag{1.1}
$$

Note that the current and charge densities satisfy the continuity relation or charge conservation

$$\frac{\partial \varrho}{\partial t} + \mathrm{div} J = 0, \tag{1.2}$$

as in fact follows also from (1.1).

Herein, we assume linear behavior laws, that is proportionality of fields and inductions,

$$D = \alpha E, \qquad B = \mu H. \tag{1.3}$$

Here $\mu$ and $\alpha$ are the magnetic permeability and electric permittivity respectively and these are assumed constant to simplify our exposition. Recall that in general, this linear behavior is not true for usual electromagnetic media and we hope to get back to this non linear aspect (in this direction see however [9]).

Moreover, we simplify Maxwell equations by considering the very special harmonic case (note that $J$ should satisfy some kind of Ohmic law). However, let us mention that most if not all of our results below could be extended easily to cover the time dependent case.

Let us recall quickly these facts, referring for much more details to standard books such as [7], [8], [9].

Since we are interested in time-periodic solutions only, we look for solutions in the form

$$D(x,t) = \Re(\exp(\mathrm{i}\omega t) D(x)), \quad H(x,t) = \Re(\exp(\mathrm{i}\omega t) H(x)), \tag{1.4}$$

$$J(x,t) = \Re(\exp(\mathrm{i}\omega t) J(x)) \tag{1.5}$$

and

$$B(x,t) = \Re(\exp(\mathrm{i}\omega t) B(x)), \quad E(x,t) = \Re(\exp(\mathrm{i}\omega t) E(x)). \tag{1.6}$$

We denote the new complex-valued functions of a variable $x$ with the same capital letters as the original real-valued functions of variables $x$, $t$. Then equation (1.1(ii)), divided by $\exp(\mathrm{i}\omega t)$, becomes (with $\mathrm{i}^2 = -1$)

$$-\mathrm{i}\omega D(x) + \mathrm{rot}_x\, H(x) = J(x) \tag{1.7}$$

and equation (1.1(ii)) becomes

$$\mathrm{i}\omega B(x) + \mathrm{rot}_x\, E(x) = 0, \tag{1.8}$$

which by applying the rot operator leads to

$$\mathrm{i}\omega \operatorname{rot}_x B(x) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.9}$$

From (1.3), one has $\operatorname{rot}_x B(x) = \mu \operatorname{rot}_x H(x)$, and thus it follows that

$$\mathrm{i}\omega\mu \operatorname{rot}_x H(x) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.10}$$

Using (1.7), one gets

$$\mathrm{i}\omega\mu(J(x) + \mathrm{i}\omega D(x)) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.11}$$

Using (1.3), one has

$$\mathrm{i}\omega\mu(J(x) + \mathrm{i}\omega\alpha E(x)) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.12}$$

Hence

$$\operatorname{rot}_x(\operatorname{rot}_x E(x)) + \mathrm{i}^2\omega^2\mu\alpha E(x) = -\mathrm{i}\omega\mu J(x).$$

Usually $J = \sigma E + J'$ (where $\sigma$ is the conductivity), so that

$$\operatorname{rot}_x(\operatorname{rot}_x E(x)) + (\mathrm{i}^2\omega^2\mu\alpha + \mathrm{i}\mu\omega\sigma)E(x) = -\mathrm{i}\omega\mu J'(x).$$

Let

$$F(x) = -\mathrm{i}\omega\mu J'(x), \quad \gamma = \mu\omega(-\omega\alpha + \mathrm{i}\sigma). \tag{1.13}$$

To avoid any mathematical problems, we will always assume that $\Re(\gamma) > 0$. In this direction, we refer to known standard difficulties associated with Maxwell equations in [9].

Then, we are finally led to the standard harmonic Maxwell equation

$$\operatorname{rot}(\operatorname{rot} E(x)) + \gamma E(x) = F(x). \tag{P}$$

Let $\Omega$ be a bounded regular open set in $\mathbb{R}^3$. Let $\varepsilon > 0$ be a small parameter and $\Omega^\varepsilon$ a bounded open set $\Omega^\varepsilon \subseteq \Omega \subseteq \mathbb{R}^3$, to be specified below.

We consider for $\gamma \in \mathbb{C}$, $\Re(\gamma) > 0$, the homogenization of problem (P) in $\Omega^\varepsilon$ with different boundary conditions on $\partial\Omega^\varepsilon$. That is, electromagnetic waves are propagating in $\Omega^\varepsilon$ and we prescribe the interactions with boundary $\partial\Omega^\varepsilon$.

More precisely, we consider the following two problems ($\mathrm{P}^\varepsilon$) and ($\mathrm{Q}^\varepsilon$), with $F^\varepsilon(x) \in L^2(\Omega^\varepsilon)^3$:

$$(\mathrm{P}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \wedge E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon \end{cases}$$

and

$$(\mathrm{Q}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \wedge \operatorname{rot} E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon. \end{cases}$$

The boundary condition in problem ($\mathrm{P}^\varepsilon$) describes the physical fact that the complementary set to $\Omega^\varepsilon$ behaves as a perfect conductor, while the boundary condition in problem ($\mathrm{Q}^\varepsilon$) could be interpreted as the absence of magnetic charges within the complementary set to $\Omega^\varepsilon$ .

A natural question arises in this electromagnetic setting as to investigate closely related problems such as

$$(\mathrm{R}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) - \operatorname{grad}(\operatorname{div} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \wedge E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon \end{cases}$$

or

$$(\mathrm{S}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) - \operatorname{grad}(\operatorname{div} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \cdot E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon. \end{cases}$$

Similarly, the last boundary condition reflects the absence of electric charges within the complementary set to $\Omega^\varepsilon$.

Such problems will not be studied herein in order to limit the length of the paper, and most importantly because test functions to be used are rather different. These important problems will be discussed in a forthcoming paper.

Let $\varepsilon > 0$ and $r_\varepsilon > 0$ be two small parameters such that $r_\varepsilon < \frac{1}{2}\varepsilon$ and consider the covering of $\mathbb{R}^3$ by cells

$$P_k^\varepsilon = \varepsilon\Big[-\frac{1}{2}, +\frac{1}{2}\Big]^3 + \varepsilon k,$$

where $k \in \mathbb{Z}^3$. In each cell $P_k^\varepsilon$ we remove the ball

$$T_k^\varepsilon = r_\varepsilon B_1 + \varepsilon k,$$

where $B_1$ denotes the unit ball in $\mathbb{R}^3$. We let

$$\Omega^\varepsilon = \Omega - T^\varepsilon$$

where $T^\varepsilon = \bigcup_{k=1}^{N(\varepsilon)} T_k^\varepsilon$ denotes the union of balls strictly included within $\Omega$. As a matter of fact, note that these balls do not intersect the boundary $\Omega$.

We will study the above problems under one of the scaling assumptions

$$\text{(HYP)}_1 \qquad \lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = 0$$

or

$$\text{(HYP)}_2 \qquad r_\varepsilon = c\varepsilon$$

where $0 < c < 1/4$ is a strictly positive fixed constant.

Finally, we will always assume (at least) that

$$\text{(HYP)}_3 \qquad \widetilde{F^\varepsilon} \rightharpoonup F \quad \text{weakly in } L^2(\Omega)^3$$

as $\varepsilon \to 0$. Above and throughout the paper the tilde over a symbol denotes the extension by zero in the holes.

We will be concerned with the asymptotic behavior as $\varepsilon \to 0$ of problems ($\mathrm{P}^\varepsilon$) and ($\mathrm{Q}^\varepsilon$).

Before stating the corresponding mathematical results, let us explain our motivation. The periodic perforated medium $\Omega^\varepsilon$ may be considered as one having two distinct di-electric constants or even holes that may be considered as charged particles.

However, let us remark that problems ($\mathrm{P}^\varepsilon$) or ($\mathrm{Q}^\varepsilon$) studied herein do not really fit with a clear (or true) physical modeling of propagation of electromagnetic waves in such non homogeneous media.

There are at least two points from the physical theory of Maxwell equations which are not really accounted for.

On the one hand, there is definitively a scaling problem between frequency and spatial scales which is not considered here. From mathematical viewpoint, this question leads to very deep technics, in progress actually. To be short, we are assuming that spatial variations are small compared to wavelenghts.

On the other hand, and this is surely one main point, if one views to the holes as charged particles, the modeling by problems ($\mathrm{P}^\varepsilon$) or ($\mathrm{Q}^\varepsilon$) is of course not true. One should for instance get back to Maxwell equations, say in harmonic form, and take care of the correct scaling and boundary conditions, involving scattering operators, see for instance [9].

Last but not least, in view of all these facts, one should note that actually the right-hand member data given are surely first order distributions, and so some if not all of our constructions in this paper would have to be modified.

Having such ideas in mind, the above problems must therefore be considered a first step in order to tackle real physical problems, maybe as a background to test mathematical tools available at present.

Even in these simple and academic problems, some constructions displayed in this paper have to be adapted to more complex situations, and this is why we have started studying them.

Some of the defects mentioned above are actually worked out.

Our results show in particular that the homogenized problems display a different output frequency with respect to $\omega$. More precisely, they are the following, see the notation below.

In section 2 we refer the following as regards to the homogenization of problem $(\mathrm{P}^\varepsilon)$.

**Theorem 1.** *Homogenization of problem* $(\mathrm{P}^\varepsilon)$.

(a) *With assumptions* $(\mathrm{HYP})_1$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{P}^\varepsilon)$, *one has (up to a subsequence)*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } H_0(\mathrm{rot}, \Omega),$$

*where* $E$ *is the variational solution of problem* $(\mathrm{P})$, *with the condition* $\vec{n} \wedge E = 0$ *on* $\partial\Omega$.

(b) *Under assumptions* $(\mathrm{HYP})_2$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{P}^\varepsilon)$, *one has (up to a subsequence)*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } H_0(\mathrm{rot}, \Omega),$$

*where* $E$ *is the (variational) solution in* $H_0(\mathrm{rot}, \Omega)$ *of*

$$\int_\Omega \eta \, \mathrm{rot}_x \, E(x) \cdot \mathrm{rot}_x \, \psi(x) \, \mathrm{d}x + \gamma\zeta \int_\Omega E(x) \cdot \psi(x) \, \mathrm{d}x = \zeta \int_\Omega F(x) \psi(x) \, \mathrm{d}x$$

*for all* $\psi \in H_0(\mathrm{rot}, \Omega)$, *where* $\eta > 0$, $\zeta$ *are two constants (given in Section 2).*

Section 3 is devoted to problem $(\mathrm{Q}^\varepsilon)$. We prove

**Theorem 2.** *Homogenization of problem* $(\mathrm{Q}^\varepsilon)$.

(a) *Under assumptions* $(\mathrm{HYP})_1$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{Q}^\varepsilon)$ *one has (up to a subsequence)*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } L^2(\Omega)^3,$$

*where* $E$ *is the variational solution of problem* $(\mathrm{P})$ *with the condition* $\vec{n} \wedge \mathrm{rot}\, E = 0$ *on* $\partial\Omega$.

(b) *With assumptions* $(\mathrm{HYP})_2$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{Q}^\varepsilon)$ *one has*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \text{weakly in } L^2(\Omega)^3,$$

*where* $E$ *is the variational solution, for all* $\psi \in H(\mathrm{rot}, \Omega)$, *of*

$$\int_\Omega \eta_1 \,\mathrm{rot}_x\, E(x) \cdot \mathrm{rot}_x\, \psi(x)\,\mathrm{d}x + \gamma \int_\Omega E(x) \cdot \psi(x)\,\mathrm{d}x = \int_\Omega F(x)\psi(x)\,\mathrm{d}x,$$

*where* $\eta_1 > 0$ *is a constant (given in Section* 3*)* .

Proofs of these results are done in Sections 2 and 3 for $(\mathrm{P}^\varepsilon)$ and $(\mathrm{Q}^\varepsilon)$ respectively. One can also find therein some remarks about the behavior of global electromagnetic energy. It is an interesting question to ask about the corresponding behavior of the *local* energy, which seems much more crucial for physical problems.

Also, the same type of results holds if we assume $r^\varepsilon/\varepsilon \to c > 0$ instead of assuming exactly that $r^\varepsilon = c\varepsilon$. However, we skip the proofs of this fact.

We end this introductory section by recalling standard materials on mathematics related to problem (P).

We introduce the following standard notation, see for instance [7], [9], [13]:

$$H^\varepsilon = L^2(\Omega^\varepsilon)^3,$$
$$H(\mathrm{rot}, \Omega^\varepsilon) = \{E \in L^2(\Omega^\varepsilon)^3;\ \mathrm{rot}\, E \in L^2(\Omega^\varepsilon)^3\},$$

with the usual norm $\|E\|_{H(\mathrm{rot},\Omega^\varepsilon)} = \|E\|_{L^2(\Omega^\varepsilon)^3} + \|\mathrm{rot}\, E\|_{L^2(\Omega^\varepsilon)^3}$.

$H(\mathrm{rot}, \Omega^\varepsilon)$ is a well known Hilbert space. In order to tackle the first boundary condition, that is problem $(\mathrm{P}^\varepsilon)$ with a perfect conductor type boundary condition, we also introduce

$$V_0^\varepsilon \equiv H_0(\mathrm{rot}, \Omega^\varepsilon) = \{E \in H(\mathrm{rot}, \Omega^\varepsilon),\ n^\varepsilon(x) \wedge E(x) = 0 \ \text{ on } \ \partial\Omega^\varepsilon\}.$$

Of course, $V_0^\varepsilon$ is a closed space in $H(\mathrm{rot}, \Omega^\varepsilon)$. The corresponding variational formulation for problem $(\mathrm{P}^\varepsilon)$ is then naturally given for all $\vec{\varphi} \in H_0(\mathrm{rot}, \Omega^\varepsilon)$ by

$$a^\varepsilon(E^\varepsilon, \vec{\varphi}) \equiv \int_{\Omega^\varepsilon} \mathrm{rot}\, E^\varepsilon \cdot \mathrm{rot}\, \vec{\varphi}\,\mathrm{d}x + \gamma \int_{\Omega^\varepsilon} E^\varepsilon \cdot \vec{\varphi}\,\mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon \cdot \vec{\varphi}\,\mathrm{d}x. \tag{1.14}$$

We recall that the sesquilinear form $a(u, v)$ is coercive on $V_0^\varepsilon$: there exists $\beta > 0$ such that $\Re a(u, u) \geqslant \beta \|u\|_{V_0^\varepsilon}$, $\forall u \in V_0^\varepsilon$.

By virtue of the above facts and the Green formula, the corresponding operator $A^\varepsilon$ is characterized by

$$A^\varepsilon E = \mathrm{rot}(\mathrm{rot}\, E) + \gamma E$$

and

$$D(A^\varepsilon) = \{E \in V^\varepsilon;\ \ \mathrm{rot}(\mathrm{rot}\, E) \in H^\varepsilon\}.$$

Since $(V_0^\varepsilon)'$ is a space of distributions and since the sesquilinear form $a$ given above is coercive on $V_0^\varepsilon$ with the constant $\beta = \inf(1, \gamma)$, there is a unique solution $E^\varepsilon$ in $V_0^\varepsilon$ of ($\mathrm{P}^\varepsilon$) thanks to the Lax-Milgram Lemma.

For problem ($\mathrm{Q}^\varepsilon$) we put $V^\varepsilon = H(\mathrm{rot}, \Omega^\varepsilon)$, and this defines the same operator $A^\varepsilon$ but with

$$D(A^\varepsilon) = \{E \in V^\varepsilon;\ \ \mathrm{rot}(\mathrm{rot}\, E) \in H^\varepsilon,\ \ n^\varepsilon(x) \wedge \mathrm{rot}\, E(x) = 0 \quad \text{on } \partial\Omega^\varepsilon\}.$$

## 2. Proof of Theorem 1

### 2.1. Proof of Theorem 1(a)

We divide the proof in several steps.

*First step: Variational formulation and uniform estimates*

The corresponding variational formulation, according to the previous section, is given for all $\vec{\varphi}^\varepsilon \in H_0(\mathrm{rot}, \Omega^\varepsilon)$ by

$$\int_{\Omega^\varepsilon} \mathrm{rot}\, E^\varepsilon \cdot \mathrm{rot}\, \vec{\varphi}^\varepsilon \,\mathrm{d}x + \gamma \int_{\Omega^\varepsilon} E^\varepsilon \cdot \vec{\varphi}^\varepsilon \,\mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon \cdot \vec{\varphi}^\varepsilon \,\mathrm{d}x. \tag{2.1}$$

One gets easily uniform estimates for $E^\varepsilon$ and $\mathrm{rot}\, E^\varepsilon$ from (2.1) by taking $\vec{\varphi} := E^\varepsilon$, hence one obtains

$$\|E^\varepsilon\|_{V^\varepsilon} \leqslant c, \tag{2.2}$$

where $c$ is a constant independent of $\varepsilon$.

