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UNCERTAIN INPUT DATA PROBLEMS AND THE WORST SCENARIO METHOD*

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Abstract. An introduction to the worst scenario method is given. We start with an example and a general abstract scheme. An analysis of the method both on the continuous and approximate levels is discussed. We show a possible incorporation of the method into the fuzzy set theory. Finally, we present a survey of applications published during the last decade.

Keywords: uncertain input data, the worst-case approach, fuzzy sets

MSC 2000: 93C20, 49J20, 74C10, 74K20

1. Introduction

The progress in computational power and the promising prospects of its further development enable us to approach reality nearer and deeper through modeling problems in technical, natural and social sciences than several decades ago. Regardless of many achievements, modeling of real world phenomena is always burdened by uncertainties which occur in the selection of a mathematical model, in the values of input data, in the correctness of computer codes and in the error of numerical results.

The accent of the present contribution is on the uncertainty in the values of *input data* and on its impact on the outputs of mathematical models. In this respect, we introduce a method of the worst scenario which is suitable even in cases when information specifying probability distribution of input data is either not available or is too expensive.

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To illustrate the leading idea of the worst scenario method, let us start with an *example.* Let us consider the following model of steady heat conduction (state problem)

$$\begin{aligned} \operatorname{div}(a(u)\operatorname{grad} u) &= f \quad \text{in } \Omega \subset \mathbb{R}^d, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $d \leqslant 3$, Ω is a bounded domain with Lipschitz boundary and $a(\cdot)$ is an *uncertain function.*

Assume that the function a belongs to a given set $\mathcal{U}_{\mathrm{ad}}$ of admissible data and that $\mathcal{U}_{\mathrm{ad}}$ is *compact* in the space $C(I)$ of continuous functions on a bounded interval I. Note that u denotes the temperature and $a(u)$ stands for the conductivity.

Let us choose a *criterion*-functional (quantity of interest)

$$\Phi(u) = \int_G u \,\mathrm{d}x/(\operatorname{meas} G),$$

where G is a given small subdomain of Ω.

Assume that for any $a \in \mathcal{U}_{\mathrm{ad}}$ there exists a unique solution $u(a)$ of the state problem (1). (See [27] for conditions guaranteeing this assumption.) Then we define the Worst Scenario Problem: find

$$a^0 = \underset{a \in \mathcal{U}_{\mathrm{ad}}}{\arg\max}\, \Phi(u(a)). \tag{2}$$

2. General abstract scheme of the Worst Scenario Method

A natural generalization of the example reads as follows. Let us consider a mathematical model (equations, differential or integral equation, eigenvalue problem, variational inequality, etc.), leading to a state problem

$$\mathcal{P}(A; u),$$

where A stands for input data (coefficients, boundary values, initial values, source terms, etc.) and u is a state variable.

Assume that

(i) a set $\mathcal{U}_{\mathrm{ad}}$ of admissible input data is given;
(ii) a criterion-functional $\Phi(A; u)$ (quantity of interest) is given;
(iii) for any $A \in \mathcal{U}_{\mathrm{ad}}$ there exists a unique solution $u(A)$ of the state problem $\mathcal{P}(A; u)$.

Then we define the *Worst Scenario Problem*: find

$$A^0 = \underset{A \in \mathcal{U}_{\rm ad}}{\arg\max}\, \Phi(A; u(A)). \tag{3}$$

In the case of *non-uniqueness* we replace assumption (iii) and definition (3) as follows:

(iiia) for any $A \in \mathcal{U}_{\rm ad}$ there exists a set $K(A)$ of solutions to problem $\mathcal{P}(A; u)$.

Then we define

$$A^0 = \underset{A \in \mathcal{U}_{\rm ad}}{\arg\max}\big(\max_{u \in K(A)} \Phi(A; u)\big). \tag{3a}$$

3. Analysis of the Worst Scenario Method on the continuous level

Assume that the input data belong to a Banach space U and the solution of the problem $\mathcal{P}(A, u)$ is to be found in another Banach space W.

