Nils Svanstedt

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MULTISCALE STOCHASTIC HOMOGENIZATION OF CONVECTION-DIFFUSION EQUATIONS

Nils Svanstedt, Göteborg

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Abstract. Multiscale stochastic homogenization is studied for convection-diffusion problems. More specifically, we consider the asymptotic behaviour of a sequence of realizations of the form $\partial u_\varepsilon^\omega/\partial t + 1/\varepsilon_3\,\mathcal{C}\big(T_3(x/\varepsilon_3)\omega_3\big)\cdot\nabla u_\varepsilon^\omega - \operatorname{div}\big(\alpha\big(T_1(x/\varepsilon_1)\omega_1, T_2(x/\varepsilon_2)\omega_2, t\big)\nabla u_\varepsilon^\omega\big) = f$. It is shown, under certain structure assumptions on the random vector field $\mathcal{C}(\omega_3)$ and the random map $\alpha(\omega_1,\omega_2,t)$, that the sequence $\{u_\varepsilon^\omega\}$ of solutions converges in the sense of G-convergence of parabolic operators to the solution u of the homogenized problem $\partial u/\partial t - \operatorname{div}(\mathcal{B}(t)\nabla u) = f$.

Keywords: multiscale, stochastic, homogenization, convection-diffusion

MSC 2000: 35B27, 35B40

1. Introduction

In this paper we consider the stochastic homogenization problem for the initial-boundary value problem

$$
(1)\qquad \begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} + \dfrac{1}{\varepsilon_3}\mathcal{C}\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big)\cdot\nabla u_\varepsilon^\omega \\ \qquad - \operatorname{div}\Big(\alpha\Big(T_1\Big(\dfrac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\dfrac{x}{\varepsilon_2}\Big)\omega_2, t\Big)\nabla u_\varepsilon^\omega\Big) = f \ \text{ in } \ Q, \\ u_\varepsilon^\omega(x,0) = u_0(x) \ \text{ in } \ Q_x, \\ u_\varepsilon^\omega(x,t) = 0 \ \text{ in } \ \partial Q_x \times (0,T). \end{cases}
$$

Here and throughout the paper we will write Q for the space-time set $Q_x \times (0,T)$ where Q_x is an open bounded set in $\mathbb{R}^n$, $n = 2,3$, and T is a positive number. Further, $\mathcal{C}$ is a divergence free random vector field in $\mathbb{R}^n$. For each fixed $\omega_i \in \Omega_i$, $i = 1,2,3$, the realization (1) is an initial-boundary value problem. Following the framework in [11], see also [4], we introduce three probability spaces $(\Omega_k, \mathcal{F}_k, \mu_k)$, $k = 1, 2, 3$. Each $\mathcal{F}_k$ is a complete σ-algebra and each μ_k is the associated countably additive non-negative probability measure on $\mathcal{F}_k$ normalized by $\mu_k(\Omega_k) = 1$. With every $x \in \mathbb{R}^n$ we associate the dynamical system

$$T_k(x)\colon\ \Omega_k \to \Omega_k.$$

For the random fields

$$\alpha(\omega_1, \omega_2, t) \quad \text{and} \quad \mathcal{C}(\omega_3)$$

we can then, for fixed ω_1, ω_2 and ω_3, consider the realizations

$$\alpha(T_1(x)\omega_1, T_2(x)\omega_2, t) \quad \text{and} \quad \mathcal{C}(T_3(x)\omega_3)$$

and the "speeded up" realizations

$$\alpha\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, t\Big) \quad \text{and} \quad \frac{1}{\varepsilon_3}\mathcal{C}\Big(T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3\Big),$$

respectively. With this construction which will be precisely defined in Section 2 the random fields become stationary due to the invariance properties of the associated probability measure and therefore the Birkhoff ergodic theorem applies and we can define limits of the speeded up realizations in terms of expectations (mean values) over the probability spaces. In the asymptotic analysis of the convection-diffusion problem a key problem is the scaling of the convection term and the diffusion term. As we will see the scaling in

$$\frac{1}{\varepsilon_3}\mathcal{C}\Big(T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3\Big) \cdot \nabla u_\varepsilon^\omega - \operatorname{div}\Big(\alpha\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, t\Big)\nabla u_\varepsilon^\omega\Big)$$

is appropriate for divergence free fields $\mathcal{C}$. We say that the random field $\mathcal{C}$ is divergence free if for every fixed $\omega_3 \in \Omega_3$ all realizations

$$\mathbb{R}^n \ni x \mapsto \mathcal{C}(T_3(x)\omega_3)$$

are divergence free. In our analysis we will have to assume that ε_1 and ε_2 are two well separated functions (scales) of $\varepsilon > 0$ which converge to zero as ε tends to zero. We say that ε_1 and ε_2 are well separated if