We recall that if $E^\varepsilon \in H_0(\mathrm{rot}, \Omega^\varepsilon)$ and

$$H_0(\mathrm{rot}, \Omega) = \{U \in (L^2(\Omega))^3,\ \ \mathrm{rot}_x\, U \in (L^2(\Omega))^3,\ \ \vec{n} \wedge \vec{U}(x) = 0 \text{ on } \partial\Omega\}$$

then $E^\varepsilon$ satisfies $\widetilde{E^\varepsilon} \in H_0(\mathrm{rot}, \Omega)$ and $\mathrm{rot}\, \widetilde{E^\varepsilon} = \widetilde{\mathrm{rot}\, E^\varepsilon}$. Hence we deduce that $\widetilde{E^\varepsilon}$ belongs to a bounded subset of $H_0(\mathrm{rot}, \Omega)$. All in all, one has (up to a subsequence)

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \text{weakly in } L^2(\Omega)^3, \tag{2.3}$$

$$\widetilde{\mathrm{rot}\, E^\varepsilon} \rightharpoonup \mathrm{rot}\, E \quad \text{weakly in } L^2(\Omega)^3 \tag{2.4}$$

and, for all $\vec{\varphi}^\varepsilon \in H_0(\mathrm{rot}, \Omega^\varepsilon)$, one has

$$\int_\Omega \mathrm{rot}\, \widetilde{E^\varepsilon}(x) \cdot \mathrm{rot}\, \vec{\varphi}^\varepsilon(x)\, \mathrm{d}x + \gamma \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \vec{\varphi}^\varepsilon(x)\, \mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x) \cdot \vec{\varphi}^\varepsilon(x)\, \mathrm{d}x. \tag{2.5}$$

Following the classical method introduced by L. Tartar, see [17], we are going to take a test function of the form

$$\vec{\varphi}^\varepsilon(x) = \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)], \tag{2.6}$$

where $x \longmapsto \vec{\psi}(x) \in (C_c^\infty(\Omega))^3$ and $x \longmapsto \mathbb{W}^\varepsilon(x)$ is a $3 \times 3$ matrix valued function given by

$$\mathbb{W}^\varepsilon(x) = (\vec{W}_1^\varepsilon(x), \vec{W}_2^\varepsilon(x), \vec{W}_3^\varepsilon(x)),$$

that is, $\vec{W}_i^{\varepsilon}(x)$ is the $i$th column of the matrix $\mathbb{W}^\varepsilon(x)$.

To simplify the exposition, we introduce some matrix operations. We set $\prod_3$ for the set of matrix of order 3.

*Second step: Definitions and formulas*

a) Let $x \longmapsto \mathbb{U}(x)$ and $x \longmapsto \mathbb{V}(x)$ be two $\prod_3$-valued functions

$$\mathbb{U}(x) = (\vec{U}_1(x), \vec{U}_2(x), \vec{U}_3(x))$$

and

$$\mathbb{V}(x) = (\vec{V}_1(x), \vec{V}_2(x), \vec{V}_3(x)).$$

Then we define an $(\mathbb{R}^3)$ vector-valued function $x \longmapsto (\mathbb{U} \wedge \mathbb{V})(x)$ by

$$(\mathbb{U} \wedge \mathbb{V})(x) = \sum_{i=1}^{3} \vec{U}_i(x) \wedge \vec{V}_i(x). \tag{2.7}$$

b) With the same notation, if $x \longmapsto \mathbb{U}(x)$ is a $\prod_3$-valued function, one defines a $\prod_3$-valued function $x \longmapsto \mathrm{rot}\, \mathbb{U}(x)$ by

$$\mathrm{rot}\, \mathbb{U}(x) = (\mathrm{rot}\, \vec{U}_1(x), \mathrm{rot}\, \vec{U}_2(x), \mathrm{rot}\, \vec{U}_3(x)). \tag{2.8}$$

c) Note that, for $x \longmapsto \mathbb{U}(x)$ and $x \longmapsto \mathbb{V}(x)$ two $\prod_3$-valued functions, we get

$$(2.9) \qquad (\operatorname{rot} \mathbb{U}) \wedge \mathbb{V}(x) = \sum_{i=1}^{3} (\operatorname{rot} \vec{U}_i(x)) \wedge \vec{V}_i(x).$$

d) Let $x \longmapsto \vec{n}(x)$ be an $(\mathbb{R}^3)$ vector-valued function. Then we define a $\prod_3$-valued function $x \longmapsto (\mathbb{U} \wedge \vec{n})(x)$ by

$$(2.10) \qquad (\mathbb{U} \wedge \vec{n})(x) = (\vec{U}_1(x) \wedge \vec{n}(x), \vec{U}_2(x) \wedge \vec{n}(x), \vec{U}_3(x) \wedge \vec{n}(x))$$

and similarly for $\vec{n} \wedge \mathbb{U}$.

Next, we state some lemmas which will be useful for the proof of Theorem 1.

**Lemma 1.** *Let* $\mathbb{U}$ *and* $\mathbb{V}$ *be as in* a) *and smooth. Then*

$$(2.11) \qquad \operatorname{div}(\mathbb{U} \wedge \mathbb{V})(x) = \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) - \mathbb{U}(x) \cdot \operatorname{rot} \mathbb{V}(x),$$

*where "$\cdot$" means the scalar product of* $(\mathbb{R}^3)$ *vectors.*

P r o o f. We use definition a), and compute

$$\begin{aligned}
\operatorname{div}(\mathbb{U} \wedge \mathbb{V})(x) &= \operatorname{div}\left(\sum_{i=1}^{3} \vec{U}_i(x) \wedge \vec{V}_i(x)\right) = \sum_{i=1}^{3} \operatorname{div}(\vec{U}_i(x) \wedge \vec{V}_i(x)) \\
&= \sum_{i=1}^{3} (\vec{V}_i(x) \cdot \operatorname{rot} \vec{U}_i(x) - \vec{U}_i(x) \cdot \operatorname{rot} \vec{V}_i(x)) \\
&= \sum_{i=1}^{3} \vec{V}_i(x) \cdot \operatorname{rot} \vec{U}_i(x) - \sum_{i=1}^{3} \vec{U}_i(x) \cdot \operatorname{rot} \vec{V}_i(x) \\
&= \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) - \mathbb{U}(x) \cdot \operatorname{rot} \mathbb{V}(x).
\end{aligned}$$

□

**Lemma 2** (Green Formula). *Let* $\mathbb{U}$ *and* $\mathbb{V}$ *be as in* a) *and smooth,* $\Omega \subseteq \mathbb{R}^3$ *a regular bounded open set in* $\mathbb{R}^3$. *Then*

$$(2.12) \qquad \int_{\partial\Omega} (\mathbb{U} \wedge \mathbb{V})(x) \cdot \vec{n}(x) \, \mathrm{d}\Gamma = \int_{\Omega} \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) \, \mathrm{d}x - \int_{\Omega} \mathbb{U}(x) \cdot \operatorname{rot} \mathbb{V}(x) \, \mathrm{d}x$$

*where "$\cdot$" denotes the scalar product of* $3 \times 3$ *matrices on the right-hand side.*

P r o o f. Using integration by parts in (2.11), one has

$$\int_{\Omega} \operatorname{div}(\mathbb{U} \wedge \mathbb{V})(x) \, \mathrm{d}x = \int_{\Omega} \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) \, \mathrm{d}x - \int_{\Omega} \mathbb{U}(x) . \operatorname{rot} \mathbb{V}(x) \, \mathrm{d}x$$

and thus (2.12) follows. □

**Lemma 3.** *Let* $\mathbb{U}$ *be as in* a) *and* $\vec{\psi}(x) = (\psi_1(x), \psi_2(x), \psi_3(x))$ *a vector-valued function* $\vec{\psi} \in (C_c^\infty(\Omega))^3$. *Then*

$$\operatorname{rot}(\mathbb{U}(x) \cdot [\vec{\psi}(x)]) = (\operatorname{rot} \mathbb{U}(x))[\vec{\psi}(x)] + \mathbb{U}(x) \wedge \nabla(\psi_1(x), \psi_2(x), \psi_3(x)). \tag{2.13}$$

Proof. Using definition b), one computes

$$\begin{aligned}
\operatorname{rot}(\mathbb{U}(x) \cdot [\vec{\psi}(x)]) &= \operatorname{rot}\Big(\sum_{i=1}^{3} \vec{U}_i(x)\psi_i(x)\Big) \\
&= \sum_{i=1}^{3} (\operatorname{rot} \vec{U}_i(x))\psi_i(x) + \sum_{i=1}^{3} \vec{U}_i(x) \wedge \nabla\psi_i(x) \\
&= (\operatorname{rot} U_1(x), \operatorname{rot} U_2(x), \operatorname{rot} U_3(x))[\vec{\psi}(x)] \\
&\quad + (\vec{U}_1(x), \vec{U}_2(x), \vec{U}_3(x)) \wedge (\nabla\psi_1(x), \nabla\psi_2(x), \nabla\psi_3(x)) \\
&= \operatorname{rot}(\mathbb{U}(x))[\vec{\psi}(x)] + \mathbb{U} \wedge \nabla\vec{\psi}(x)
\end{aligned}$$

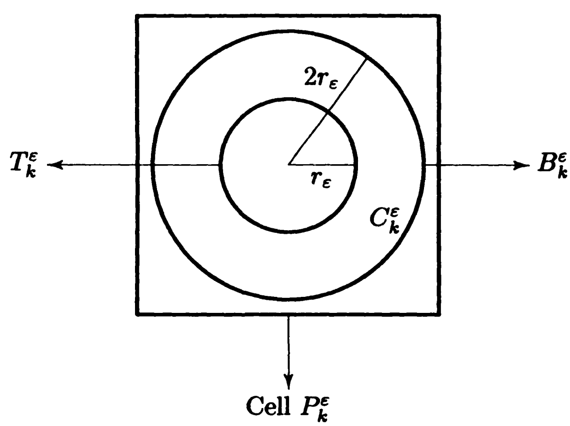
with $\nabla\vec{\psi}(x) = (\nabla\psi_1(x), \nabla\psi_2(x), \nabla\psi_3(x))$. □

*Third step: Oscillating function*

We are going to construct a $\prod_3$-valued function $x \longmapsto \mathbb{W}^\varepsilon(x)$, appearing in (2.6), as follows.

For each cell $P_k^\varepsilon$ included in $\Omega$, we construct $\mathbb{W}^\varepsilon$ in one typical cell and repeat the process by $\varepsilon$ periodicity. In the cells not strictly included in $\Omega$, we simply set $\mathbb{W}^\varepsilon$ as equal to Id (Identity matrix of order 3).

Let $P_k^\varepsilon$ be a cell strictly included in $\Omega$. Recall that $\partial T_k^\varepsilon$ denotes the boundary of the hole and has radius $r_\varepsilon$. We consider the hole centered at $\varepsilon k$ and radius $2r_\varepsilon$, denoted by $B_k^\varepsilon$.



Therefore, we have divided the cell $P_k^\varepsilon$ in three subregions: the ball $T_k^\varepsilon$ centered at $\varepsilon k$ and radius $r_\varepsilon$, the circular annulus $C_k^\varepsilon$ of small radius $r_\varepsilon$ and large radius $2r_\varepsilon$ and the exterior region $P_k^\varepsilon - (T_k^\varepsilon \cup C_k^\varepsilon)$. In $T_k^\varepsilon$, we set $\mathbb{W}^\varepsilon = 0$ (null matrix). In $P_k^\varepsilon - (T_k^\varepsilon \cup C_k^\varepsilon)$ we set $\mathbb{W}^\varepsilon = \mathrm{Id}$.

It remains to specify $\mathbb{W}^\varepsilon$ in $C_k^\varepsilon$ and we look (for reasons to be explained below) for $\mathbb{W}^\varepsilon$ such that

$$\begin{cases} \operatorname{rot}_x \mathbb{W}^\varepsilon = 0, & x \in C_k^\varepsilon, \\ n^\varepsilon \wedge \mathbb{W}^\varepsilon = 0, & x \in \partial T_k^\varepsilon, \\ n^\varepsilon \wedge \mathbb{W}^\varepsilon = n^\varepsilon \wedge \mathrm{Id}, & x \in \partial B_k^\varepsilon. \end{cases} \tag{2.14}$$

Since $C_k^\varepsilon$ has exactly a scale of $r_\varepsilon$ units, it is enough to look for $\mathbb{W} = \mathbb{W}(y)$ (matrix-valued) defined for $y \in C = \{y \in \mathbb{R}^3,\ 1 \leqslant |y| \leqslant 2\}$, and satisfying

$$\begin{cases} \operatorname{rot}_y \mathbb{W} = 0, & y \in C, \\ n \wedge \mathbb{W} = 0, & |y| = 1, \\ n \wedge \mathbb{W} = n \wedge \mathrm{Id}, & |y| = 2, \end{cases} \tag{2.15}$$

and then in each $C_k^\varepsilon$ define $\mathbb{W}^\varepsilon$ by $\mathbb{W}^\varepsilon(x) = \mathbb{W}(x - \varepsilon k / r_\varepsilon)$, which clearly satisfies (2.14) if $\mathbb{W}$ satisfies (2.15). In fact, we shall display explicitly $\mathbb{W}$. First, note that $\operatorname{rot} \mathrm{Id} = 0$.

On the other hand, let $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $|y|^2 = y_1^2 + y_2^2 + y_3^2$. Then

$$\operatorname{rot}(|y|^2 \mathrm{Id}) = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \wedge \begin{pmatrix} |y|^2 & 0 & 0 \\ 0 & |y|^2 & 0 \\ 0 & 0 & |y|^2 \end{pmatrix} = \begin{pmatrix} 0 & -2y_3 & 2y_2 \\ 2y_3 & 0 & -2y_1 \\ -2y_2 & 2y_1 & 0 \end{pmatrix}$$

and as $(\vec{n}(y) \otimes \vec{n}(y)) \wedge \vec{n}(y) = 0$, where $\vec{n}(y) = y/|y|$, we find adjusting coefficients that

$$\mathbb{W}(y) = \frac{1}{3}(|y|^2 - 1)\,\mathrm{Id} + \frac{2}{3}\vec{n}(y) \otimes \vec{n}(y) \tag{2.16}$$

does the job.

*Fourth step: Properties of* $\mathbb{W}^\varepsilon$

First, let us show that $\mathbb{W}^\varepsilon \longrightarrow \mathrm{Id}$ strongly in $L^2(\Omega)$ as $\varepsilon \longrightarrow 0$. Indeed, $C^\varepsilon$ being the union of the $C_k^\varepsilon$'s and $T^\varepsilon$ the union of the $T_k^\varepsilon$, one has

$$\begin{aligned}
\|\mathbb{W}^\varepsilon - \mathrm{Id}\|_{L^2(\Omega)} &= \|\mathbb{W}^\varepsilon - \mathrm{Id}\|_{L^2(T^\varepsilon)} + \|\mathbb{W}^\varepsilon - \mathrm{Id}\|_{L^2(C^\varepsilon)} \\
&= \|\mathrm{Id}\|_{L^2(T^\varepsilon)} + \|\mathbb{W}^\varepsilon - \mathrm{Id}\|_{L^2(C^\varepsilon)} \\
&\leqslant c_1 \,\mathrm{mes}(T^\varepsilon) + \int_{C^\varepsilon} |\mathbb{W}^\varepsilon(x) - \mathrm{Id}|^2 \,\mathrm{d}x \\
&\leqslant c_1 (r_\varepsilon)^3 \frac{\mathrm{mes}(\Omega)^\varepsilon}{\varepsilon^3} + \int_{C^\varepsilon} |\mathbb{W}^\varepsilon(x) - \mathrm{Id}|^2 \,\mathrm{d}x \\
&\leqslant c_2 \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3 + \frac{c_3}{\varepsilon^3} \int_{C_0^\varepsilon} \Big|\mathbb{W}\Big(\frac{x}{r_\varepsilon}\Big) - \mathrm{Id}\Big|^2 \,\mathrm{d}x.
\end{aligned}$$

Setting $x/r_\varepsilon = y$, thus $\mathrm{d}x = (r_\varepsilon)^3 \,\mathrm{d}y$, one has

$$\|\mathbb{W}^\varepsilon - \mathrm{Id}\|_{L^2(\Omega)} \leqslant c_2 \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3 + c_3 \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3 \int_C |\mathbb{W}(y) - \mathrm{Id}|^2 \,\mathrm{d}y \leqslant c \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3.$$

As $\varepsilon \to 0$, we get $\lim\limits_{\varepsilon\to 0} \|\mathbb{W}^\varepsilon - \mathrm{Id}\|_{L^2(\Omega)} = 0$.

Next, we wish to show that $\mathrm{rot}_x \,\mathbb{W}^\varepsilon$ is bounded uniformly in $L^2(\Omega)$. In fact, we shall show that $\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0$. Note that there are no discontinuities for $\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x)$ across the various boundaries.