To prove the existence of a solution to problem (3), we have to establish some continuity of the map $A \mapsto u(A)$ and $\{A; u\} \mapsto \Phi(A; u)$. Thus, for instance, if

$$A_n \in \mathcal{U}_{\rm ad},\ \ A_n \to A \ \text{ in } U \ \text{ as } n \to \infty \Rightarrow u(A_n) \rightharpoonup u(A) \ \text{(weakly) in } W, \tag{4}$$

$$A_n \in \mathcal{U}_{\rm ad},\ \ A_n \to A \ \text{ in } U \ \ \&\ \ v_n \rightharpoonup v \ \text{(weakly) in } W \Rightarrow \limsup \Phi(A_n, v_n) \leqslant \Phi(A, v), \tag{5}$$

then there exists at least one solution of problem (3) (for detailed proofs for problems (3) and (3a), see [12] or [25], [30]).

4. Numerical realization and analysis of approximate Worst Scenario Method

Very often we have no chance to find an exact solution $u(A)$ of the state problem, so that approximate solutions $u_h(A)$ by finite elements, boundary elements, finite differences, etc., are inevitable. If the input data A involve continuous functions (like in Example (1)), the set $\mathcal{U}_{\rm ad}$ has to be discretized, as well, by sets $\mathcal{U}_{\rm ad}^M$ via finite elements, for instance. We are led to the *Approximate Worst Scenario Problem*: find

$$A_M^0(h) = \underset{A_M \in \mathcal{U}_{\rm ad}^M}{\arg\max}\, \Phi_h(A_M, u_h(A_M)). \tag{6}$$

A typical result of convergence analysis: Under some assumption on uniform convergence of approximations with respect to $A \in \mathcal{U}_{\rm ad}$ there exists a function $\varphi\colon [0,1] \to$

$[0,1]$, $\lim \varphi(h) = 0$ as $h \to 0_+$, such that given a sequence $\{A^0_M(h)\}$ of solutions to problem (6), where $h \to 0_+$, $M \to +\infty$, $h \leqslant \varphi(1/M)$, there exists a subsequence $\{A^0_{M'}(h')\}$ such that

$$A^0_{M'}(h') \to A^0 \quad \text{in } U, \tag{7}$$

$$u_{h'}(A^0_{M'}(h')) \to u(A^0) \quad \text{in } W, \tag{8}$$

$$\Phi_{h'}(A^0_{M'}(h'); u_{h'}(A^0_{M'}(h'))) \to \Phi(A^0; u(A^0)). \tag{9}$$

In practice, (9) is the most important result.

Since the structure of problem (6) is the same as that of problems in Optimal Design/Control, efficient algorithms and software are available for the solution of problem (6). We can employ classical tools of *sensitivity analysis*, i.e., we compute $\operatorname{grad} \Psi(A_M)$, where $\Psi(A_M) := \Phi_h(A_M; u_h(A_M))$, by means of an *adjoint problem* (see e.g. [25, Section 25], [10], [33], [36]).

5. Incorporation of Worst Scenario Method into the fuzzy set theory

In contrast to the concept of a classical set, with sharp distinction between $x \in B$ or $x \notin B$, a fuzzy set is related to the degree of truth of the statement "$x \in X$ belongs to B" via a *membership function* $\mu_B\colon X \to [0,1]$ (see Fig. 1 and the book [38]).

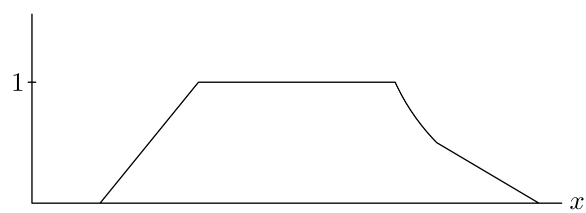

Fig. 1. A membership function $\mu_B(x)$.

Since $B \equiv \mathcal{U}_{\rm ad}$ is given in the worst scenario method and since μ_B can also be interpreted as a weight function, it is convenient to restrict ourselves to $\mu_B\colon B \to [0,1]$ in our approach. Consequently, $\mu_B(x) \geqslant 0$ will simply mean that $x \in B$.