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_2}{\varepsilon_1} = 0.$$

This means that ε_2 is a finer scale than ε_1. For instance if $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \varepsilon^2$, then ε_1 and ε_2 are well separated scales. In this paper we also assume that the scale ε_3 is well separated from the other two scales and that ε_3 is the fastest scale. For instance $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$ and $\varepsilon_3 = \varepsilon^3$ meet this assumption. The multiscale stochastic homogenization problem for (1) consists in studying the asymptotic behavior of the solutions u_ε^ω as ε tends to zero.

Homogenization problems with more than one oscillating scale in the periodic setting was first introduced in [1] for linear elliptic problems. Recently the monotone elliptic case has been studied in [7]. The multiscale monotone stochastic elliptic and parabolic cases are recently studied in [11].

In the present work we will use the classical framework of G-convergence, which can be thought of as a non-periodic "homogenization" or stabilization of sequences of operator equations. We refer to [10] concerning the G-convergence results for elliptic and parabolic operators needed in this report. Here we show that the general theory also applies to the situation of multiple scales and multiscale stochastic homogenization of a class of nonlinear convection-diffusion problems. The main result (Theorem 5) says that the sequence of solutions $\{u_\varepsilon^\omega\}$ to (1) converges to the solution u to a homogenized problem of the form

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \operatorname{div}(\mathcal{B}(t)\nabla u) = f \ \text{ in } \ Q, \\
u(x,0) = u_0(x) \ \text{ in } \ Q_x, \\
u(x,t) = 0 \ \text{ in } \ \partial Q_x \times (0,T),
\end{cases} \tag{2}
$$

where the *convection enhanced* effective diffusion matrix $\mathcal{B}$ depends on t but is no longer oscillating in space with ε. A motivation for the present work is that it allows to homogenize structures which have periodic oscillations in some scales and random oscillations in other. A typical situation where periodic and random scales occur is the modeling of porous media. A meso-scale can be modeled as a periodic distribution of solid parts whereas a sub-scale on a finer level can be modeled by a certain random distribution. The homogenization problem for monotone operators in the random setting has been studied by Efendiev and Pankov, see [4] and the references therein. They consider single spatial and temporal scales but consider oscillations also in time. The corresponding multiscale situation is studied in [11]. For a careful study of convection enhanced diffusion for periodic flows we refer to [6].

2. Some basic notation

Let $(\Omega, \mathcal{F}, \mu)$ denote a probability space, where $\mathcal{F}$ is a complete σ-algebra and μ is a probability measure. With every $x \in \mathbb{R}^n$ we associate the dynamical system

$$T(x)\colon \ \Omega \to \Omega$$

where both $T(x)$ and $T(x)^{-1}$ are assumed to be μ-measurable. Moreover, we assume that the following (measure preserving) properties are satisfied:

- $T(0)\omega = \omega$ for each $\omega \in \Omega$.
- $T(x+y) = T(x)T(y)$ for $x, y \in \mathbb{R}^n$.
- The set $\{(x,\omega) \in \mathbb{R}^n \times \Omega\colon\ T(x)\omega \in F\}$ is a $\mathrm{d}x \times \mathrm{d}\mu(\omega)$ measurable subset of $\mathbb{R}^n \times \Omega$ for each $F \in \mathcal{F}$ where $\mathrm{d}x$ denotes the Lebesgue measure.
- For any measurable function $f(\omega)$ defined on Ω, the function $f(T(x)\omega)$ defined on $\mathbb{R}^n \times \Omega$ is also measurable where $\mathbb{R}^n$ is endowed with the Lebesgue measure.

The dynamical system T is said to be *ergodic* if every invariant function f (i.e. a function f which satisfies $f(T(x)\omega) = f(\omega)$) is constant almost everywhere in Ω.

Example 1 (periodic case). As a special case we recover the periodic functions by letting

$$\Omega = \{\omega \in \mathbb{R}^n:\ 0 \leqslant \omega_k \leqslant 1,\ k = 1, \ldots, n\} \quad \text{and} \quad T(x)\colon\ \Omega \to \Omega$$

given by

$$T(x)\omega = x + \omega \pmod 1.$$

For a random field $f(x,\omega)$ the "periodic" realization is given by $f(x+\omega)$.