Clearly

$$\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0, \quad x \in T_k^\varepsilon$$

and

$$\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0, \quad x \in P_k^\varepsilon - (T_k^\varepsilon \cup C_k^\varepsilon),$$

and we recall that in $C_k^\varepsilon$,

$$\mathbb{W}^\varepsilon(x) = \mathbb{W}\Big(\frac{x - \varepsilon k}{r_\varepsilon}\Big).$$

Then $\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0$ follows from $\mathrm{rot}_y \,\mathbb{W}(y) = 0$ which is the case as

$$\mathrm{rot}_y \,\mathbb{W}(y) = \mathrm{rot}_y \frac{1}{3}(|y|^2 - 1)\,\mathrm{Id} + \frac{2}{3}\vec{n}(y) \otimes \vec{n}(y) = 0$$

since

$$\mathrm{rot}_y\Big(\frac{1}{3}(|y|^2\,\mathrm{Id})\Big) = -\frac{2}{3}\,\mathrm{Id} \quad \text{and} \quad \frac{2}{3}\,\mathrm{rot}_y(\vec{n}(y) \otimes \vec{n}(y)) = \frac{2}{3}\,\mathrm{Id}.$$

Thus $\mathrm{rot}_x \,\mathbb{W}^\varepsilon$ being null is of course bounded uniformly in $L^2(\Omega)$.

*Fifth step: Passing to the limit*

We return to the variational formulation (2.5) and replace $\vec{\varphi}^\varepsilon$ by

$$\vec{\varphi}^\varepsilon(x) = \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]$$

for all but fixed $\vec{\psi} \in (C_c^\infty(\Omega))^3$. After extending $E^\varepsilon$ by zero in the holes, and using the matrix formulas and convergence properties given above, one obtains

$$\begin{aligned}(2.17)\qquad & \int_\Omega \operatorname{rot} \widetilde{E^\varepsilon} \cdot \operatorname{rot} \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x \\ & + \int_\Omega \operatorname{rot} \widetilde{E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon(x) \wedge (\partial\psi_1(x), \partial\psi_2(x), \partial\psi_3(x)) \\ & + \gamma \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x.\end{aligned}$$

As $\varepsilon \to 0$, we get

$$\begin{aligned}- & \int_\Omega \operatorname{rot} E(x) \cdot \mathrm{Id} \wedge (\partial\psi_1(x), \partial\psi_2(x), \partial\psi_3(x))\,\mathrm{d}x + \gamma \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x \\ & = \int_\Omega \widetilde{F}(x) . \vec{\psi}(x)\,\mathrm{d}x,\end{aligned}$$

thus

$$\int_\Omega \operatorname{rot} E(x) \cdot \operatorname{rot} \vec{\psi}(x)\,\mathrm{d}x + \gamma \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \int_\Omega F(x) \cdot \vec{\psi}(x)\,\mathrm{d}x$$

for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$, and this is exactly the conclusion of Theorem 1(a).

## 2.2. Proof of Theorem 1(b)

*First step: Uniform estimates*

As in the proof of Theorem 1(a) above, one has $\|E^\varepsilon\|_{V^\varepsilon} \leqslant c$.

*Second step: Two-Scale convergence*

We use standard notation and set $Y = \left[-\frac{1}{2}, +\frac{1}{2}\right]^3$ (identified with the periodic cell), and $Y^* = Y - T$ where $T = cB_1$. Since $(\widetilde{E^\varepsilon})$ and $(\widetilde{\operatorname{rot} E^\varepsilon})$ are bounded in $L^2(\Omega)^3$, see [1], [12], [14], up to a subsequence, they two-scale converge respectively to $E^0(x,y)$ and $V(x,y)$ belonging to $(L^2(\Omega \times Y))^3$. That is, for any $\vec{\psi}(x,y) \in$

$D(\Omega; C_p^\infty(Y))^3$ we have

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \text{weakly in } (L^2(\Omega))^3, \tag{2.18}$$

$$\operatorname{rot} \widetilde{E^\varepsilon} \rightharpoonup \operatorname{rot} E \quad \text{weakly in } (L^2(\Omega))^3, \tag{2.19}$$

$$\lim_{\varepsilon \to 0} \int_\Omega \widetilde{E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_\Omega \int_Y E^0(x,y) \cdot \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y, \tag{2.20}$$

$$\int_Y E^0(x,y)\, \mathrm{d}y = E(x),$$

$$\lim_{\varepsilon \to 0} \int_\Omega \operatorname{rot} \widetilde{E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_\Omega \int_{Y^*} V(x,y) \cdot \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y, \tag{2.21}$$

with

$$\int_{Y^*} V(x,y)\, \mathrm{d}y = \operatorname{rot} E. \tag{2.22}$$

We note that $\widetilde{E^\varepsilon}$ and $\widetilde{\operatorname{rot} E^\varepsilon}$ are equal to zero in $\Omega - \Omega^\varepsilon$.

Therefore, their two-scale limits $E^0(x,y)$ and $V(x,y)$ are equal to zero if $y \in Y - Y^*$.

Next, let $\vec{\psi} = \vec{\psi}(x,y)$ be a smooth function with support in $\overline{Y}^*$ with respect to the variable $y$. It follows that

$$\int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_{\Omega^\varepsilon} E^\varepsilon \cdot \Big[\operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \frac{1}{\varepsilon} \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big]\, \mathrm{d}x,$$

thus

$$\varepsilon \int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_{\Omega^\varepsilon} E^\varepsilon \cdot \Big[\varepsilon \operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big]\, \mathrm{d}x,$$

and extending by zero in the holes we obtain

$$\varepsilon \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_\Omega \widetilde{E^\varepsilon}.\Big[\varepsilon \operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big]\, \mathrm{d}x.$$

Sending $\varepsilon \longrightarrow 0$ and using two-scale convergence, one has

$$0 = \int_\Omega \int_{Y^*} E^0(x,y) \operatorname{rot}_y \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

Hence

$$\operatorname{rot}_y E^0(x,y) = 0 \quad \text{in } D'(\Omega \times Y^*)$$

and one finally gets

$$\begin{cases} \operatorname{rot}_y E^0(x,y) = 0, & y \in Y^*, \\ \vec{n}(y) \wedge E^0(x,y) = 0, & y \in \partial Y^*. \end{cases}$$

From the results of [2], [3], [4], [5], [9], [13] one deduces that ($\chi_{Y^*}$ being the characteristic function of $Y^*$)

$$E^0(x,y) = \chi_{Y^*}(y)\mathbb{U}(y)[E(x)], \tag{2.23}$$

where $\mathbb{U}$ is the $\prod_3$-valued function defined for $y \in Y^*$ by

$$\mathbb{U}(y) = (\vec{U}_1(y), \vec{U}_2(y), \vec{U}_3(y)) \tag{2.24}$$

and

$$\vec{U}_i(y) = (\nabla\Phi_i)(y), \tag{2.25}$$

each $\Phi_i$ being a solution in $H^1(Y^*)$ of

$$\begin{cases} \Phi_i = y_i - \Theta_i, \quad \Theta_i \ Y\text{-periodic}, \\ \Delta\Phi_i = 0 \quad \text{in } Y^*, \\ \Phi_i|_{\partial T} = 0. \end{cases} \tag{2.26}$$

Note that $\operatorname{rot}_y \mathbb{U} = 0$ and $\vec{n}(y) \wedge \mathbb{U} = 0$.

We make the first passage to the limit in the variational formulation still given by (2.5)

$$\int_\Omega \operatorname{rot}_x \widetilde{E^\varepsilon}(x) \cdot \operatorname{rot}_x \vec{\varphi}^\varepsilon(x)\,\mathrm{d}x + \gamma \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \vec{\varphi}^\varepsilon(x)\,\mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x).\vec{\varphi}^\varepsilon(x)\,\mathrm{d}x$$

for all $\vec{\varphi}^\varepsilon \in H_0(\operatorname{rot}, \Omega^\varepsilon)$, by making the precise choice

$$\vec{\varphi}^\varepsilon = \mathbb{U}\Big(\frac{x}{\varepsilon}\Big)[\vec{\psi}(x)], \tag{2.27}$$

where $\mathbb{U}$ is defined by (2.24) for any $\vec{\psi} \in (C_c^\infty(\Omega))^3$ and we obtain

$$\begin{aligned} &\int_\Omega \operatorname{rot}_x \widetilde{E^\varepsilon}(x) \cdot \operatorname{rot}_x\{\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\}\,\mathrm{d}x + \gamma \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x \\ &\quad = \int_\Omega \widetilde{F^\varepsilon}(x) \cdot \mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x. \end{aligned} \tag{2.28}$$

For the first term on the left hand side and the term on the right-hand side of (2.28) we use two-scale convergence, noticing that

$$\begin{aligned} \operatorname{rot}_x\{\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\} &= (\operatorname{rot}_x \mathbb{U}^\varepsilon(x))[\vec{\psi}(x)] + \mathbb{U}^\varepsilon(x) \wedge \nabla\vec{\psi}(x) \\ &= \frac{1}{\varepsilon}(\operatorname{rot}_y \mathbb{U})^\varepsilon[\vec{\psi}] + \mathbb{U}^\varepsilon \wedge \nabla\vec{\psi} = \mathbb{U}^\varepsilon \wedge \nabla\psi \end{aligned}$$

and that the two-scale limit associated with $\operatorname{rot}_x \widetilde{E^\varepsilon}$ is $V(x,y)$.

Let us deal with the second term on the left hand side of (2.28). For this purpose we use simply the div-rot lemma of L. Tartar, F. Murat, see for instance [18], since $\operatorname{rot}_x \widetilde{E^\varepsilon}$ as well as $\operatorname{div}(\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)])$ are (uniformly) bounded in $(L^2(\Omega))^3$ since

$$\begin{aligned}\operatorname{div}(\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]) &= \operatorname{div}\Big(\sum_i \vec{U}^{\varepsilon i}\psi_i\Big) = \sum_i \operatorname{div}(\vec{U}^{\varepsilon i})\psi_i + \sum_i \vec{U}^{\varepsilon i}\cdot\nabla\psi_i(x)\\ &= 0 + \sum_i \vec{U}^{\varepsilon i}\cdot\nabla\psi_i(x).\end{aligned}$$

Therefore one obtains, for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$,

$$\begin{aligned}\int_\Omega\int_{Y^*} V(x,y)\mathbb{U}(y)\wedge\nabla\vec{\psi}(x)\,\mathrm{d}x\,\mathrm{d}y + \gamma\int_\Omega E(x)\,\mathrm{d}x\int_{Y^*}\mathbb{U}(y)[\vec{\psi}]\,\mathrm{d}y\\ = \int_\Omega F(x)\,\mathrm{d}x\int_{Y^*}\mathbb{U}(y)[\vec{\psi}]\,\mathrm{d}y.\end{aligned}$$

Setting $\int_{Y^*}\mathbb{U}(y)\,\mathrm{d}y = \zeta\,\mathrm{Id}$, where $\zeta \in \mathbb{R}$ is a fixed constant, one obtains

$$\int_\Omega\int_{Y^*} V(x,y)\cdot\mathbb{U}(y)\wedge\nabla\vec{\psi}(x)\,\mathrm{d}x\,\mathrm{d}y + \zeta\gamma\int_\Omega E(x)\cdot\vec{\psi}\,\mathrm{d}x = \zeta\int_\Omega F(x)\cdot\vec{\psi}\,\mathrm{d}x. \tag{2.29}$$

It remains to analyze the first term on the left hand side of expression (2.29).

By definition, for all smooth $\vec{\psi}(x,y)$ we have

$$\int_\Omega \operatorname{rot}\widetilde{E^\varepsilon}(x)\cdot\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \longrightarrow \int_\Omega\int_{Y^*} V(x,y)\cdot\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Choose any $\vec{\psi}$ such that $\operatorname{rot}_y\vec{\psi} = 0$ and $\operatorname{supp}_y\vec{\psi}\subset\overline{Y}^*$. Then the left-hand side can also be written as

$$\int_\Omega \widetilde{E^\varepsilon}\cdot(\operatorname{rot}_x\vec{\psi})\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \widetilde{E^\varepsilon}\cdot\operatorname{rot}_x\Big[\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big]\,\mathrm{d}x.$$

We can again use the div-rot lemma. As $\operatorname{rot}\widetilde{E^\varepsilon}$ is bounded and since obviously

$$\operatorname{div}\Big\{\operatorname{rot}_x\Big[\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big]\Big\} = 0 \quad\text{and}\quad \operatorname{rot}_x\Big[\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big] = (\operatorname{rot}_x\vec{\psi})\Big(x,\frac{x}{\varepsilon}\Big)$$

are bounded uniformly in $(L^2(\Omega))^3$,

$$\int_\Omega \widetilde{E^\varepsilon}\cdot(\operatorname{rot}_x\vec{\psi})\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \longrightarrow \int_\Omega E(x)\,\mathrm{d}x\int_{Y^*}\operatorname{rot}_x\vec{\psi}(x,y)\,\mathrm{d}y.$$

Alltogether we have obtained that for all $\vec{\psi}(x,y)$, $\operatorname{supp}_y \vec{\psi} \subset \overline{Y}^*$, $\operatorname{rot}_y \vec{\psi}(x,y) = 0$, one has

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega E(x)\,\mathrm{d}x \int_{Y^*} \operatorname{rot}_x \vec{\psi}(x,y)\,\mathrm{d}y$$
$$= \int_\Omega \mathrm{d}x \int_{Y^*} \operatorname{rot}_x E(x) \cdot \vec{\psi}(x,y)\,\mathrm{d}y,$$

thus

$$\int_\Omega \int_{Y^*} [V(x,y) - \operatorname{rot}_x E(x)] \cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

From [2], [3], [4], [9], [5], [13] we deduce that there exists $V_1 \equiv V_1(x,y)$ such that

$$V(x,y) = \operatorname{rot}_x E(x) + \operatorname{rot}_y V_1(x,y). \tag{2.30}$$

Therefore (2.29) becomes, for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$,

$$\int_\Omega \int_{Y^*} \{\operatorname{rot}_x E(x) + \operatorname{rot}_y V_1(x,y)\} \cdot \mathbb{U}(y) \wedge \nabla_x \psi(x)\,\mathrm{d}x\,\mathrm{d}y + \zeta\gamma \int_\Omega E(x) \cdot \vec{\psi}\,\mathrm{d}x = \zeta \int_\Omega F(x) \cdot \vec{\psi}(x,y)\,\mathrm{d}x. \tag{2.31}$$

In the second step, we precise further the term $\operatorname{rot}_y V_1(x,y)$. For this purpose, we get back again to the variational formulation (2.5), take $\vec{\varphi}^\varepsilon = \vec{\psi}(x, x/\varepsilon)$, multiply by $\varepsilon$ and send it to zero. In this way we obtain

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \operatorname{rot}_y \vec{\psi} = 0$$

for all $\vec{\psi}$ such that $\operatorname{supp}_y \vec{\psi} \subset \overline{Y}^*$ with $\vec{n} \wedge \vec{\psi}|_{\partial T} = 0$. On the other hand, one has also easily $\operatorname{div}_y V(x,y) = 0$ in $D'(\Omega \times Y^*)$. We obtain in particular from the facts already mentioned, that $V$ is given by

$$V(x,y) = \mathbb{U}(y)[\operatorname{rot}_x E(x)]. \tag{2.32}$$

Therefore (2.31) becomes

$$\int_\Omega \int_{Y^*} \mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot (\mathbb{U}(y) \wedge \nabla_x \vec{\psi}(x)) + \zeta\gamma \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \zeta \int_\Omega F(x) \cdot \vec{\psi}(x)\,\mathrm{d}x. \tag{2.33}$$

Next, one has

$$\begin{aligned}
&\mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot (\mathbb{U}(y) \wedge \nabla_x \vec{\psi}(x)) \\
&\qquad = \sum_i \mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot \vec{U}_i \wedge \nabla_x \psi_i(x) \\
&\qquad = -\sum_i (\vec{U}_i \wedge \mathbb{U}(y)[\operatorname{rot}_x E(x)]) \cdot \nabla_x \psi_i(x) \\
&\qquad = -\sum_i \sum_j (\vec{U}_i(y) \wedge \vec{U}_j(y)(\operatorname{rot}_x E(x))_j) \cdot \nabla_x \psi_i(x).
\end{aligned}$$

Define $\eta \in \mathbb{R}$ such that

$$\int_{Y^*} (\vec{U}_1 \wedge \vec{U}_2)\, \mathrm{d}y = \eta \vec{e_3}, \quad \int_{Y^*} (\vec{U}_2 \wedge \vec{U}_3)\, \mathrm{d}y = \eta \vec{e_1}, \quad \int_{Y^*} (\vec{U}_3 \wedge \vec{U}_1)\, \mathrm{d}y = \eta \vec{e_2}.$$

Then, one obtains that

$$\begin{aligned}
&\int_{Y^*} \mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot (\mathbb{U}(y) \wedge \nabla_x \vec{\psi}(x))\, \mathrm{d}y \\
&\qquad = \eta\{(\operatorname{rot}_x E)_2 \cdot \partial_3 \psi_1 - (\operatorname{rot}_x E)_1 \partial_3 \psi_2\} \\
&\qquad\quad + \eta\{(\operatorname{rot}_x E)_3 \cdot \partial_1 \psi_2 - (\operatorname{rot}_x E)_2 \partial_1 \psi_3\} \\
&\qquad\quad + \eta\{(\operatorname{rot}_x E)_1 \cdot \partial_2 \psi_3 - (\operatorname{rot}_x E)_3 \partial_2 \psi_1\} \\
&\qquad = \eta(\operatorname{rot}_x E)_1(\partial_2 \psi_3 - \partial_3 \psi_2) + \eta(\operatorname{rot}_x E)_2(\partial_3 \psi_1 - \partial_1 \psi_3) \\
&\qquad\quad + \eta(\operatorname{rot}_x E)_3(\partial_1 \psi_2 - \partial_2 \psi_1) \\
&\qquad = \eta \operatorname{rot}_x E \cdot \operatorname{rot}_x \psi.
\end{aligned}$$

Consequently, for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$ we obtain

$$\eta \int_\Omega \operatorname{rot}_x E(x) \cdot \operatorname{rot}_x \vec{\psi}(x)\, \mathrm{d}x + \gamma\zeta \int_\Omega E(x) \cdot \vec{\psi}(x)\, \mathrm{d}x = \zeta \int_\Omega F(x) \cdot \vec{\psi}(x)\, \mathrm{d}x,$$

where $\eta$, $\zeta$ are constants, which is the claim of Theorem 1(b).