A natural question arises: how a fuzzy set of *input* data propagates into a fuzzy set of outputs (values of the criterion-functional)? Answers are proposed in [4], [8]. Let us illustrate the approach by a simple example. Assume that A is a single scalar and let $\mu_I(A)$ be a given membership function of a fuzzy set $I \equiv \mathcal{U}_{\rm ad}$ of input data. We define α-cuts I^α of the set I as (see Fig. 2)

$$I^\alpha = \{A\colon\ \mu_I(A) \geqslant \alpha\}, \quad \alpha \in [0,1];$$

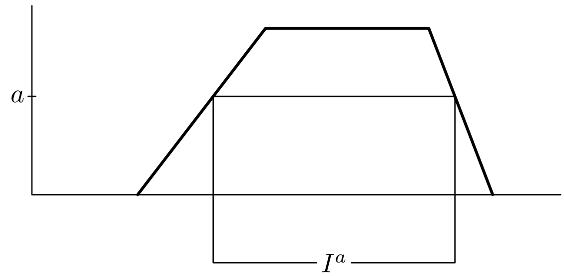

Fig. 2. An example of a-cut I^a.

and the Maximum Range Problem: find

$$
\begin{aligned}
(10) \qquad A^0 &= \operatorname*{arg\,max}_{A\in\mathcal{U}_{\mathrm{ad}}} \Phi(A; u(A)),\\
A_0 &= \operatorname*{arg\,min}_{A\in\mathcal{U}_{\mathrm{ad}}} \Phi(A; u(A)).
\end{aligned}
$$

Let us solve problem (10) for a sequence $\mathcal{U}^i_{\mathrm{ad}} = I^{\alpha_i}$, where $\alpha_1 = 1$, $\alpha_2 = 0.9$, …, $\alpha_{10} = 0.1$, $\alpha_{11} = 0$. We obtain a sequence of intervals (ranges)

$$J_{\alpha_i} = [\Psi(A^i_0), \Psi(A^0_i)],$$

where $\Psi(A) := \Phi(A; u(A))$. Then we can construct an approximate membership function μ_J of the *fuzzy set J of outputs* by the formula

$$\mu_J(\Psi(A)) = \max_{i\leqslant 11} \min\{\alpha_i, \chi(J_{\alpha_i}; \Psi(A))\},$$

where $\chi(J_{\alpha_i}; \cdot)$ denotes the characteristic function of J_{α_i} (see Fig. 3).

6. History of the worst scenario method

It seems that the first publication involving the Worst Scenario Method was the paper [5] by Bulgakov. Fifty years later, a monograph [3] by Ben-Haim and I. Elishakoff appeared, which contains the concept of the Worst Scenario Method with *convex* sets $\mathcal{U}_{\mathrm{ad}}$ and numerous examples.

Other terms used for the Worst Scenario Method are: unknown but bounded uncertainty approach; guaranteed performance approach; anti-optimization. Theory of interval matrices in linear algebra has been employed to solve the worst scenario problems from economics or structural mechanics.

At a suggestion of Professor Ivo Babuška, who initiated and published numerous papers on uncertainties in mathematical modeling, on stochastic differential

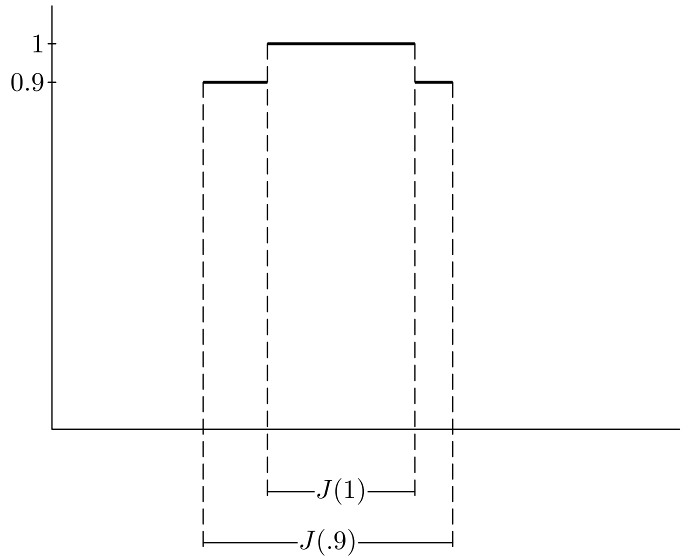

Fig. 3. Construction of approximate membership function of the fuzzy set of outputs.