Definition 1. We say that a vector field f is a potential field if there exists a function $g \in W_0^{1,p}(\mathbb{R}^n)$ such that $f = Dg$.

By $L^p(\Omega)$ we denote the equivalence class of all μ-integrable functions (with exponent $p \geqslant 1$). For each $f \in L^p(\Omega)$ the dynamical system $T(x)$ yields a function $x \mapsto f(T(x)\omega)$ on $\mathbb{R}^n$. We call this function the realization of f.

Definition 2. We say that a random vector field $f \in [L^p(\Omega)]^n$ is a potential field if almost all its realizations $f(T(x)\omega)$ are potential fields in the sense of Definition 1. We denote this field by $L^p_{\mathrm{pot}}(\Omega)$.

Definition 3. We also define the space of vector fields with mean value zero:

$$V_{\mathrm{pot}}(\Omega) = \left\{ f \in [L^p(\Omega)]^n:\ \langle f\rangle = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) = 0 \right\}.$$

We observe that by the Fubini Theorem it follows that if $f \in L^p(\Omega)$ then almost all realizations $f(T(\cdot)\omega) \in L^p(\mathbb{R}^n)$.

Definition 4. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The number $M(f)$ is called the mean value of f if

$$\lim_{\varepsilon\to 0} \int_K f(x/\varepsilon)\,\mathrm{d}x = |K|M(f)$$

for any Lebesgue measurable bounded set $K \in \mathbb{R}^n$. Alternatively the mean can be expressed in terms of weak convergence. If the family $\{f(\cdot/\varepsilon)\}$ is in $L^p(Q_x)$, $p \geqslant 1$ then $M(f)$ is called the mean value of f if

$$\{f(\cdot/\varepsilon)\} \rightharpoonup M(f) \quad \text{in } L^p(Q_x).$$

We can now formulate the important

Theorem 1 (Birkhoff Ergodic Theorem)**.** *Let $f \in L^p(\Omega)$, $p \geqslant 1$. Then for almost all $\omega \in \Omega$ the realization $f(T(x)\omega)$ possesses a mean value $M(f(T(x)\omega))$. Moreover, as a function of $\omega \in \Omega$, this mean value $M(f(T(x)\omega))$ is invariant and*

$$\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega M(f(T(x)\omega))\,\mathrm{d}\mu(\omega).$$

If the system $T(x)$ is ergodic then

$$\int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) = M(f(T(x)\omega)).$$

Now let $\{(\Omega_k, \mathcal{F}_k, \mu_k)\}_{k=1}^M$ denote a family of probability spaces, where each $\mathcal{F}_k$ is a complete σ-algebra and each μ_k is the associated probability measure. With every $x \in \mathbb{R}^n$ we also associate a dynamical system

$$T_k(x)\colon\ \Omega_k \to \Omega_k.$$

With this we can now formulate a multiscale extension of the Birkhoff ergodic theorem:

Theorem 2. *Let $f \in L^p(\Omega_1 \times \ldots \times \Omega_M)$, $p \geqslant 1$. Then for almost all $\omega_k \in \Omega_k$ the realization $f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)$ possesses a mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$. Moreover, as a function of $\omega_k \in \Omega_k$, this mean value $M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))$ is invariant and*

$$\begin{aligned}\langle f \rangle &\equiv \int_{\Omega_1} \ldots \int_{\Omega_M} f(\omega_1, \ldots, \omega_M)\,\mathrm{d}\mu_1(\omega_1) \ldots \mathrm{d}\mu_M(\omega_M) \\ &= \int_{\Omega_1} \ldots \int_{\Omega_M} M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M))\,\mathrm{d}\mu_1(\omega_1) \ldots \mathrm{d}\mu_M(\omega_M).\end{aligned}$$

If the systems $T_k(x)$ are ergodic then

$$\langle f \rangle = M(f(T_1(x)\omega_1, \ldots, T_M(x)\omega_M)).$$

We continue by setting the appropriate structure conditions:

Definition 5. Let $(\Omega_k, \mathcal{F}_k, \mu_k)$, $k = 1, 2, 3$, be three probability spaces. Given $0 < \delta \leqslant 1$, $2 \leqslant p < \infty$ and three positive real constants c_0, c_1 and c_2, we define the class $S^\omega = S^\omega(c_0, c_1, c_2, \delta)$ of maps

$$a\colon\ \Omega_1 \times \Omega_2 \times \Omega_3 \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfying

(i) $|a(\omega_1, \omega_2, \omega_3, t, 0)| \leqslant c_0$ a.e. in $\Omega_1 \times \Omega_2 \times \Omega_3 \times (0, T)$,

(ii) almost all realizations $a(\cdot, \cdot, \cdot, \cdot, \xi)$ are Lebesgue measurable for every $\xi \in \mathbb{R}^n$,

(iii) $|(a(\omega_1, \omega_2, \omega_3, t, \xi_1) - a(\omega_1, \omega_2, \omega_3, t, \xi_2)| \leqslant c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\delta}|\xi_1 - \xi_2|^\delta$, a.e. in $\Omega_1 \times \Omega_2 \times \Omega_3 \times (0, T)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$,

(iv) $(a(\omega_1, \omega_2, \omega_3, t, \xi_1) - a(\omega_1, \omega_2, \omega_3, t, \xi_2), \xi_1 - \xi_2) \geqslant c_2|\xi_1 - \xi_2|^p$, a.e. in $\Omega_1 \times \Omega_2 \times \Omega_3 \times (0, T)$ for all $\xi_1, \xi_2 \in \mathbb{R}^n$.

It easily follows that

$$|a(\omega_1, \omega_2, \omega_3, t, \xi)| \leqslant C(1 + |\xi|)^{p-1}, \tag{3}$$

$$|\xi|^p \leqslant C(1 + (a(\omega_1, \omega_2, \omega_3, t, \xi), \xi)). \tag{4}$$

Let us introduce some function spaces related to the differential equations studied in this paper. Let V be a reflexive real Banach space with dual V' and let H be a real Hilbert space. We introduce the triple

$$V \subseteq H \subseteq V'$$

with dense embeddings. Further, for positive T and for $2 \leqslant p < \infty$ let us introduce spaces $\mathcal{V} = L^p(0, T; V)$, $\mathcal{H} = L^2(0, T; H)$ and $\mathcal{V}' = L^q(0, T; V')$, where $1/p + 1/q = 1$. Then we can consider the corresponding evolution triple

$$\mathcal{V} \subseteq \mathcal{H} \subseteq \mathcal{V}'$$

also with dense embeddings where the duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ between $\mathcal{V}$ and $\mathcal{V}'$ is given by

$$\langle f, u \rangle_{\mathcal{V}} = \int_0^T \langle f(t), u(t) \rangle_V \,\mathrm{d}t, \quad \text{for } u \in \mathcal{V},\ \ f \in \mathcal{V}'.$$

We define spaces $\mathcal{W}$ and $\mathcal{W}_0$ as

$$\mathcal{W} = \{v \in \mathcal{V}\colon\ v' \in \mathcal{V}'\} \quad \text{and} \quad \mathcal{W}_0 = \{v \in \mathcal{W}\colon\ v(0) = 0\}.$$

Here v' denotes the time derivative of v, where this derivative is taken in distributional sense. Equipped with the graph norm

$$\|v\|_{\mathcal{W}_0} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}'}$$

$\mathcal{W}_0$ becomes a real reflexive Banach space. Moreover, since the embedding $\mathcal{W}_0 \to C(0,T;H)$ is continuous, every function in $\mathcal{W}_0$, with a possible modification on a set of measure zero, can be considered as a continuous function with values in H. Let us define the operator $\mathrm{d}/\mathrm{d}t\colon\ \mathcal{V} \to \mathcal{V}'$ given by

$$\frac{\mathrm{d}}{\mathrm{dt}}u = u' \quad \text{for } \ u \in \mathcal{W}_0.$$

We denote $V = W_0^{1,p}(Q_x)$ with the norm $\|u\|_V^p = \int_{Q_x} |Du|^p \,\mathrm{d}x$, $H = L^2(Q_x)$ and $V' = W^{-1,q}(Q_x)$. Then the evolution triples considered above are well-defined with dense embeddings. We define spaces

$$U = L^p(Q_x; \mathbb{R}^n) \quad \text{and} \quad U' = L^q(Q_x; \mathbb{R}^n)$$

and spaces

$$\mathcal{U} = L^p(0,T;U) \quad \text{and} \quad \mathcal{U}' = L^q(0,T;U').$$

3. Elliptic and parabolic homogenization

We are interested in the asymptotic behaviour (as $\varepsilon \to 0$) of the sequence $\{u_\varepsilon^\omega\}$ of solutions to the initial-boundary value problems