Remark 1. If we replace the boundary condition

$$\vec{n}^\epsilon(x) \wedge \vec{E}^\epsilon(x) = 0$$

by the non homogeneous condition

$$\vec{n}^\epsilon(x) \wedge \vec{E}^\epsilon(x) = \vec{n}^\epsilon \wedge \vec{g}(x),$$

written also as

$$\vec{n}^\epsilon \wedge (\vec{E}^\epsilon - \vec{g}(x)) = 0,$$

where $\vec{g}$ is a given (smooth) vector field, we have the same type of result by setting

$$\vec{E}^\varepsilon - \vec{g}(x) = \vec{V}^\varepsilon(x).$$

Remark 2. Under the assumption $(\mathrm{HYP})_1$ and if we suppose furthermore the strong convergence of the sequence $\widetilde{F^\varepsilon}$ in $L^2(\Omega)$, it is not difficult to check that the global electromagnetic "energy"

$$\int_\Omega |\operatorname{rot} \widetilde{E^\varepsilon}|^2 + \gamma \int_\Omega |\widetilde{E^\varepsilon}|^2$$

converges towards the corresponding homogenized one (involving $E$).

Under the case of assumption $(\mathrm{HYP})_2$, corrector results can eventually give a clue.

## 3. Proof of Theorems 2

### 3.1. Proof of Theorem 2(a)

*First step: Variational formulation and uniform estimates*

Problem $(\mathrm{Q}^\varepsilon)$ has the variational formulation

$$\int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon \cdot \operatorname{rot} \vec{\varphi}(x)\,\mathrm{d}x + \int_{\Omega^\varepsilon} E^\varepsilon \cdot \vec{\varphi}(x)\,\mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon \cdot \vec{\varphi}(x)\,\mathrm{d}x \tag{3.1}$$

for all $\vec{\varphi} \in H(\operatorname{rot}, \Omega^\varepsilon)$.

Let us note that $\widetilde{\operatorname{rot} E^\varepsilon} \neq \operatorname{rot} \widetilde{E^\varepsilon}$ in general. Extending by zero, we now get

$$\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \operatorname{rot} \vec{\varphi}(x)\,\mathrm{d}x + \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \vec{\varphi}(x)\,\mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x) \cdot \vec{\varphi}(x)\,\mathrm{d}x. \tag{3.2}$$

Letting $\vec{\varphi} := E^\varepsilon$, one has

$$\int_{\Omega^\varepsilon} |\operatorname{rot} E^\varepsilon|^2\,\mathrm{d}x + \int_{\Omega^\varepsilon} |E^\varepsilon|^2\,\mathrm{d}x \leqslant \frac{1}{2} \int_{\Omega^\varepsilon} |F^\varepsilon|^2\,\mathrm{d}x + \frac{1}{2} \int_{\Omega^\varepsilon} |E^\varepsilon|^2\,\mathrm{d}x,$$

thus

$$\begin{aligned} \|E^\varepsilon\|^2_{H(\operatorname{rot},\Omega^\varepsilon)} &\leqslant \int_{\Omega^\varepsilon} |\operatorname{rot} E^\varepsilon|^2\,\mathrm{d}x + \int_{\Omega^\varepsilon} |E^\varepsilon|^2\,\mathrm{d}x \\ &\leqslant \int_{\Omega^\varepsilon} |\operatorname{rot} E^\varepsilon|^2\,\mathrm{d}x + \frac{1}{2} \int_{\Omega^\varepsilon} |E^\varepsilon|^2\,\mathrm{d}x \leqslant C \end{aligned}$$

and consequently

$$\|E^\varepsilon\|_{H(\operatorname{rot},\Omega^\varepsilon)} \leqslant C.$$

*Second step: Passing to the limit*

Up to a subsequence we can assume

$$\widetilde{E^\varepsilon}(x) \rightharpoonup E(x) \quad \text{weakly in} \quad (L^2(\Omega))^3, \tag{3.3}$$

$$\widetilde{\operatorname{rot} E^\varepsilon} \rightharpoonup V(x) \quad \text{weakly in} \quad (L^2(\Omega))^3. \tag{3.4}$$

Then from (3.2) one has, for all $\vec{\varphi} \in (C^\infty(\bar{\Omega}))^3$,

$$\int_\Omega V(x) \cdot \operatorname{rot} \vec{\varphi}(x)\, \mathrm{d}x + \int_\Omega E(x) \cdot \vec{\varphi}(x)\, \mathrm{d}x = \int_\Omega F(x) \cdot \vec{\varphi}(x)\, \mathrm{d}x.$$

It remains to identify $V$. Let $\vec{\psi} \in (C_c^\infty(\bar{\Omega}))^3$. Then of course

$$\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \vec{\psi}(x)\, \mathrm{d}x \longrightarrow \int_\Omega V(x) \cdot \vec{\psi}(x)\, \mathrm{d}x.$$

On the other hand,

$$\begin{aligned}\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \vec{\psi}(x)\, \mathrm{d}x &= \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot [\vec{\psi}(x) - \mathbb{W}^\varepsilon[\vec{\psi}](x)]\, \mathrm{d}x \\ &\quad + \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\, \mathrm{d}x,\end{aligned}$$

where $\mathbb{W}^\varepsilon$ is the test (matrix-valued) function given in Section 2.

Since $\vec{\psi} - \mathbb{W}^\varepsilon(\vec{\psi}) \longrightarrow 0$ strongly in $(L^2(\Omega))^3$, one has

$$\int_\Omega V(x) \cdot \vec{\psi}(x)\, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\, \mathrm{d}x.$$

Further,

$$\begin{aligned}\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\, \mathrm{d}x &= \int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\, \mathrm{d}x \\ &= \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \operatorname{rot}\{\mathbb{W}^\varepsilon[\vec{\psi}](x)\}\, \mathrm{d}x,\end{aligned}$$

since $\vec{n}^\varepsilon \wedge \mathbb{W}^\varepsilon = 0$ on $\partial\Omega^\varepsilon$.

As

$$\begin{aligned}&\int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \operatorname{rot}\{\mathbb{W}^\varepsilon[\vec{\psi}](x)\}\, \mathrm{d}x \\ &\quad = \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot (\operatorname{rot} \mathbb{W}^\varepsilon)[\vec{\psi}](x)\, \mathrm{d}x + \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \mathbb{W}^\varepsilon \wedge \nabla\vec{\psi}(x)\, \mathrm{d}x \\ &\quad = 0 + \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \mathbb{W}^\varepsilon \wedge \nabla\vec{\psi}(x)\, \mathrm{d}x \longrightarrow \int_\Omega E(x) \cdot \operatorname{rot} \vec{\psi}(x)\, \mathrm{d}x,\end{aligned}$$

we have finally obtained that

$$\int_\Omega V(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \int_\Omega E(x) \cdot \operatorname{rot} \vec{\psi}(x)\,\mathrm{d}x,$$

and thus

$$V = \operatorname{rot} E.$$

This concludes the proof of Theorem 2(a).

### 3.2. Proof of Theorem 2(b)

*First step: Uniform estimates in* $H(\operatorname{rot}, \Omega)$

Repeating the same lines of calculus, we get again similar uniform estimates, and thus we can assume that weak convergences (3.3) and (3.4) hold true.

*Second step: Two-scale convergence*

Since $(\widetilde{E^\varepsilon})$ as well as $(\widetilde{\operatorname{rot} E^\varepsilon})$ are bounded in $(L^2(\Omega))^3$, there exist two two-scale limits, $E^0 \in (L^2(\Omega \times Y))^3$, $V \in (L^2(\Omega \times Y))^3$ such that for any $\vec{\psi}(x,y) \in (D(\Omega, C_p^\infty(Y)))^3$ one has

$$\lim_{\varepsilon \to 0} \int_\Omega \widetilde{E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_{Y^*} E^0(x,y) \cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.5}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_{Y^*} V(x,y) \cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6}$$

By integration by parts, we obtain for all $\vec{\psi}(x,y) \in (D(\Omega, C_p^\infty(Y^*)))^3$ with compact support (in $y$) in $Y^*$

$$\varepsilon \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \widetilde{E^\varepsilon} \cdot \Big[\varepsilon \operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big]\,\mathrm{d}x.$$

Passing to the limit in both terms with the help of two-scale convergence leads to

$$\int_\Omega \int_{Y^*} E^0(x,y) \operatorname{rot}_y \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{3.7}$$

that is

$$\operatorname{rot}_y E^0(x,y) = 0 \quad \text{in } (D'(\Omega \times Y^*))^3. \tag{3.8}$$

Now, to the assumption on $\vec{\psi}(x,y)$ we add the condition $\operatorname{rot}_y \vec{\psi}(x,y) = 0$ with compact support in $x$ as well as in $y$ in $Y^*$. Since

$$\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \widetilde{E^\varepsilon}(x) \operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x,$$

using the two-scale convergence we are led to

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_\Omega \int_{Y^*} \widetilde{E^0}(x,y) \cdot \mathrm{rot}_x\, \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y,$$

and thus integrating by parts in $\Omega$ we get

$$\int_\Omega \int_{Y^*} E^0(x,y) \cdot \mathrm{rot}_x\, \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_\Omega \int_{Y^*} \mathrm{rot}_x\, E^0(x,y) \cdot \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

Hence

$$\int_\Omega \int_{Y^*} [V(x,y) - \mathrm{rot}_x\, E^0(x,y)] \cdot \vec{\psi}(x,y)\, \mathrm{d}x\, \mathrm{d}y = 0. \tag{3.9}$$

We will use all these facts to identify $E^0$ and $V$.

According to [2], [3], [4], [9] there exist unique $\varphi \in H^1(Y^*)$ and $h_1 \in H_1(Y^*)$ with

$$H_1(Y^*) = \{u \in L^2(Y^*)^3,\ \ \mathrm{rot}_y\, u = 0,\ \ \mathrm{div}_y\, u = 0,\ \ \vec{n}(y) \cdot \vec{u}(y) = 0\}$$

such that

$$E^0(x,y) = \nabla_y \varphi(x,y) + h_1(x,y), \tag{3.10}$$

and finally, one can write

$$E^0(x,y) = \chi_{Y^*}(y) \mathbb{G}(y)[E(x)] \tag{3.11}$$

and

$$V(x,y) = \chi_{Y^*}(y) \mathbb{G}(y) \times \mathbb{G}(y)[\mathrm{rot}\, E(x)], \tag{3.12}$$

where $\mathbb{G}$ is the $\prod_3$-valued function defined for $y \in Y^*$ by

$$\mathbb{G}(y) = (\vec{G}_1(y), \vec{G}_2(y), \vec{G}_3(y)) \tag{3.13}$$

and

$$\vec{G}_i(y) = (\nabla \widetilde{\Phi}_i)(y),$$

each $\widetilde{\Phi}_i$ being solution in $H^1(Y^*)$ of

$$\begin{cases} \widetilde{\Phi}_i = y_i - \Upsilon_i, \quad \Upsilon_i\ Y\text{-periodic}, \\ \Delta \widetilde{\Phi}_i = 0 \quad \text{in } Y^*, \\ \dfrac{\partial \Phi_i}{\partial n}\Big|_{\partial T} = 0. \end{cases} \tag{3.14}$$

Note that $\mathrm{rot}_y\, \mathbb{G} = 0$ and $\vec{n}(y) \cdot \mathbb{G}(y) = 0$.

From the variational formulation (3.2), one gets $\forall \vec{\psi} \in (C^\infty(\Omega))^3$

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \mathrm{rot}\, \vec{\psi}(x)\, \mathrm{d}x\, \mathrm{d}y + \int_\Omega E(x) \cdot \vec{\psi}(x)\, \mathrm{d}x = \int_\Omega F(x) \cdot \vec{\psi}(x)\, \mathrm{d}x,$$

which by virtue of (3.12) yields

$$\int_\Omega \chi_{Y^*}(y) \mathbb{G}(y) \times \mathbb{G}(y) [\mathrm{rot}_x\, E(x)] \cdot \mathrm{rot}_x\, \vec{\psi}(x)\, \mathrm{d}x + \int_\Omega E(x) \cdot \vec{\psi}(x)\, \mathrm{d}x = \int_\Omega F(x) \cdot \vec{\psi}(x)\, \mathrm{d}x.$$

Since we can set $\int_{Y^*} \mathbb{G}(y) \times \mathbb{G}(y)\, \mathrm{d}y = \eta_1 I$, this concludes the proof of Theorem 2(b).

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*Authors' address*: *R. Alexandre, H. Taha*, MAPMO-UMR 6628, BP 6759, Université d'Orléans, 45067 Orleans Cedex, France, e-mail: **alexandr@labomath.univ-orleans.fr**; **hassan.taha@labomath.univ-orleans.fr**.

# PROPAGATION OF ELECTROMAGNETIC WAVES IN NON-HOMOGENEOUS MEDIA

Radjesvarane Alexandre, Hassan Taha, Orleans Cedex

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*Abstract.* We consider electromagnetic waves propagating in a periodic medium characterized by two small scales. We perform the corresponding homogenization process, relying on the modelling by Maxwell partial differential equations.

*Keywords*: Maxwell equations, homogenization, two-scale convergence, oscillating test functions

*MSC 2000*: 35Q60, 35B27

## 1. Introduction

Homogenization results for Maxwell equations, by the classical method of asymptotic expansions or more recently by two-scale convergence, are well known, see for instance [1], [3], [4], [5], [6], [19]. Let us first recall Maxwell equations framework, see [7], [8], [9]. Let $\Omega \subset \mathbb{R}^3$ be a physical domain. Then the electromagnetic propagation of waves in $\Omega$ is described by four $(\mathbb{R}^3)$ vector valued functions $D$, $E$, $B$, $H$ of $(x,t) \in \Omega \times \mathbb{R}$. Here $D$ is the electric induction, $E$ the electric field, $B$ the magnetic induction and $H$ the magnetic field. Introducing furthermore the charge density $\varrho = \varrho(t,x)$ and the current density $J = J(t,x)$ of charges inside $\Omega$, one has Maxwell equations in the form

$$
\begin{aligned}
&\text{(i)} && -\frac{\partial D}{\partial t} + \operatorname{rot} H = J && \text{Ampere law,}\\
&\text{(ii)} && \frac{\partial B}{\partial t} + \operatorname{rot} E = 0 && \text{Faraday law,}\\
&\text{(iii)} && \operatorname{div} D = \varrho && \text{Gauss electrical law,}\\
&\text{(iv)} && \operatorname{div} B = 0 && \text{Gauss magnetic law.}
\end{aligned}
\tag{1.1}
$$

Note that the current and charge densities satisfy the continuity relation or charge conservation

$$\frac{\partial \varrho}{\partial t} + \mathrm{div} J = 0, \tag{1.2}$$

as in fact follows also from (1.1).

Herein, we assume linear behavior laws, that is proportionality of fields and inductions,

$$D = \alpha E, \qquad B = \mu H. \tag{1.3}$$

Here $\mu$ and $\alpha$ are the magnetic permeability and electric permittivity respectively and these are assumed constant to simplify our exposition. Recall that in general, this linear behavior is not true for usual electromagnetic media and we hope to get back to this non linear aspect (in this direction see however [9]).