equations and on validation and verification (cf. the literature and Appendix in the book [25]), I studied the Worst Scenario Method from a general abstract viewpoint [12].

The Worst Scenario Method was used in the book [2] by Y. Ben-Haim as a tool for prediction and decision in practice.

A variety of information about approaches to uncertainty and numerous applications can be found in the book [25].

7. A survey of applications during 1996–2006

We display a series of branches of science and technology where the Worst Scenario Method has been applied during the last decade.

- *Linear elliptic PDE*:
  [1] uncertain coefficients;
- *Quasilinear elliptic boundary value problems* (like example (1)):
  [12], [13], [6], [7]—uncertain coefficients and source terms;
  [32] uncertain boundary condition;
- *Parabolic problems*:
  [14] uncertain time-dependent coefficients in PDE;
  [34] uncertain coefficients and uncertain obstacle in a variational inequality;

- *Elastic beams—Timoshenko model*:
  [24] transverse vibrations, uncertain shear correction factor;
  [22] buckling, elastic foundation with uncertain stiffness and shear correction factor;
- *Thermoelastic beams—coupled model*:
  [37] uncertain coupling coefficient;
- *Elastic pseudoplates*:
  [26] unilateral boundary value problems, uncertain loads, stiffness of the unilateral foundation, slip limits on the boundary;
- *Elastic plates—linear theory*:
  [23] classical boundary value problems, uncertain material coefficients and loading;
- *Buckling of elastic plates—von Kármán model*:
  [21] uncertain initial geometric imperfections;
- *Shallow elastic shells*:
  [28] semi-coercive variational inequalities, general abstract result, applications with uncertain coefficients, loads, slip limits;
- *Unilateral contact with friction of elastic bodies*:
  [16] Signorini problem with given friction, uncertain Lamé coefficients, slip limit;
  [30] Coulomb friction model, non-uniqueness, uncertain coefficients;
  [29] quasi-coupled thermoelasticity, uncertain conductivity, stress-strain law, loading, slip limits;
- *Hencky's model of plasticity*:
  [15] Timoshenko beam, uncertain yield function;
  [17] torsion problem, uncertain stress-strain law and yield function;
- *Deformation theory of plasticity*:
  [11] monotone operator theory, uncertain material function;
- *Perfect plasticity*:
  [18] Prandtl-Reuss model $3D$-orthotropic, uncertain stress-strain law and yield function;
- *Plasticity with isotropic hardening*:
  [19] $3D$-body, uncertain stress-strain law, yield function, initial hardening;
  [20] $3D$-body, formulation in strain space, uncertain stress-strain law, initial hardening and loading;
- *Plasticity with combined linear kinematic and isotropic hardening*:
  [25] Section 21, $3D$-body, uncertain coefficients;

- *Validation of 3 elasto-plastic models*:

  [31] (i) perfect plasticity, (ii) isotropic hardening, (iii) kinematic hardening models are computed with uncertain moduli E, ν, yield stress;

- *Clinical biochemistry*:

  [9] uncertain parameters and measurements reading;

- *Homogenization of elliptic boundary value problems*:

  [35] problems in $\mathbb{R}^N$, uncertain periodic coefficients.

8. Conclusions

The worst scenario and maximum range methods are applicable even in cases when information specifying the probability distribution or the membership function of the input data are either not available or too expensive.

The structure of the worst scenario problem is the same as that in problems of Optimal Design/Control. As a consequence, efficient algorithms are available.

The Worst Scenario Method can be incorporated in the fuzzy set theory or in the probabilistic approach as a building brick.

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