$$\text{(5)} \qquad \begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}\Big(a\Big(T_1\Big(\dfrac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\dfrac{x}{\varepsilon_2}\Big)\omega_2, T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3, t, \nabla u_\varepsilon^\omega\Big)\Big) = f_\varepsilon \ \text{ in } \ Q, \\ u_\varepsilon^\omega(0) = u_0^\omega \ \text{ in } \ Q_x, \\ u_\varepsilon^\omega \in L^p(0,T;V). \end{cases}$$

We assume for technical reasons that $2 \leqslant p < \infty$. The results can also be obtained for $1 < p < 2$, see [10]. We follow the idea in [11] and present first the homogenization problem for the corresponding elliptic problem and then use a comparison result. We now define an operator $A_\varepsilon^\omega\colon\ \mathcal{V} \to \mathcal{U}'$ as

$$\text{(6)} \qquad A_\varepsilon^\omega(x,t,\xi) = a\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3, t, \xi\Big).$$

With some abuse of notation we will say that A_ε^ω belongs to S^ω if the corresponding map a does. Then, (5) can be written as

$$\text{(7)} \qquad \begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}(A_\varepsilon^\omega(x,t,\nabla u_\varepsilon^\omega)) = f \ \text{ in } \ Q, \\ u_\varepsilon^\omega \in \mathcal{W}_0. \end{cases}$$

It is a standard result, see [12, Chapter 30], in the theory of monotone operators that (7) possesses a unique weak solution for a.e. $(\omega_1, \omega_2, \omega_3) \in \Omega_1 \times \Omega_2 \times \Omega_3$. Moreover, by Theorem 3.1 in [10] there exist subsequences such that

$$u_\varepsilon^\omega \rightharpoonup u \quad \text{in } \mathcal{W}_0$$

and

$$A_\varepsilon^\omega(x, t, \nabla u_\varepsilon^\omega) \rightharpoonup b(x, t, \nabla u) \quad \text{in } \mathcal{U}'.$$

We will now use the technique of an auxiliary local problem to construct b explicitly.

We begin with

Theorem 3. *Let us consider the sequence of parameter-dependent elliptic boundary value problems*

$$\begin{cases} -\operatorname{div}(A_\varepsilon^\omega(x, t, \nabla u_\varepsilon^\omega)) = f \ \text{ in } Q_x, \\ u_\varepsilon^\omega(\cdot, t) \in V, \ \ t \in [0, T]. \end{cases} \tag{8}$$

Assume that $A_\varepsilon^\omega \in S^\omega$ and that

$$|A_\varepsilon^\omega(x, t, \xi) - A_\varepsilon^\omega(x, s, \xi)| \leqslant \eta(t - s)(1 + |\xi|^{p-1})$$

where η is the modulus of continuity. Also assume that the underlying dynamical systems $T_1(x)$ and $T_2(x)$ are ergodic. Then

$$u_\varepsilon^\omega(\cdot, t) \rightharpoonup u \quad \text{in } V$$

and

$$A_\varepsilon^\omega(\cdot, t, \nabla u_\varepsilon^\omega) \rightharpoonup b(t, \nabla u) \quad \text{in } U',$$

where u is the solution to the homogenized problem

$$\begin{cases} -\operatorname{div}(b(t, \nabla u)) = f \ \text{ in } Q_x, \\ u(\cdot, t) \in V, \ \ t \in [0, T]. \end{cases} \tag{9}$$

The operator b is defined as

$$b(t, \xi) = \int_{\Omega_1} b_1(\omega_1, t, \xi + z_1^\xi(\omega_1, t)) \, \mathrm{d}\mu_1(\omega_1),$$

where $z_1^\xi(\omega_1, t) \in V_{\text{pot}}(\Omega_1)$ is the solution to the ε_1-scale local problem

$$\langle b_1(\omega_1, t, \xi + z_1^\xi(\omega_1, t), \Phi_1(\omega_1)\rangle = 0$$

for all $\Phi_1(\omega_1) \in V_{\text{pot}}(\Omega_1)$, $t \in [0, T]$. *The operator* b_1 *is defined as*

$$b_1(\omega_1, t, \xi) = \int_{\Omega_2} a(\omega_1, \omega_2, t, \xi + z_2^\xi(\omega_1, \omega_2, t))\,\mathrm{d}\mu_2(\omega_2),$$