Moreover, we simplify Maxwell equations by considering the very special harmonic case (note that $J$ should satisfy some kind of Ohmic law). However, let us mention that most if not all of our results below could be extended easily to cover the time dependent case.

Let us recall quickly these facts, referring for much more details to standard books such as [7], [8], [9].

Since we are interested in time-periodic solutions only, we look for solutions in the form

$$D(x,t) = \Re(\exp(\mathrm{i}\omega t) D(x)), \quad H(x,t) = \Re(\exp(\mathrm{i}\omega t) H(x)), \tag{1.4}$$

$$J(x,t) = \Re(\exp(\mathrm{i}\omega t) J(x)) \tag{1.5}$$

and

$$B(x,t) = \Re(\exp(\mathrm{i}\omega t) B(x)), \quad E(x,t) = \Re(\exp(\mathrm{i}\omega t) E(x)). \tag{1.6}$$

We denote the new complex-valued functions of a variable $x$ with the same capital letters as the original real-valued functions of variables $x$, $t$. Then equation (1.1(ii)), divided by $\exp(\mathrm{i}\omega t)$, becomes (with $\mathrm{i}^2 = -1$)

$$-\mathrm{i}\omega D(x) + \mathrm{rot}_x H(x) = J(x) \tag{1.7}$$

and equation (1.1(ii)) becomes

$$\mathrm{i}\omega B(x) + \mathrm{rot}_x E(x) = 0, \tag{1.8}$$

which by applying the rot operator leads to

$$\mathrm{i}\omega \operatorname{rot}_x B(x) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.9}$$

From (1.3), one has $\operatorname{rot}_x B(x) = \mu \operatorname{rot}_x H(x)$, and thus it follows that

$$\mathrm{i}\omega\mu \operatorname{rot}_x H(x) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.10}$$

Using (1.7), one gets

$$\mathrm{i}\omega\mu(J(x) + \mathrm{i}\omega D(x)) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.11}$$

Using (1.3), one has

$$\mathrm{i}\omega\mu(J(x) + \mathrm{i}\omega\alpha E(x)) + \operatorname{rot}_x(\operatorname{rot}_x E(x)) = 0. \tag{1.12}$$

Hence

$$\operatorname{rot}_x(\operatorname{rot}_x E(x)) + \mathrm{i}^2\omega^2\mu\alpha E(x) = -\mathrm{i}\omega\mu J(x).$$

Usually $J = \sigma E + J'$ (where $\sigma$ is the conductivity), so that

$$\operatorname{rot}_x(\operatorname{rot}_x E(x)) + (\mathrm{i}^2\omega^2\mu\alpha + \mathrm{i}\mu\omega\sigma)E(x) = -\mathrm{i}\omega\mu J'(x).$$

Let

$$F(x) = -\mathrm{i}\omega\mu J'(x), \quad \gamma = \mu\omega(-\omega\alpha + \mathrm{i}\sigma). \tag{1.13}$$

To avoid any mathematical problems, we will always assume that $\Re(\gamma) > 0$. In this direction, we refer to known standard difficulties associated with Maxwell equations in [9].

Then, we are finally led to the standard harmonic Maxwell equation

$$\operatorname{rot}(\operatorname{rot} E(x)) + \gamma E(x) = F(x). \tag{P}$$

Let $\Omega$ be a bounded regular open set in $\mathbb{R}^3$. Let $\varepsilon > 0$ be a small parameter and $\Omega^\varepsilon$ a bounded open set $\Omega^\varepsilon \subseteq \Omega \subseteq \mathbb{R}^3$, to be specified below.

We consider for $\gamma \in \mathbb{C}$, $\Re(\gamma) > 0$, the homogenization of problem (P) in $\Omega^\varepsilon$ with different boundary conditions on $\partial\Omega^\varepsilon$. That is, electromagnetic waves are propagating in $\Omega^\varepsilon$ and we prescribe the interactions with boundary $\partial\Omega^\varepsilon$.

More precisely, we consider the following two problems ($\mathrm{P}^\varepsilon$) and ($\mathrm{Q}^\varepsilon$), with $F^\varepsilon(x) \in L^2(\Omega^\varepsilon)^3$:

$$(\mathrm{P}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \wedge E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon \end{cases}$$

and

$$(\mathrm{Q}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \wedge \operatorname{rot} E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon. \end{cases}$$

The boundary condition in problem ($\mathrm{P}^\varepsilon$) describes the physical fact that the complementary set to $\Omega^\varepsilon$ behaves as a perfect conductor, while the boundary condition in problem ($\mathrm{Q}^\varepsilon$) could be interpreted as the absence of magnetic charges within the complementary set to $\Omega^\varepsilon$ .

A natural question arises in this electromagnetic setting as to investigate closely related problems such as

$$(\mathrm{R}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) - \operatorname{grad}(\operatorname{div} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \wedge E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon \end{cases}$$

or

$$(\mathrm{S}^\varepsilon) \qquad \begin{cases} \operatorname{rot}(\operatorname{rot} E^\varepsilon(x)) - \operatorname{grad}(\operatorname{div} E^\varepsilon(x)) + \gamma E^\varepsilon(x) = F^\varepsilon(x), & x \in \Omega^\varepsilon, \\ \vec{n}^\varepsilon(x) \cdot E^\varepsilon(x) = 0, & x \in \partial\Omega^\varepsilon. \end{cases}$$

Similarly, the last boundary condition reflects the absence of electric charges within the complementary set to $\Omega^\varepsilon$.

Such problems will not be studied herein in order to limit the length of the paper, and most importantly because test functions to be used are rather different. These important problems will be discussed in a forthcoming paper.

Let $\varepsilon > 0$ and $r_\varepsilon > 0$ be two small parameters such that $r_\varepsilon < \frac{1}{2}\varepsilon$ and consider the covering of $\mathbb{R}^3$ by cells

$$P_k^\varepsilon = \varepsilon\Big[-\frac{1}{2}, +\frac{1}{2}\Big]^3 + \varepsilon k,$$

where $k \in \mathbb{Z}^3$. In each cell $P_k^\varepsilon$ we remove the ball

$$T_k^\varepsilon = r_\varepsilon B_1 + \varepsilon k,$$

where $B_1$ denotes the unit ball in $\mathbb{R}^3$. We let

$$\Omega^\varepsilon = \Omega - T^\varepsilon$$

where $T^\varepsilon = \bigcup_{k=1}^{N(\varepsilon)} T_k^\varepsilon$ denotes the union of balls strictly included within $\Omega$. As a matter of fact, note that these balls do not intersect the boundary $\Omega$.

We will study the above problems under one of the scaling assumptions

$$\text{(HYP)}_1 \qquad \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = 0$$

or

$$\text{(HYP)}_2 \qquad r_\varepsilon = c\varepsilon$$

where $0 < c < 1/4$ is a strictly positive fixed constant.

Finally, we will always assume (at least) that

$$\text{(HYP)}_3 \qquad \widetilde{F^\varepsilon} \rightharpoonup F \quad \text{weakly in } \ L^2(\Omega)^3$$

as $\varepsilon \to 0$. Above and throughout the paper the tilde over a symbol denotes the extension by zero in the holes.

We will be concerned with the asymptotic behavior as $\varepsilon \to 0$ of problems $(\mathrm{P}^\varepsilon)$ and $(\mathrm{Q}^\varepsilon)$.

Before stating the corresponding mathematical results, let us explain our motivation. The periodic perforated medium $\Omega^\varepsilon$ may be considered as one having two distinct di-electric constants or even holes that may be considered as charged particles.

However, let us remark that problems $(\mathrm{P}^\varepsilon)$ or $(\mathrm{Q}^\varepsilon)$ studied herein do not really fit with a clear (or true) physical modeling of propagation of electromagnetic waves in such non homogeneous media.

There are at least two points from the physical theory of Maxwell equations which are not really accounted for.

On the one hand, there is definitively a scaling problem between frequency and spatial scales which is not considered here. From mathematical viewpoint, this question leads to very deep technics, in progress actually. To be short, we are assuming that spatial variations are small compared to wavelenghts.

On the other hand, and this is surely one main point, if one views to the holes as charged particles, the modeling by problems $(\mathrm{P}^\varepsilon)$ or $(\mathrm{Q}^\varepsilon)$ is of course not true. One should for instance get back to Maxwell equations, say in harmonic form, and take care of the correct scaling and boundary conditions, involving scattering operators, see for instance [9].

Last but not least, in view of all these facts, one should note that actually the right-hand member data given are surely first order distributions, and so some if not all of our constructions in this paper would have to be modified.

Having such ideas in mind, the above problems must therefore be considered a first step in order to tackle real physical problems, maybe as a background to test mathematical tools available at present.

Even in these simple and academic problems, some constructions displayed in this paper have to be adapted to more complex situations, and this is why we have started studying them.

Some of the defects mentioned above are actually worked out.

Our results show in particular that the homogenized problems display a different output frequency with respect to $\omega$. More precisely, they are the following, see the notation below.

In section 2 we refer the following as regards to the homogenization of problem $(\mathrm{P}^\varepsilon)$.

**Theorem 1.** *Homogenization of problem* $(\mathrm{P}^\varepsilon)$.

(a) *With assumptions* $(\mathrm{HYP})_1$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{P}^\varepsilon)$, *one has (up to a subsequence)*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } H_0(\mathrm{rot}, \Omega),$$

*where* $E$ *is the variational solution of problem* $(\mathrm{P})$, *with the condition* $\vec{n} \wedge E = 0$ *on* $\partial\Omega$.

(b) *Under assumptions* $(\mathrm{HYP})_2$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{P}^\varepsilon)$, *one has (up to a subsequence)*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } H_0(\mathrm{rot}, \Omega),$$

*where* $E$ *is the (variational) solution in* $H_0(\mathrm{rot}, \Omega)$ *of*

$$\int_\Omega \eta \,\mathrm{rot}_x\, E(x) \cdot \mathrm{rot}_x\, \psi(x)\,\mathrm{d}x + \gamma\zeta \int_\Omega E(x) \cdot \psi(x)\,\mathrm{d}x = \zeta \int_\Omega F(x)\psi(x)\,\mathrm{d}x$$

*for all* $\psi \in H_0(\mathrm{rot}, \Omega)$, *where* $\eta > 0$, $\zeta$ *are two constants (given in Section* 2*).*

Section 3 is devoted to problem $(\mathrm{Q}^\varepsilon)$. We prove

**Theorem 2.** *Homogenization of problem* $(\mathrm{Q}^\varepsilon)$.

(a) *Under assumptions* $(\mathrm{HYP})_1$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{Q}^\varepsilon)$ *one has (up to a subsequence)*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } L^2(\Omega)^3,$$

*where* $E$ *is the variational solution of problem* $(\mathrm{P})$ *with the condition* $\vec{n} \wedge \mathrm{rot}\, E = 0$ *on* $\partial\Omega$.

(b) *With assumptions* $(\mathrm{HYP})_2$ *and* $(\mathrm{HYP})_3$ *for* $(\mathrm{Q}^\varepsilon)$ *one has*

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \textit{weakly in } L^2(\Omega)^3,$$

*where* $E$ *is the variational solution, for all* $\psi \in H(\mathrm{rot}, \Omega)$, *of*

$$\int_\Omega \eta_1 \,\mathrm{rot}_x\, E(x) \cdot \mathrm{rot}_x\, \psi(x)\,\mathrm{d}x + \gamma \int_\Omega E(x) \cdot \psi(x)\,\mathrm{d}x = \int_\Omega F(x)\psi(x)\,\mathrm{d}x,$$

*where* $\eta_1 > 0$ *is a constant (given in Section* 3*)* .

Proofs of these results are done in Sections 2 and 3 for $(\mathrm{P}^\varepsilon)$ and $(\mathrm{Q}^\varepsilon)$ respectively. One can also find therein some remarks about the behavior of global electromagnetic energy. It is an interesting question to ask about the corresponding behavior of the *local* energy, which seems much more crucial for physical problems.

Also, the same type of results holds if we assume $r^\varepsilon/\varepsilon \to c > 0$ instead of assuming exactly that $r^\varepsilon = c\varepsilon$. However, we skip the proofs of this fact.

We end this introductory section by recalling standard materials on mathematics related to problem (P).

We introduce the following standard notation, see for instance [7], [9], [13]:

$$H^\varepsilon = L^2(\Omega^\varepsilon)^3,$$
$$H(\mathrm{rot}, \Omega^\varepsilon) = \{E \in L^2(\Omega^\varepsilon)^3;\ \mathrm{rot}\, E \in L^2(\Omega^\varepsilon)^3\},$$

with the usual norm $\|E\|_{H(\mathrm{rot},\Omega^\varepsilon)} = \|E\|_{L^2(\Omega^\varepsilon)^3} + \|\mathrm{rot}\, E\|_{L^2(\Omega^\varepsilon)^3}$.

$H(\mathrm{rot}, \Omega^\varepsilon)$ is a well known Hilbert space. In order to tackle the first boundary condition, that is problem $(\mathrm{P}^\varepsilon)$ with a perfect conductor type boundary condition, we also introduce

$$V_0^\varepsilon \equiv H_0(\mathrm{rot}, \Omega^\varepsilon) = \{E \in H(\mathrm{rot}, \Omega^\varepsilon),\ \ n^\varepsilon(x) \wedge E(x) = 0 \ \text{ on } \ \partial\Omega^\varepsilon\}.$$

Of course, $V_0^\varepsilon$ is a closed space in $H(\mathrm{rot}, \Omega^\varepsilon)$. The corresponding variational formulation for problem $(\mathrm{P}^\varepsilon)$ is then naturally given for all $\vec{\varphi} \in H_0(\mathrm{rot}, \Omega^\varepsilon)$ by

$$a^\varepsilon(E^\varepsilon, \vec{\varphi}) \equiv \int_{\Omega^\varepsilon} \mathrm{rot}\, E^\varepsilon \cdot \mathrm{rot}\, \vec{\varphi}\,\mathrm{d}x + \gamma \int_{\Omega^\varepsilon} E^\varepsilon \cdot \vec{\varphi}\,\mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon \cdot \vec{\varphi}\,\mathrm{d}x. \tag{1.14}$$

We recall that the sesquilinear form $a(u, v)$ is coercive on $V_0^\varepsilon$: there exists $\beta > 0$ such that $\Re a(u, u) \geqslant \beta\|u\|_{V_0^\varepsilon}$, $\forall u \in V_0^\varepsilon$.

By virtue of the above facts and the Green formula, the corresponding operator $A^\varepsilon$ is characterized by

$$A^\varepsilon E = \operatorname{rot}(\operatorname{rot} E) + \gamma E$$

and

$$D(A^\varepsilon) = \{E \in V^\varepsilon;\ \operatorname{rot}(\operatorname{rot} E) \in H^\varepsilon\}.$$

Since $(V_0^\varepsilon)'$ is a space of distributions and since the sesquilinear form $a$ given above is coercive on $V_0^\varepsilon$ with the constant $\beta = \inf(1, \gamma)$, there is a unique solution $E^\varepsilon$ in $V_0^\varepsilon$ of ($\mathrm{P}^\varepsilon$) thanks to the Lax-Milgram Lemma.

For problem ($\mathrm{Q}^\varepsilon$) we put $V^\varepsilon = H(\operatorname{rot}, \Omega^\varepsilon)$, and this defines the same operator $A^\varepsilon$ but with

$$D(A^\varepsilon) = \{E \in V^\varepsilon;\ \operatorname{rot}(\operatorname{rot} E) \in H^\varepsilon,\ \ n^\varepsilon(x) \wedge \operatorname{rot} E(x) = 0 \quad \text{on}\ \ \partial\Omega^\varepsilon\}.$$

## 2. Proof of Theorem 1

### 2.1. Proof of Theorem 1(a)

We divide the proof in several steps.

*First step: Variational formulation and uniform estimates*

The corresponding variational formulation, according to the previous section, is given for all $\vec{\varphi}^\varepsilon \in H_0(\operatorname{rot}, \Omega^\varepsilon)$ by

$$\int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon \cdot \operatorname{rot} \vec{\varphi}^\varepsilon \,\mathrm{d}x + \gamma \int_{\Omega^\varepsilon} E^\varepsilon \cdot \vec{\varphi}^\varepsilon \,\mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon \cdot \vec{\varphi}^\varepsilon \,\mathrm{d}x. \tag{2.1}$$

One gets easily uniform estimates for $E^\varepsilon$ and $\operatorname{rot} E^\varepsilon$ from (2.1) by taking $\vec{\varphi} := E^\varepsilon$, hence one obtains

$$\|E^\varepsilon\|_{V^\varepsilon} \leqslant c, \tag{2.2}$$

where $c$ is a constant independent of $\varepsilon$.

We recall that if $E^\varepsilon \in H_0(\operatorname{rot}, \Omega^\varepsilon)$ and

$$H_0(\operatorname{rot}, \Omega) = \{U \in (L^2(\Omega))^3,\ \ \operatorname{rot}_x U \in (L^2(\Omega))^3,\ \ \vec{n} \wedge \vec{U}(x) = 0 \ \text{on}\ \ \partial\Omega\}$$

then $E^\varepsilon$ satisfies $\widetilde{E^\varepsilon} \in H_0(\mathrm{rot}, \Omega)$ and $\mathrm{rot}\, \widetilde{E^\varepsilon} = \widetilde{\mathrm{rot}\, E^\varepsilon}$. Hence we deduce that $\widetilde{E^\varepsilon}$ belongs to a bounded subset of $H_0(\mathrm{rot}, \Omega)$. All in all, one has (up to a subsequence)

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \text{weakly in } L^2(\Omega)^3, \tag{2.3}$$

$$\widetilde{\mathrm{rot}\, E^\varepsilon} \rightharpoonup \mathrm{rot}\, E \quad \text{weakly in } L^2(\Omega)^3 \tag{2.4}$$

and, for all $\vec{\varphi}^\varepsilon \in H_0(\mathrm{rot}, \Omega^\varepsilon)$, one has

$$\int_\Omega \mathrm{rot}\, \widetilde{E^\varepsilon}(x) \cdot \mathrm{rot}\, \vec{\varphi}^\varepsilon(x)\, \mathrm{d}x + \gamma \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \vec{\varphi}^\varepsilon(x)\, \mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x) \cdot \vec{\varphi}^\varepsilon(x)\, \mathrm{d}x. \tag{2.5}$$

Following the classical method introduced by L. Tartar, see [17], we are going to take a test function of the form

$$\vec{\varphi}^\varepsilon(x) = \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)], \tag{2.6}$$

where $x \longmapsto \vec{\psi}(x) \in (C_c^\infty(\Omega))^3$ and $x \longmapsto \mathbb{W}^\varepsilon(x)$ is a $3 \times 3$ matrix valued function given by

$$\mathbb{W}^\varepsilon(x) = (\vec{W}_1^\varepsilon(x), \vec{W}_2^\varepsilon(x), \vec{W}_3^\varepsilon(x)),$$

that is, $\vec{W}_i^{\,\varepsilon}(x)$ is the $i$th column of the matrix $\mathbb{W}^\varepsilon(x)$.

To simplify the exposition, we introduce some matrix operations. We set $\prod_3$ for the set of matrix of order 3.

*Second step: Definitions and formulas*

a) Let $x \longmapsto \mathbb{U}(x)$ and $x \longmapsto \mathbb{V}(x)$ be two $\prod_3$-valued functions

$$\mathbb{U}(x) = (\vec{U}_1(x), \vec{U}_2(x), \vec{U}_3(x))$$

and

$$\mathbb{V}(x) = (\vec{V}_1(x), \vec{V}_2(x), \vec{V}_3(x)).$$

Then we define an ($\mathbb{R}^3$) vector-valued function $x \longmapsto (\mathbb{U} \wedge \mathbb{V})(x)$ by

$$(\mathbb{U} \wedge \mathbb{V})(x) = \sum_{i=1}^{3} \vec{U}_i(x) \wedge \vec{V}_i(x). \tag{2.7}$$

b) With the same notation, if $x \longmapsto \mathbb{U}(x)$ is a $\prod_3$-valued function, one defines a $\prod_3$-valued function $x \longmapsto \mathrm{rot}\, \mathbb{U}(x)$ by

$$\mathrm{rot}\, \mathbb{U}(x) = (\mathrm{rot}\, \vec{U}_1(x), \mathrm{rot}\, \vec{U}_2(x), \mathrm{rot}\, \vec{U}_3(x)). \tag{2.8}$$

c) Note that, for $x \longmapsto \mathbb{U}(x)$ and $x \longmapsto \mathbb{V}(x)$ two $\prod_3$-valued functions, we get

$$(2.9) \qquad (\operatorname{rot} \mathbb{U}) \wedge \mathbb{V}(x) = \sum_{i=1}^{3} (\operatorname{rot} \vec{U}_i(x)) \wedge \vec{V}_i(x).$$

d) Let $x \longmapsto \vec{n}(x)$ be an ($\mathbb{R}^3$) vector-valued function. Then we define a $\prod_3$-valued function $x \longmapsto (\mathbb{U} \wedge \vec{n})(x)$ by

$$(2.10) \qquad (\mathbb{U} \wedge \vec{n})(x) = (\vec{U}_1(x) \wedge \vec{n}(x), \vec{U}_2(x) \wedge \vec{n}(x), \vec{U}_3(x) \wedge \vec{n}(x))$$

and similarly for $\vec{n} \wedge \mathbb{U}$.

Next, we state some lemmas which will be useful for the proof of Theorem 1.

**Lemma 1.** *Let $\mathbb{U}$ and $\mathbb{V}$ be as in* a) *and smooth. Then*

$$(2.11) \qquad \operatorname{div}(\mathbb{U} \wedge \mathbb{V})(x) = \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) - \mathbb{U}(x) \cdot \operatorname{rot} \mathbb{V}(x),$$

*where "$\cdot$" means the scalar product of ($\mathbb{R}^3$) vectors.*

P r o o f. We use definition a), and compute

$$\begin{aligned}
\operatorname{div}(\mathbb{U} \wedge \mathbb{V})(x) &= \operatorname{div}\left(\sum_{i=1}^{3} \vec{U}_i(x) \wedge \vec{V}_i(x)\right) = \sum_{i=1}^{3} \operatorname{div}(\vec{U}_i(x) \wedge \vec{V}_i(x)) \\
&= \sum_{i=1}^{3} (\vec{V}_i(x) \cdot \operatorname{rot} \vec{U}_i(x) - \vec{U}_i(x) \cdot \operatorname{rot} \vec{V}_i(x)) \\
&= \sum_{i=1}^{3} \vec{V}_i(x) \cdot \operatorname{rot} \vec{U}_i(x) - \sum_{i=1}^{3} \vec{U}_i(x) \cdot \operatorname{rot} \vec{V}_i(x) \\
&= \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) - \mathbb{U}(x) \cdot \operatorname{rot} \mathbb{V}(x).
\end{aligned}$$

$\square$

**Lemma 2** (Green Formula)**.** *Let $\mathbb{U}$ and $\mathbb{V}$ be as in* a) *and smooth, $\Omega \subseteq \mathbb{R}^3$ a regular bounded open set in $\mathbb{R}^3$. Then*

$$(2.12) \qquad \int_{\partial\Omega} (\mathbb{U} \wedge \mathbb{V})(x) \cdot \vec{n}(x) \,\mathrm{d}\Gamma = \int_{\Omega} \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) \,\mathrm{d}x - \int_{\Omega} \mathbb{U}(x) \cdot \operatorname{rot} \mathbb{V}(x) \,\mathrm{d}x$$

*where "$\cdot$" denotes the scalar product of* $3 \times 3$ *matrices on the right-hand side.*

P r o o f. Using integration by parts in (2.11), one has

$$\int_{\Omega} \operatorname{div}(\mathbb{U} \wedge \mathbb{V})(x) \,\mathrm{d}x = \int_{\Omega} \mathbb{V}(x) \cdot \operatorname{rot} \mathbb{U}(x) \,\mathrm{d}x - \int_{\Omega} \mathbb{U}(x). \operatorname{rot} \mathbb{V}(x) \,\mathrm{d}x$$

and thus (2.12) follows. $\square$

**Lemma 3.** *Let* $\mathbb{U}$ *be as in* a) *and* $\vec{\psi}(x) = (\psi_1(x), \psi_2(x), \psi_3(x))$ *a vector-valued function* $\vec{\psi} \in (C_c^\infty(\Omega))^3$*. Then*

$$\operatorname{rot}(\mathbb{U}(x) \cdot [\vec{\psi}(x)]) = (\operatorname{rot} \mathbb{U}(x))[\vec{\psi}(x)] + \mathbb{U}(x) \wedge \nabla(\psi_1(x), \psi_2(x), \psi_3(x)). \tag{2.13}$$

Proof. Using definition b), one computes

$$\begin{aligned}
\operatorname{rot}(\mathbb{U}(x) \cdot [\vec{\psi}(x)]) &= \operatorname{rot}\left(\sum_{i=1}^{3} \vec{U}_i(x)\psi_i(x)\right) \\
&= \sum_{i=1}^{3} (\operatorname{rot} \vec{U}_i(x))\psi_i(x) + \sum_{i=1}^{3} \vec{U}_i(x) \wedge \nabla\psi_i(x) \\
&= (\operatorname{rot} U_1(x), \operatorname{rot} U_2(x), \operatorname{rot} U_3(x))[\vec{\psi}(x)] \\
&\quad + (\vec{U}_1(x), \vec{U}_2(x), \vec{U}_3(x)) \wedge (\nabla\psi_1(x), \nabla\psi_2(x), \nabla\psi_3(x)) \\
&= \operatorname{rot}(\mathbb{U}(x))[\vec{\psi}(x)] + \mathbb{U} \wedge \nabla\vec{\psi}(x)
\end{aligned}$$

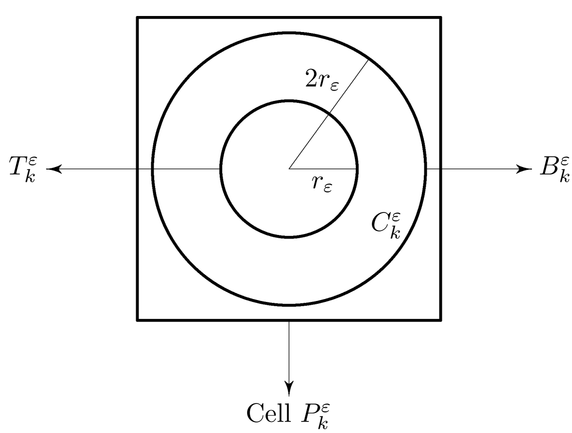
with $\nabla\vec{\psi}(x) = (\nabla\psi_1(x), \nabla\psi_2(x), \nabla\psi_3(x))$. □

*Third step: Oscillating function*

We are going to construct a $\prod_3$-valued function $x \longmapsto \mathbb{W}^\varepsilon(x)$, appearing in (2.6), as follows.

For each cell $P_k^\varepsilon$ included in $\Omega$, we construct $\mathbb{W}^\varepsilon$ in one typical cell and repeat the process by $\varepsilon$ periodicity. In the cells not strictly included in $\Omega$, we simply set $\mathbb{W}^\varepsilon$ as equal to Id (Identity matrix of order 3).

Let $P_k^\varepsilon$ be a cell strictly included in $\Omega$. Recall that $\partial T_k^\varepsilon$ denotes the boundary of the hole and has radius $r_\varepsilon$. We consider the hole centered at $\varepsilon k$ and radius $2r_\varepsilon$, denoted by $B_k^\varepsilon$.



Therefore, we have divided the cell $P_k^\varepsilon$ in three subregions: the ball $T_k^\varepsilon$ centered at $\varepsilon k$ and radius $r_\varepsilon$, the circular annulus $C_k^\varepsilon$ of small radius $r_\varepsilon$ and large radius $2r_\varepsilon$ and the exterior region $P_k^\varepsilon - (T_k^\varepsilon \cup C_k^\varepsilon)$. In $T_k^\varepsilon$, we set $\mathbb{W}^\varepsilon = 0$ (null matrix). In $P_k^\varepsilon - (T_k^\varepsilon \cup C_k^\varepsilon)$ we set $\mathbb{W}^\varepsilon = \mathrm{Id}$.

It remains to specify $\mathbb{W}^\varepsilon$ in $C_k^\varepsilon$ and we look (for reasons to be explained below) for $\mathbb{W}^\varepsilon$ such that

$$(2.14) \qquad \begin{cases} \operatorname{rot}_x \mathbb{W}^\varepsilon = 0, & x \in C_k^\varepsilon, \\ n^\varepsilon \wedge \mathbb{W}^\varepsilon = 0, & x \in \partial T_k^\varepsilon, \\ n^\varepsilon \wedge \mathbb{W}^\varepsilon = n^\varepsilon \wedge \mathrm{Id}, & x \in \partial B_k^\varepsilon. \end{cases}$$

Since $C_k^\varepsilon$ has exactly a scale of $r_\varepsilon$ units, it is enough to look for $\mathbb{W} = \mathbb{W}(y)$ (matrix-valued) defined for $y \in C = \{y \in \mathbb{R}^3,\ 1 \leqslant |y| \leqslant 2\}$, and satisfying

$$(2.15) \qquad \begin{cases} \operatorname{rot}_y \mathbb{W} = 0, & y \in C, \\ n \wedge \mathbb{W} = 0, & |y| = 1, \\ n \wedge \mathbb{W} = n \wedge \mathrm{Id}, & |y| = 2, \end{cases}$$

and then in each $C_k^\varepsilon$ define $\mathbb{W}^\varepsilon$ by $\mathbb{W}^\varepsilon(x) = \mathbb{W}(x - \varepsilon k/r_\varepsilon)$, which clearly satisfies (2.14) if $\mathbb{W}$ satisfies (2.15). In fact, we shall display explicitly $\mathbb{W}$. First, note that $\operatorname{rot} \mathrm{Id} = 0$.

On the other hand, let $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $|y|^2 = y_1^2 + y_2^2 + y_3^2$. Then

$$\operatorname{rot}(|y|^2 \mathrm{Id}) = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \wedge \begin{pmatrix} |y|^2 & 0 & 0 \\ 0 & |y|^2 & 0 \\ 0 & 0 & |y|^2 \end{pmatrix} = \begin{pmatrix} 0 & -2y_3 & 2y_2 \\ 2y_3 & 0 & -2y_1 \\ -2y_2 & 2y_1 & 0 \end{pmatrix}$$

and as $(\vec{n}(y) \otimes \vec{n}(y)) \wedge \vec{n}(y) = 0$, where $\vec{n}(y) = y/|y|$, we find adjusting coefficients that

$$(2.16) \qquad \mathbb{W}(y) = \frac{1}{3}(|y|^2 - 1)\,\mathrm{Id} + \frac{2}{3}\vec{n}(y) \otimes \vec{n}(y)$$

does the job.

*Fourth step: Properties of* $\mathbb{W}^\varepsilon$

First, let us show that $\mathbb{W}^\varepsilon \longrightarrow \mathrm{Id}$ strongly in $L^2(\Omega)$ as $\varepsilon \longrightarrow 0$. Indeed, $C^\varepsilon$ being the union of the $C_k^\varepsilon$'s and $T^\varepsilon$ the union of the $T_k^\varepsilon$, one has

$$\begin{aligned}
\|\mathbb{W}^\varepsilon - \mathrm{Id}\,\|_{L^2(\Omega)} &= \|\mathbb{W}^\varepsilon - \mathrm{Id}\,\|_{L^2(T^\varepsilon)} + \|\mathbb{W}^\varepsilon - \mathrm{Id}\,\|_{L^2(C^\varepsilon)} \\
&= \|\,\mathrm{Id}\,\|_{L^2(T^\varepsilon)} + \|\mathbb{W}^\varepsilon - \mathrm{Id}\,\|_{L^2(C^\varepsilon)} \\
&\leqslant c_1 \,\mathrm{mes}(T^\varepsilon) + \int_{C^\varepsilon} |\mathbb{W}^\varepsilon(x) - \mathrm{Id}\,|^2 \,\mathrm{d}x \\
&\leqslant c_1 (r_\varepsilon)^3 \frac{\mathrm{mes}(\Omega)^\varepsilon}{\varepsilon^3} + \int_{C^\varepsilon} |\mathbb{W}^\varepsilon(x) - \mathrm{Id}\,|^2 \,\mathrm{d}x \\
&\leqslant c_2 \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3 + \frac{c_3}{\varepsilon^3} \int_{C_0^\varepsilon} \Big|\mathbb{W}\Big(\frac{x}{r_\varepsilon}\Big) - \mathrm{Id}\,\Big|^2 \,\mathrm{d}x.
\end{aligned}$$

Setting $x/r_\varepsilon = y$, thus $\mathrm{d}x = (r_\varepsilon)^3 \,\mathrm{d}y$, one has

$$\|\mathbb{W}^\varepsilon - \mathrm{Id}\,\|_{L^2(\Omega)} \leqslant c_2 \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3 + c_3 \Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3 \int_C |\mathbb{W}(y) - \mathrm{Id}\,|^2 \,\mathrm{d}y \leqslant c\Big(\frac{r_\varepsilon}{\varepsilon}\Big)^3.$$

As $\varepsilon \to 0$, we get $\lim\limits_{\varepsilon \to 0} \|\mathbb{W}^\varepsilon - \mathrm{Id}\,\|_{L^2(\Omega)} = 0$.

Next, we wish to show that $\mathrm{rot}_x \,\mathbb{W}^\varepsilon$ is bounded uniformly in $L^2(\Omega)$. In fact, we shall show that $\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0$. Note that there are no discontinuities for $\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x)$ across the various boundaries.