where $z_2^\xi(\omega_1, \omega_2, t) \in V_{\text{pot}}(\Omega_2)$ *is the solution to the* ε_2*-scale local problem*

$$\langle b_2(\omega_1, \omega_2, t, \xi + z_2^\xi(\omega_1, \omega_2, t), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{\text{pot}}(\Omega_2)$, a.e. $\omega_1 \in \Omega_1$, $t \in [0, T]$. *The operator* b_2 *is defined as*

$$b_2(\omega_1, \omega_2, t, \xi) = \int_{\Omega_3} a(\omega_1, \omega_2, \omega_3, t, \xi + z_3^\xi(\omega_1, \omega_2, \omega_3, t))\,\mathrm{d}\mu_3(\omega_3),$$

where $z_3^\xi(\omega_1, \omega_2, \omega_3, t) \in V_{\text{pot}}(\Omega_3)$ *is the solution to the* ε_3*-scale local problem*

$$\langle a(\omega_1, \omega_2, \omega_3, t, \xi + z_3^\xi(\omega_1, \omega_2, \omega_3, t), \Phi_3(\omega_3)\rangle = 0$$

for all $\Phi_3(\omega_3) \in V_{\text{pot}}(\Omega_3)$, a.e. $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, $t \in [0, T]$.

Proof. For the proof we refer to [11]. □

We can now also state the following reiterated homogenization result:

Theorem 4. *Consider the initial-boundary value problem* (7)*:*

$$\begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}(A_\varepsilon^\omega(x, t, \nabla u_\varepsilon^\omega)) = f \ \text{ in } Q, \\ u_\varepsilon^\omega \in \mathcal{W}_0. \end{cases}$$

Under the same assumptions as in Theorem 3 *it holds true that as* $\varepsilon \to 0$,

$$u_\varepsilon^\omega \rightharpoonup u \quad \text{in } \mathcal{W}_0$$

and

$$A_\varepsilon^\omega(x, t, \nabla u_\varepsilon^\omega) \rightharpoonup b(t, \nabla u) \quad \text{in } \mathcal{U}',$$

where u *is the unique solution to the homogenized problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(b(t, \nabla u)) = f \ \text{ in } Q, \\ u(0) = u_0 \ \text{ in } Q_x, \\ u \in \mathcal{W}_0, \end{cases} \tag{10}$$

Proof. The proof follows by combining Theorem 3 above and the comparison result Theorem 8.2 in [10]. □

We are interested in the asymptotic behaviour (as $\varepsilon \to 0$) of the sequence of initial-boundary value problems

$$\text{(11)} \qquad \begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} + \dfrac{1}{\varepsilon_3}\mathcal{C}\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) \cdot \nabla u_\varepsilon^\omega \\ \qquad - \operatorname{div}\Big(\alpha\Big(T_1\Big(\dfrac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\dfrac{x}{\varepsilon_2}\Big)\omega_2, t\Big)\nabla u_\varepsilon^\omega\Big) = f_\varepsilon \text{ in } Q, \\ u_\varepsilon^\omega(0) = u_0^\omega \text{ in } \Omega, \\ u_\varepsilon^\omega \in L^2(0,T; H_0^1(\Omega)), \end{cases}$$

where $\mathcal{C}$ is a divergence free random vector field. We have chosen to consider in this presentation the linear case for the exposition of explicit homogenization formulae. The coefficients α are expected to satisfy the assumptions in Definition 5 with

$$\alpha(\omega_1, \omega_2, t)\xi = a(\omega_1, \omega_2, t, \xi),$$

where the variable ω_3 is just removed here. Theorem 5 below also holds true for monotone vector fields $\mathcal{C}$ and a. Since $\operatorname{div}\mathcal{C} = 0$ it is well known that there exists a skew-symmetric matrix S such that $\operatorname{div} S = \mathcal{C}$. In space dimension two we have

$$\text{(12)} \qquad S\Big(T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3\Big) = \begin{pmatrix} 0 & s\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) \\ -s\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) & 0 \end{pmatrix},$$

where s is the stream function corresponding to the field $\mathcal{C}$. In space dimension three there exists a vector potential $\mathbf{s} = (s_1, s_2, s_3)$ and a skew-symmetric matrix

$$\text{(13)} \quad S\Big(T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3\Big) = \begin{pmatrix} 0 & -s_3\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\Big)\omega_3 & s_2\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) \\ s_3\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) & 0 & -s_1\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) \\ -s_2\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) & s_1\Big(T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3\Big) & 0 \end{pmatrix}.$$

In the rest of this section we will consider the two-dimensional case. By using the stream function we can write the convection-diffusion equation as a parabolic divergence form initial-boundary value problem:

$$\text{(14)} \qquad \begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}\Big(\mathcal{A}\Big(T_1\Big(\dfrac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\dfrac{x}{\varepsilon_2}\Big)\omega_2, T_3\Big(\dfrac{x}{\varepsilon_3}\Big)\omega_3, t\Big)\nabla u_\varepsilon^\omega\Big) = f_\varepsilon \text{ in } Q, \\ u_\varepsilon^\omega(0) = u_0^\omega \text{ in } Q_x, \\ u_\varepsilon^\omega \in L^2(0,T; H_0^1(Q_x)), \end{cases}$$

where

$$\mathcal{A}\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3, t\Big)$$
$$= \begin{pmatrix} \alpha\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, t\Big) & -s\Big(T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3\Big) \\ s\Big(T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3\Big) & \alpha\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, t\Big) \end{pmatrix}.$$

We also define the realizations $\mathcal{A}_\varepsilon^\omega$ as

$$\mathcal{A}_\varepsilon^\omega(x,t) = \mathcal{A}\Big(T_1\Big(\frac{x}{\varepsilon_1}\Big)\omega_1, T_2\Big(\frac{x}{\varepsilon_2}\Big)\omega_2, T_3\Big(\frac{x}{\varepsilon_3}\Big)\omega_3, t\Big).$$

Before we state the main theorem we also define the appropriate class of coefficients:

Definition 6. We say that a matrix function $M(\omega_1,\omega_2,\omega_3,t) \in S^2$ if for $0 < \beta_1 < \beta_2 < \infty$ we have

$$\beta_1|\xi|^2 \leqslant \xi^T M\xi \leqslant \beta_2|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, with the transpose ξ^T, a.e. in $\Omega_1 \times \Omega_2 \times \Omega_3 \times (0,T)$.

We can now state the following theorem:

Theorem 5. *Consider the sequence of convection-diffusion equations*

$$\text{(15)} \qquad \begin{cases} \dfrac{\partial u_\varepsilon^\omega}{\partial t} - \operatorname{div}(\mathcal{A}_\varepsilon^\omega(x,t)\nabla u_\varepsilon^\omega) = f_\varepsilon \ \text{ in } \ Q, \\ u_\varepsilon^\omega(0) = u_0^\omega \ \text{ in } \ Q_x, \\ u_\varepsilon^\omega \in L^2(0,T; H_0^1(Q_x)). \end{cases}$$

Assume that $\mathcal{A}_\varepsilon^\omega \in S^2$ *and that*

$$|\mathcal{A}_\varepsilon^\omega(x,t) - \mathcal{A}_\varepsilon^\omega(x,s)| \leqslant \eta(t-s),$$

where η *is the modulus of continuity. Also assume that the underlying dynamical systems* $T_1(x)$, $T_2(x)$ *and* $T_3(x)$ *are ergodic. Then for every compact set* $\{f_\varepsilon\}$ *in* $L^2(0,T;H^{-1}(\Omega))$ *and for every* $\varepsilon > 0$, *there exists a unique solution* $u_\varepsilon^\omega \in L^2(0,T;H_0^1(Q_x))$ *to* (15) *and moreover, as* $\varepsilon \to 0$,

$$u_\varepsilon^\omega \rightharpoonup u \quad \text{in} \ \ L^2(0,T;H_0^1(Q_x))$$

and

$$\mathcal{A}_\varepsilon^\omega(\cdot,t)\nabla u_\varepsilon^\omega \rightharpoonup \mathcal{B}(t)\nabla u \quad \text{in} \ \ L^2(0,T;[L^2(Q_x)]^n),$$

where u is the solution to the homogenized problem

$$\text{(16)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(\mathcal{B}(t)\nabla u) = f & \textit{in } Q, \\ u \in L^2(0,T;H_0^1(Q_x)). \end{cases}$$