Clearly

$$\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0, \quad x \in T_k^\varepsilon$$

and

$$\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0, \quad x \in P_k^\varepsilon - (T_k^\varepsilon \cup C_k^\varepsilon),$$

and we recall that in $C_k^\varepsilon$,

$$\mathbb{W}^\varepsilon(x) = \mathbb{W}\Big(\frac{x - \varepsilon k}{r_\varepsilon}\Big).$$

Then $\mathrm{rot}_x \,\mathbb{W}^\varepsilon(x) = 0$ follows from $\mathrm{rot}_y \,\mathbb{W}(y) = 0$ which is the case as

$$\mathrm{rot}_y \,\mathbb{W}(y) = \mathrm{rot}_y \frac{1}{3}(|y|^2 - 1)\,\mathrm{Id} + \frac{2}{3}\vec{n}(y) \otimes \vec{n}(y) = 0$$

since

$$\mathrm{rot}_y \Big(\frac{1}{3}(|y|^2 \,\mathrm{Id})\Big) = -\frac{2}{3}\,\mathrm{Id} \quad \text{and} \quad \frac{2}{3}\,\mathrm{rot}_y(\vec{n}(y) \otimes \vec{n}(y)) = \frac{2}{3}\,\mathrm{Id}\,.$$

Thus $\mathrm{rot}_x \,\mathbb{W}^\varepsilon$ being null is of course bounded uniformly in $L^2(\Omega)$.

*Fifth step: Passing to the limit*

We return to the variational formulation (2.5) and replace $\vec{\varphi}^\varepsilon$ by

$$\vec{\varphi}^\varepsilon(x) = \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]$$

for all but fixed $\vec{\psi} \in (C_c^\infty(\Omega))^3$. After extending $E^\varepsilon$ by zero in the holes, and using the matrix formulas and convergence properties given above, one obtains

$$\begin{aligned}(2.17)\qquad &\int_\Omega \operatorname{rot} \widetilde{E^\varepsilon} \cdot \operatorname{rot} \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x \\ &\quad + \int_\Omega \operatorname{rot} \widetilde{E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon(x) \wedge (\partial\psi_1(x), \partial\psi_2(x), \partial\psi_3(x)) \\ &\quad + \gamma \int_\Omega \widetilde{E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x.\end{aligned}$$

As $\varepsilon \to 0$, we get

$$\begin{aligned}&-\int_\Omega \operatorname{rot} E(x) \cdot \operatorname{Id} \wedge(\partial\psi_1(x), \partial\psi_2(x), \partial\psi_3(x))\,\mathrm{d}x + \gamma \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x \\ &\qquad = \int_\Omega \widetilde{F}(x).\vec{\psi}(x)\,\mathrm{d}x,\end{aligned}$$

thus

$$\int_\Omega \operatorname{rot} E(x) \cdot \operatorname{rot} \vec{\psi}(x)\,\mathrm{d}x + \gamma \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \int_\Omega F(x) \cdot \vec{\psi}(x)\,\mathrm{d}x$$

for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$, and this is exactly the conclusion of Theorem 1(a).

### 2.2. Proof of Theorem 1(b)

*First step: Uniform estimates*

As in the proof of Theorem 1(a) above, one has $\|E^\varepsilon\|_{V^\varepsilon} \leqslant c$.

*Second step: Two-Scale convergence*

We use standard notation and set $Y = \left[-\frac{1}{2}, +\frac{1}{2}\right]^3$ (identified with the periodic cell), and $Y^* = Y - T$ where $T = cB_1$. Since $(\widetilde{E^\varepsilon})$ and $(\widetilde{\operatorname{rot} E^\varepsilon})$ are bounded in $L^2(\Omega)^3$, see [1], [12], [14], up to a subsequence, they two-scale converge respectively to $E^0(x,y)$ and $V(x,y)$ belonging to $(L^2(\Omega \times Y))^3$. That is, for any $\vec{\psi}(x,y) \in$

$D(\Omega; C_p^\infty(Y))^3$ we have

$$\widetilde{E^\varepsilon} \rightharpoonup E \quad \text{weakly in } (L^2(\Omega))^3, \tag{2.18}$$

$$\operatorname{rot} \widetilde{E^\varepsilon} \rightharpoonup \operatorname{rot} E \quad \text{weakly in } (L^2(\Omega))^3, \tag{2.19}$$

$$\lim_{\varepsilon \to 0} \int_\Omega \widetilde{E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) \, \mathrm{d}x = \int_\Omega \int_Y E^0(x,y) \cdot \vec{\psi}(x,y) \, \mathrm{d}x \, \mathrm{d}y, \tag{2.20}$$

$$\int_Y E^0(x,y) \, \mathrm{d}y = E(x),$$

$$\lim_{\varepsilon \to 0} \int_\Omega \operatorname{rot} \widetilde{E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) \, \mathrm{d}x = \int_\Omega \int_{Y^*} V(x,y) \cdot \vec{\psi}(x,y) \, \mathrm{d}x \, \mathrm{d}y, \tag{2.21}$$

with

$$\int_{Y^*} V(x,y) \, \mathrm{d}y = \operatorname{rot} E. \tag{2.22}$$

We note that $\widetilde{E^\varepsilon}$ and $\widetilde{\operatorname{rot} E^\varepsilon}$ are equal to zero in $\Omega - \Omega^\varepsilon$.

Therefore, their two-scale limits $E^0(x,y)$ and $V(x,y)$ are equal to zero if $y \in Y - Y^*$.

Next, let $\vec{\psi} = \vec{\psi}(x,y)$ be a smooth function with support in $\overline{Y}^*$ with respect to the variable $y$. It follows that

$$\int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) \, \mathrm{d}x = \int_{\Omega^\varepsilon} E^\varepsilon \cdot \Big[\operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \frac{1}{\varepsilon} \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big] \, \mathrm{d}x,$$

thus

$$\varepsilon \int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) \, \mathrm{d}x = \int_{\Omega^\varepsilon} E^\varepsilon \cdot \Big[\varepsilon \operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big] \, \mathrm{d}x,$$

and extending by zero in the holes we obtain

$$\varepsilon \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon} \cdot \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) \, \mathrm{d}x = \int_\Omega \widetilde{E^\varepsilon} . \Big[\varepsilon \operatorname{rot}_x \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big) + \operatorname{rot}_y \vec{\psi}\Big(x, \frac{x}{\varepsilon}\Big)\Big] \, \mathrm{d}x.$$

Sending $\varepsilon \longrightarrow 0$ and using two-scale convergence, one has

$$0 = \int_\Omega \int_{Y^*} E^0(x,y) \operatorname{rot}_y \vec{\psi}(x,y) \, \mathrm{d}x \, \mathrm{d}y.$$

Hence

$$\operatorname{rot}_y E^0(x,y) = 0 \quad \text{in } D'(\Omega \times Y^*)$$

and one finally gets

$$\begin{cases} \operatorname{rot}_y E^0(x,y) = 0, & y \in Y^*, \\ \vec{n}(y) \wedge E^0(x,y) = 0, & y \in \partial Y^*. \end{cases}$$

From the results of [2], [3], [4], [5], [9], [13] one deduces that ($\chi_{Y^*}$ being the characteristic function of $Y^*$)

$$E^0(x,y) = \chi_{Y^*}(y)\mathbb{U}(y)[E(x)], \tag{2.23}$$

where $\mathbb{U}$ is the $\prod_3$-valued function defined for $y \in Y^*$ by

$$\mathbb{U}(y) = (\vec{U}_1(y), \vec{U}_2(y), \vec{U}_3(y)) \tag{2.24}$$

and

$$\vec{U}_i(y) = (\nabla\Phi_i)(y), \tag{2.25}$$

each $\Phi_i$ being a solution in $H^1(Y^*)$ of

$$\begin{cases} \Phi_i = y_i - \Theta_i, \quad \Theta_i \ Y\text{-periodic}, \\ \Delta\Phi_i = 0 \quad \text{in } Y^*, \\ \Phi_i|_{\partial T} = 0. \end{cases} \tag{2.26}$$

Note that $\mathrm{rot}_y\,\mathbb{U} = 0$ and $\vec{n}(y) \wedge \mathbb{U} = 0$.

We make the first passage to the limit in the variational formulation still given by (2.5)

$$\int_\Omega \mathrm{rot}_x\,\widetilde{E^\varepsilon}(x)\cdot \mathrm{rot}_x\,\vec{\varphi}^\varepsilon(x)\,\mathrm{d}x + \gamma\int_\Omega \widetilde{E^\varepsilon}(x)\cdot\vec{\varphi}^\varepsilon(x)\,\mathrm{d}x = \int_\Omega \widetilde{F^\varepsilon}(x).\vec{\varphi}^\varepsilon(x)\,\mathrm{d}x$$

for all $\vec{\varphi}^\varepsilon \in H_0(\mathrm{rot}, \Omega^\varepsilon)$, by making the precise choice

$$\vec{\varphi}^\varepsilon = \mathbb{U}\Big(\frac{x}{\varepsilon}\Big)[\vec{\psi}(x)], \tag{2.27}$$

where $\mathbb{U}$ is defined by (2.24) for any $\vec{\psi} \in (C_c^\infty(\Omega))^3$ and we obtain

$$\begin{aligned} &\int_\Omega \mathrm{rot}_x\,\widetilde{E^\varepsilon}(x)\cdot\mathrm{rot}_x\{\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\}\,\mathrm{d}x + \gamma\int_\Omega \widetilde{E^\varepsilon}(x)\cdot\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x \\ &\quad = \int_\Omega \widetilde{F^\varepsilon}(x)\cdot\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\,\mathrm{d}x. \end{aligned} \tag{2.28}$$

For the first term on the left hand side and the term on the right-hand side of (2.28) we use two-scale convergence, noticing that

$$\begin{aligned} \mathrm{rot}_x\{\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]\} &= (\mathrm{rot}_x\,\mathbb{U}^\varepsilon(x))[\vec{\psi}(x)] + \mathbb{U}^\varepsilon(x)\wedge\nabla\vec{\psi}(x) \\ &= \frac{1}{\varepsilon}(\mathrm{rot}_y\,\mathbb{U})^\varepsilon[\vec{\psi}] + \mathbb{U}^\varepsilon\wedge\nabla\vec{\psi} = \mathbb{U}^\varepsilon\wedge\nabla\psi \end{aligned}$$

and that the two-scale limit associated with $\mathrm{rot}_x\,\widetilde{E^\varepsilon}$ is $V(x,y)$.

Let us deal with the second term on the left hand side of (2.28). For this purpose we use simply the div-rot lemma of L. Tartar, F. Murat, see for instance [18], since $\operatorname{rot}_x \widetilde{E^\varepsilon}$ as well as $\operatorname{div}(\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)])$ are (uniformly) bounded in $(L^2(\Omega))^3$ since

$$\begin{aligned}\operatorname{div}(\mathbb{U}^\varepsilon(x)[\vec{\psi}(x)]) &= \operatorname{div}\Big(\sum_i \vec{U}^{\varepsilon i}\psi_i\Big) = \sum_i \operatorname{div}(\vec{U}^{\varepsilon i})\psi_i + \sum_i \vec{U}^{\varepsilon i}\cdot\nabla\psi_i(x)\\ &= 0 + \sum_i \vec{U}^{\varepsilon i}\cdot\nabla\psi_i(x).\end{aligned}$$

Therefore one obtains, for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$,

$$\begin{aligned}\int_\Omega\int_{Y^*} V(x,y)\mathbb{U}(y)\wedge\nabla\vec{\psi}(x)\,\mathrm{d}x\,\mathrm{d}y &+ \gamma\int_\Omega E(x)\,\mathrm{d}x\int_{Y^*}\mathbb{U}(y)[\vec{\psi}]\,\mathrm{d}y\\ &= \int_\Omega F(x)\,\mathrm{d}x\int_{Y^*}\mathbb{U}(y)[\vec{\psi}]\,\mathrm{d}y.\end{aligned}$$

Setting $\int_{Y^*}\mathbb{U}(y)\,\mathrm{d}y = \zeta\,\mathrm{Id}$, where $\zeta\in\mathbb{R}$ is a fixed constant, one obtains

$$\int_\Omega\int_{Y^*} V(x,y)\cdot\mathbb{U}(y)\wedge\nabla\vec{\psi}(x)\,\mathrm{d}x\,\mathrm{d}y + \zeta\gamma\int_\Omega E(x)\cdot\vec{\psi}\,\mathrm{d}x = \zeta\int_\Omega F(x)\cdot\vec{\psi}\,\mathrm{d}x. \tag{2.29}$$

It remains to analyze the first term on the left hand side of expression (2.29).

By definition, for all smooth $\vec{\psi}(x,y)$ we have

$$\int_\Omega \operatorname{rot}\widetilde{E^\varepsilon}(x)\cdot\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \longrightarrow \int_\Omega\int_{Y^*} V(x,y)\cdot\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Choose any $\vec{\psi}$ such that $\operatorname{rot}_y\vec{\psi} = 0$ and $\operatorname{supp}_y\vec{\psi}\subset\overline{Y}^*$. Then the left-hand side can also be written as

$$\int_\Omega \widetilde{E^\varepsilon}\cdot(\operatorname{rot}_x\vec{\psi})\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \widetilde{E^\varepsilon}\cdot\operatorname{rot}_x\Big[\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big]\,\mathrm{d}x.$$

We can again use the div-rot lemma. As $\operatorname{rot}\widetilde{E^\varepsilon}$ is bounded and since obviously

$$\operatorname{div}\Big\{\operatorname{rot}_x\Big[\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big]\Big\} = 0 \quad\text{and}\quad \operatorname{rot}_x\Big[\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big] = (\operatorname{rot}_x\vec{\psi})\Big(x,\frac{x}{\varepsilon}\Big)$$

are bounded uniformly in $(L^2(\Omega))^3$,

$$\int_\Omega \widetilde{E^\varepsilon}\cdot(\operatorname{rot}_x\vec{\psi})\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \longrightarrow \int_\Omega E(x)\,\mathrm{d}x\int_{Y^*}\operatorname{rot}_x\vec{\psi}(x,y)\,\mathrm{d}y.$$

Alltogether we have obtained that for all $\vec{\psi}(x,y)$, $\operatorname{supp}_y \vec{\psi} \subset \overline{Y}^*$, $\operatorname{rot}_y \vec{\psi}(x,y) = 0$, one has

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega E(x)\,\mathrm{d}x \int_{Y^*} \operatorname{rot}_x \vec{\psi}(x,y)\,\mathrm{d}y$$
$$= \int_\Omega \mathrm{d}x \int_{Y^*} \operatorname{rot}_x E(x) \cdot \vec{\psi}(x,y)\,\mathrm{d}y,$$

thus

$$\int_\Omega \int_{Y^*} [V(x,y) - \operatorname{rot}_x E(x)] \cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

From [2], [3], [4], [9], [5], [13] we deduce that there exists $V_1 \equiv V_1(x,y)$ such that

$$V(x,y) = \operatorname{rot}_x E(x) + \operatorname{rot}_y V_1(x,y). \tag{2.30}$$

Therefore (2.29) becomes, for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$,

$$\int_\Omega \int_{Y^*} \{\operatorname{rot}_x E(x) + \operatorname{rot}_y V_1(x,y)\} \cdot \mathbb{U}(y) \wedge \nabla_x \psi(x)\,\mathrm{d}x\,\mathrm{d}y + \zeta\gamma \int_\Omega E(x) \cdot \vec{\psi}\,\mathrm{d}x = \zeta \int_\Omega F(x) \cdot \vec{\psi}(x,y)\,\mathrm{d}x. \tag{2.31}$$

In the second step, we precise further the term $\operatorname{rot}_y V_1(x,y)$. For this purpose, we get back again to the variational formulation (2.5), take $\vec{\varphi}^\varepsilon = \vec{\psi}(x, x/\varepsilon)$, multiply by $\varepsilon$ and send it to zero. In this way we obtain

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \operatorname{rot}_y \vec{\psi} = 0$$

for all $\vec{\psi}$ such that $\operatorname{supp}_y \vec{\psi} \subset \overline{Y}^*$ with $\vec{n} \wedge \vec{\psi}|_{\partial T} = 0$. On the other hand, one has also easily $\operatorname{div}_y V(x,y) = 0$ in $D'(\Omega \times Y^*)$. We obtain in particular from the facts already mentioned, that $V$ is given by

$$V(x,y) = \mathbb{U}(y)[\operatorname{rot}_x E(x)]. \tag{2.32}$$

Therefore (2.31) becomes

$$\int_\Omega \int_{Y^*} \mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot (\mathbb{U}(y) \wedge \nabla_x \vec{\psi}(x)) + \zeta\gamma \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \zeta \int_\Omega F(x) \cdot \vec{\psi}(x)\,\mathrm{d}x. \tag{2.33}$$