For a fixed $\xi \in \mathbb{R}^n$ the operator $\mathcal{B}(t)$ is defined as

$$\mathcal{B}(t)\xi = \int_{\Omega_1} \mathcal{B}_1(\omega_1,t)(\xi + z_1^{\xi}(\omega_1,t))\,\mathrm{d}\mu_1(\omega_1),$$

where $z_1^{\xi}(\omega_1,t) \in V_{\mathrm{pot}}(\Omega_1)$ is the solution to the ε_1-scale local problem

$$\langle \mathcal{B}_1(\omega_1,t)(\xi + z_1^{\xi}(\omega_1,t)), \Phi_1(\omega_1)\rangle = 0$$

for all $\Phi_1(\omega_1) \in V_{\mathrm{pot}}(\Omega_1)$, $t \in [0,T]$. The operator $\mathcal{B}_1(\omega_1,t)$ is defined as

$$\mathcal{B}_1(\omega_1,t)\xi = \int_{\Omega_2} \mathcal{B}_2(\omega_1,\omega_2,t,\xi + z_2^{\omega_1,\xi}(\omega_2,t))\,\mathrm{d}\mu_2(\omega_2),$$

where $z_2^{\omega_1,\xi}(\omega_2,t) \in V_{\mathrm{pot}}(\Omega_2)$ is the solution to the ε_2-scale local problem

$$\langle \mathcal{B}_2(\omega_1,\omega_2,t)(\xi + z_2^{\omega_1,\xi}(\omega_2,t)), \Phi_2(\omega_2)\rangle = 0$$

for all $\Phi_2(\omega_2) \in V_{\mathrm{pot}}(\Omega_2)$, a.e. $\omega_1 \in \Omega_1$, $t \in [0,T]$. The operator $\mathcal{B}_2(\omega_1,\omega_2,t)$ is defined as

$$\mathcal{B}_2(\omega_1,\omega_2,t)\xi = \int_{\Omega_3} \mathcal{A}(\omega_1,\omega_2,\omega_3,t,\xi + z_3^{\omega_1,\omega_2,\xi}(\omega_3,t))\,\mathrm{d}\mu_3(\omega_3),$$

where $z_3^{\omega_1,\omega_2,\xi}(\omega_3,t) \in V_{\mathrm{pot}}(\Omega_3)$ is the solution to the ε_3-scale local problem

$$\langle \mathcal{A}(\omega_1,\omega_2,\omega_3,t)(\xi + z_3^{\omega_1,\omega_2,\xi}(\omega_3,t)), \Phi_3(\omega_3)\rangle = 0$$

for all $\Phi_3(\omega_3) \in V_{\mathrm{pot}}(\Omega_3)$, a.e. $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, $t \in [0,T]$.

Proof. The proof follows to a large extent the proof of Theorem 7 in [11] (this is just Theorem 3 above). Just choose $p = 2$. □

Remark 1. The effective diffusion matrix $\mathcal{B}$ is called the convection enhanced diffusion matrix. We note that $\mathcal{A}_\varepsilon^\omega$ and $\mathcal{B}$ are both skew symmetric. This means that in order to practically satisfy the S^2-condition the convection field and the diffusion must balance. If the convection is far to large the existence breaks down.

R e m a r k 2. The homogenized matrices have the explicit form

$$\mathcal{B}_2(\omega_1,\omega_2,t)\xi = \int_{\Omega_3} \begin{pmatrix} \alpha(\omega_1,\omega_2,t) & -s(\omega_3) \\ s(\omega_3) & \alpha(\omega_1,\omega_2,t) \end{pmatrix} (\xi + z_3^{\omega_1,\omega_2,\xi}(\omega_3,t))\,\mathrm{d}\mu_3(\omega_3),$$

$$\mathcal{B}_1(\omega_1,t)\xi = \int_{\Omega_2} \begin{pmatrix} \alpha(\omega_1,\omega_2,t) & -\overline{s} \\ \overline{s} & \alpha(\omega_1,\omega_2,t) \end{pmatrix} (\xi + z_2^{\omega_1,\xi}(\omega_2,t))\,\mathrm{d}\mu_2(\omega_2),$$

$$\mathcal{B}(t)\xi = \int_{\Omega_1} \begin{pmatrix} b_1(\omega_1,t) & -\overline{s} \\ \overline{s} & b_1(\omega_1,t) \end{pmatrix} (\xi + z_1^{\xi}(\omega_1,t))\,\mathrm{d}\mu_1(\omega_1),$$

where $\overline{s}$ denotes the average over Ω_3.

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Author's address: *N. Svanstedt*, Department of Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden, e-mail: `nilss@math.chalmers.se`.