Next, one has

$$\begin{aligned}
&\mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot (\mathbb{U}(y) \wedge \nabla_x \vec{\psi}(x)) \\
&\qquad = \sum_i \mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot \vec{U}_i \wedge \nabla_x \psi_i(x) \\
&\qquad = -\sum_i (\vec{U}_i \wedge \mathbb{U}(y)[\operatorname{rot}_x E(x)]) \cdot \nabla_x \psi_i(x) \\
&\qquad = -\sum_i \sum_j (\vec{U}_i(y) \wedge \vec{U}_j(y)(\operatorname{rot}_x E(x))_j) \cdot \nabla_x \psi_i(x).
\end{aligned}$$

Define $\eta \in \mathbb{R}$ such that

$$\int_{Y^*} (\vec{U_1} \wedge \vec{U_2})\,\mathrm{d}y = \eta \vec{e_3}, \quad \int_{Y^*} (\vec{U_2} \wedge \vec{U_3})\,\mathrm{d}y = \eta \vec{e_1}, \quad \int_{Y^*} (\vec{U_3} \wedge \vec{U_1})\,\mathrm{d}y = \eta \vec{e_2}.$$

Then, one obtains that

$$\begin{aligned}
&\int_{Y^*} \mathbb{U}(y)[\operatorname{rot}_x E(x)] \cdot (\mathbb{U}(y) \wedge \nabla_x \vec{\psi}(x))\,\mathrm{d}y \\
&\qquad = \eta\{(\operatorname{rot}_x E)_2 \cdot \partial_3 \psi_1 - (\operatorname{rot}_x E)_1 \partial_3 \psi_2\} \\
&\qquad\quad + \eta\{(\operatorname{rot}_x E)_3 \cdot \partial_1 \psi_2 - (\operatorname{rot}_x E)_2 \partial_1 \psi_3\} \\
&\qquad\quad + \eta\{(\operatorname{rot}_x E)_1 \cdot \partial_2 \psi_3 - (\operatorname{rot}_x E)_3 \partial_2 \psi_1\} \\
&\qquad = \eta(\operatorname{rot}_x E)_1(\partial_2 \psi_3 - \partial_3 \psi_2) + \eta(\operatorname{rot}_x E)_2(\partial_3 \psi_1 - \partial_1 \psi_3) \\
&\qquad\quad + \eta(\operatorname{rot}_x E)_3(\partial_1 \psi_2 - \partial_2 \psi_1) \\
&\qquad = \eta \operatorname{rot}_x E \cdot \operatorname{rot}_x \psi.
\end{aligned}$$

Consequently, for all $\vec{\psi} \in (C_c^\infty(\Omega))^3$ we obtain

$$\eta \int_\Omega \operatorname{rot}_x E(x) \cdot \operatorname{rot}_x \vec{\psi}(x)\,\mathrm{d}x + \gamma\zeta \int_\Omega E(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \zeta \int_\Omega F(x) \cdot \vec{\psi}(x)\,\mathrm{d}x,$$

where $\eta$, $\zeta$ are constants, which is the claim of Theorem 1(b).

Remark 1. If we replace the boundary condition

$$\vec{n}^\varepsilon(x) \wedge \vec{E}^\varepsilon(x) = 0$$

by the non homogeneous condition

$$\vec{n}^\varepsilon(x) \wedge \vec{E}^\varepsilon(x) = \vec{n}^\varepsilon \wedge \vec{g}(x),$$

written also as

$$\vec{n}^\varepsilon \wedge (\vec{E}^\varepsilon - \vec{g}(x)) = 0,$$

where $\vec{g}$ is a given (smooth) vector field, we have the same type of result by setting

$$\vec{E}^{\varepsilon} - \vec{g}(x) = \vec{V}^{\varepsilon}(x).$$

R e m a r k 2. Under the assumption $(\mathrm{HYP})_1$ and if we suppose furthermore the strong convergence of the sequence $\widetilde{F^{\varepsilon}}$ in $L^2(\Omega)$, it is not difficult to check that the global electromagnetic "energy"

$$\int_{\Omega} |\mathrm{rot}\, \widetilde{E^{\varepsilon}}|^2 + \gamma \int_{\Omega} |\widetilde{E^{\varepsilon}}|^2$$

converges towards the corresponding homogenized one (involving $E$).

Under the case of assumption $(\mathrm{HYP})_2$, corrector results can eventually give a clue.

## 3. Proof of Theorems 2

### 3.1. Proof of Theorem 2(a)

*First step: Variational formulation and uniform estimates*

Problem $(\mathrm{Q}^{\varepsilon})$ has the variational formulation

$$\int_{\Omega^{\varepsilon}} \mathrm{rot}\, E^{\varepsilon} \cdot \mathrm{rot}\, \vec{\varphi}(x)\,\mathrm{d}x + \int_{\Omega^{\varepsilon}} E^{\varepsilon} \cdot \vec{\varphi}(x)\,\mathrm{d}x = \int_{\Omega^{\varepsilon}} F^{\varepsilon} \cdot \vec{\varphi}(x)\,\mathrm{d}x \tag{3.1}$$

for all $\vec{\varphi} \in H(\mathrm{rot}, \Omega^{\varepsilon})$.

Let us note that $\widetilde{\mathrm{rot}\, E^{\varepsilon}} \neq \mathrm{rot}\, \widetilde{E^{\varepsilon}}$ in general. Extending by zero, we now get

$$\int_{\Omega} \widetilde{\mathrm{rot}\, E^{\varepsilon}}(x) \cdot \mathrm{rot}\, \vec{\varphi}(x)\,\mathrm{d}x + \int_{\Omega} \widetilde{E^{\varepsilon}}(x) \cdot \vec{\varphi}(x)\,\mathrm{d}x = \int_{\Omega} \widetilde{F^{\varepsilon}}(x) \cdot \vec{\varphi}(x)\,\mathrm{d}x. \tag{3.2}$$

Letting $\vec{\varphi} := E^{\varepsilon}$, one has

$$\int_{\Omega^{\varepsilon}} |\mathrm{rot}\, E^{\varepsilon}|^2\,\mathrm{d}x + \int_{\Omega^{\varepsilon}} |E^{\varepsilon}|^2\,\mathrm{d}x \leqslant \frac{1}{2}\int_{\Omega^{\varepsilon}} |F^{\varepsilon}|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^{\varepsilon}} |E^{\varepsilon}|^2\,\mathrm{d}x,$$

thus

$$\begin{aligned} \|E^{\varepsilon}\|^2_{H(\mathrm{rot},\Omega^{\varepsilon})} &\leqslant \int_{\Omega^{\varepsilon}} |\mathrm{rot}\, E^{\varepsilon}|^2\,\mathrm{d}x + \int_{\Omega^{\varepsilon}} |E^{\varepsilon}|^2\,\mathrm{d}x \\ &\leqslant \int_{\Omega^{\varepsilon}} |\mathrm{rot}\, E^{\varepsilon}|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^{\varepsilon}} |E^{\varepsilon}|^2\,\mathrm{d}x \leqslant C \end{aligned}$$

and consequently

$$\|E^{\varepsilon}\|_{H(\mathrm{rot},\Omega^{\varepsilon})} \leqslant C.$$

*Second step: Passing to the limit*

Up to a subsequence we can assume

$$\widetilde{E^\varepsilon}(x) \rightharpoonup E(x) \quad \text{weakly in } (L^2(\Omega))^3, \tag{3.3}$$

$$\widetilde{\operatorname{rot} E^\varepsilon} \rightharpoonup V(x) \quad \text{weakly in } (L^2(\Omega))^3. \tag{3.4}$$

Then from (3.2) one has, for all $\vec{\varphi} \in (C^\infty(\overline{\Omega}))^3$,

$$\int_\Omega V(x) \cdot \operatorname{rot} \vec{\varphi}(x)\,\mathrm{d}x + \int_\Omega E(x) \cdot \vec{\varphi}(x)\,\mathrm{d}x = \int_\Omega F(x) \cdot \vec{\varphi}(x)\,\mathrm{d}x.$$

It remains to identify $V$. Let $\vec{\psi} \in (C_c^\infty(\overline{\Omega}))^3$. Then of course

$$\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \vec{\psi}(x)\,\mathrm{d}x \longrightarrow \int_\Omega V(x) \cdot \vec{\psi}(x)\,\mathrm{d}x.$$

On the other hand,

$$\begin{aligned}\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = & \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot [\vec{\psi}(x) - \mathbb{W}^\varepsilon[\vec{\psi}](x)]\,\mathrm{d}x \\ & + \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\,\mathrm{d}x,\end{aligned}$$

where $\mathbb{W}^\varepsilon$ is the test (matrix-valued) function given in Section 2.

Since $\vec{\psi} - \mathbb{W}^\varepsilon(\vec{\psi}) \longrightarrow 0$ strongly in $(L^2(\Omega))^3$, one has

$$\int_\Omega V(x) \cdot \vec{\psi}(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\,\mathrm{d}x.$$

Further,

$$\begin{aligned}\int_\Omega \widetilde{\operatorname{rot} E^\varepsilon}(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\,\mathrm{d}x &= \int_{\Omega^\varepsilon} \operatorname{rot} E^\varepsilon(x) \cdot \mathbb{W}^\varepsilon[\vec{\psi}](x)\,\mathrm{d}x \\ &= \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \operatorname{rot}\{\mathbb{W}^\varepsilon[\vec{\psi}](x)\}\,\mathrm{d}x,\end{aligned}$$

since $\vec{n}^\varepsilon \wedge \mathbb{W}^\varepsilon = 0$ on $\partial\Omega^\varepsilon$.

As

$$\begin{aligned}&\int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \operatorname{rot}\{\mathbb{W}^\varepsilon[\vec{\psi}](x)\}\,\mathrm{d}x \\ &\qquad = \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot (\operatorname{rot} \mathbb{W}^\varepsilon)[\vec{\psi}](x)\,\mathrm{d}x + \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \mathbb{W}^\varepsilon \wedge \nabla\vec{\psi}(x)\,\mathrm{d}x \\ &\qquad = 0 + \int_{\Omega^\varepsilon} E^\varepsilon(x) \cdot \mathbb{W}^\varepsilon \wedge \nabla\vec{\psi}(x)\,\mathrm{d}x \longrightarrow \int_\Omega E(x) \cdot \operatorname{rot}\vec{\psi}(x)\,\mathrm{d}x,\end{aligned}$$

we have finally obtained that

$$\int_\Omega V(x)\cdot\vec{\psi}(x)\,\mathrm{d}x=\int_\Omega E(x)\cdot\operatorname{rot}\vec{\psi}(x)\,\mathrm{d}x,$$

and thus

$$V=\operatorname{rot}E.$$

This concludes the proof of Theorem 2(a).

### 3.2. Proof of Theorem 2(b)

*First step: Uniform estimates in* $H(\operatorname{rot},\Omega)$

Repeating the same lines of calculus, we get again similar uniform estimates, and thus we can assume that weak convergences (3.3) and (3.4) hold true.

*Second step: Two-scale convergence*

Since $(\widetilde{E^\varepsilon})$ as well as $(\widetilde{\operatorname{rot}E^\varepsilon})$ are bounded in $(L^2(\Omega))^3$, there exist two two-scale limits, $E^0\in(L^2(\Omega\times Y))^3$, $V\in(L^2(\Omega\times Y))^3$ such that for any $\vec{\psi}(x,y)\in(D(\Omega,C_p^\infty(Y)))^3$ one has

$$\lim_{\varepsilon\to0}\int_\Omega\widetilde{E^\varepsilon}\cdot\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x=\int_\Omega\int_{Y^*}E^0(x,y)\cdot\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.5}$$

and

$$\lim_{\varepsilon\to0}\int_\Omega\widetilde{\operatorname{rot}E^\varepsilon}\cdot\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x=\int_\Omega\int_{Y^*}V(x,y)\cdot\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6}$$

By integration by parts, we obtain for all $\vec{\psi}(x,y)\in(D(\Omega,C_p^\infty(Y^*)))^3$ with compact support (in $y$) in $Y^*$

$$\varepsilon\int_\Omega\widetilde{\operatorname{rot}E^\varepsilon}\cdot\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x=\int_\Omega\widetilde{E^\varepsilon}\cdot\Big[\varepsilon\operatorname{rot}_x\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)+\operatorname{rot}_y\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\Big]\,\mathrm{d}x.$$

Passing to the limit in both terms with the help of two-scale convergence leads to

$$\int_\Omega\int_{Y^*}E^0(x,y)\operatorname{rot}_y\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y=0, \tag{3.7}$$

that is

$$\operatorname{rot}_y E^0(x,y)=0\quad\text{in }\ (D'(\Omega\times Y^*))^3. \tag{3.8}$$

Now, to the assumption on $\vec{\psi}(x,y)$ we add the condition $\operatorname{rot}_y\vec{\psi}(x,y)=0$ with compact support in $x$ as well as in $y$ in $Y^*$. Since

$$\int_\Omega\widetilde{\operatorname{rot}E^\varepsilon}(x)\cdot\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x=\int_\Omega\widetilde{E^\varepsilon}(x)\operatorname{rot}_x\vec{\psi}\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x,$$

using the two-scale convergence we are led to

$$\int_\Omega \int_{Y^*} V(x,y)\cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \int_{Y^*} \widetilde{E^0}(x,y)\cdot \mathrm{rot}_x\,\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

and thus integrating by parts in $\Omega$ we get

$$\int_\Omega \int_{Y^*} E^0(x,y)\cdot \mathrm{rot}_x\,\vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \int_{Y^*} \mathrm{rot}_x\,E^0(x,y)\cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Hence

$$\int_\Omega \int_{Y^*} [V(x,y) - \mathrm{rot}_x\,E^0(x,y)]\cdot \vec{\psi}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.9}$$

We will use all these facts to identify $E^0$ and $V$.

According to [2], [3], [4], [9] there exist unique $\varphi \in H^1(Y^*)$ and $h_1 \in H_1(Y^*)$ with

$$H_1(Y^*) = \{u \in L^2(Y^*)^3,\ \ \mathrm{rot}_y\,u = 0,\ \ \mathrm{div}_y\,u = 0,\ \ \vec{n}(y)\cdot\vec{u}(y) = 0\}$$

such that

$$E^0(x,y) = \nabla_y\varphi(x,y) + h_1(x,y), \tag{3.10}$$

and finally, one can write

$$E^0(x,y) = \chi_{Y^*}(y)\mathbb{G}(y)[E(x)] \tag{3.11}$$

and

$$V(x,y) = \chi_{Y^*}(y)\mathbb{G}(y)\times\mathbb{G}(y)[\mathrm{rot}\,E(x)], \tag{3.12}$$

where $\mathbb{G}$ is the $\prod_3$-valued function defined for $y \in Y^*$ by

$$\mathbb{G}(y) = (\vec{G}_1(y), \vec{G}_2(y), \vec{G}_3(y)) \tag{3.13}$$

and

$$\vec{G}_i(y) = (\nabla\widetilde{\Phi}_i)(y),$$

each $\widetilde{\Phi}_i$ being solution in $H^1(Y^*)$ of

$$\begin{cases} \widetilde{\Phi}_i = y_i - \Upsilon_i, \quad \Upsilon_i\ Y\text{-periodic},\\ \Delta\widetilde{\Phi}_i = 0 \quad \text{in } Y^*,\\ \left.\dfrac{\partial\Phi_i}{\partial n}\right|_{\partial T} = 0. \end{cases} \tag{3.14}$$

Note that $\mathrm{rot}_y\, \mathbb{G} = 0$ and $\vec{n}(y) \cdot \mathbb{G}(y) = 0$.

From the variational formulation (3.2), one gets $\forall \vec{\psi} \in (C^\infty(\Omega))^3$

$$\int_\Omega \int_{Y^*} V(x,y) \cdot \mathrm{rot}\, \vec{\psi}(x)\, \mathrm{d}x\, \mathrm{d}y + \int_\Omega E(x) \cdot \vec{\psi}(x)\, \mathrm{d}x = \int_\Omega F(x) \cdot \vec{\psi}(x)\, \mathrm{d}x,$$

which by virtue of (3.12) yields

$$\int_\Omega \chi_{Y^*}(y) \mathbb{G}(y) \times \mathbb{G}(y) [\mathrm{rot}_x\, E(x)] \cdot \mathrm{rot}_x\, \vec{\psi}(x)\, \mathrm{d}x + \int_\Omega E(x) \cdot \vec{\psi}(x)\, \mathrm{d}x = \int_\Omega F(x) \cdot \vec{\psi}(x)\, \mathrm{d}x.$$

Since we can set $\int_{Y^*} \mathbb{G}(y) \times \mathbb{G}(y)\, \mathrm{d}y = \eta_1 I$, this concludes the proof of Theorem 2(b).

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*Authors' address*: *R. Alexandre, H. Taha*, MAPMO-UMR 6628, BP 6759, Université d'Orléans, 45067 Orleans Cedex, France, e-mail: `alexandr@labomath.univ-orleans.fr`; `hassan.taha@labomath.univ-orleans.fr`.